Mixed-characteristic shtukas

Shtukas are objects introduced by Drinfeld to prove the local Langlands correspondence for GL_2 over function fields (equicharacteristic case). This was later extended by others to all GL_n. Scholze has defined shtukas in mixed characteristic, i.e. p-adic shtukas, which one could expect to be used to give a proof of the local Langlands correspondence for GL_n over p-adic fields (this was proven earlier by Harris and Taylor by different methods). This definition of p-adic shtukas relies on the notion of diamonds.

1 Motivation: rank-1 shtukas and class field theory

What follows is an informal discussion of the case of GL_1, which is simply class field theory. The geometric viewpoint on this theory is due to Lang, Rosenlicht, Deligne, and others.

Let $X/\mathbf{F}_p$ a smooth projective curve and $K := K(X)$ its function field, with adèle ring $\mathbf{A}$. Unramified class field theory gives a canonical isomorphism

$$\left\{ \begin{array}{c} \text{idèle class characters} \\ K^\times \backslash \mathbf{A}^\times / \widehat{\mathcal{O}}^\times \to \mathbf{C}^\times \end{array} \right\} \simeq \left\{ \begin{array}{c} \text{Galois characters} \\ \mathrm{Gal}(K^{\mathrm{ur}}/K) \to \mathbf{C}^\times \end{array} \right\}$$

satisfying a compatibility condition: if idèle class character χ corresponds to Galois character ρ then

$$\chi(1, \dots, 1, \pi_x, 1, \dots) = \rho(\mathrm{Frob}_x)$$

for all closed points $x \in X$. This statement is naturally given a geometric interpretation: there is an isomorphism

$$\{\text{characters } \mathrm{Pic}(X) \to \overline{\mathbf{Q}}_\ell^\times\} \simeq \{\text{characters } \pi_1(X) \to \overline{\mathbf{Q}}_\ell^\times\}$$

also satisfying compatibility condition $\chi(\mathcal{O}(x)) = \rho(\mathrm{Frob}_x)$. The RHS of this equation can be interpreted as rank-1 étale local systems on X.

Drinfeld, reinterpreting work of the aformentioned figures, tells us how to assign a character of $\mathrm{Pic}\, X$ to a character of $\pi_1(X)$ using shtukas.

Definition 1.1. Let $S/\mathbf{F}_p$ a scheme and $\alpha, \beta \in X(S)$. A *Drinfeld shtuka of rank* 1 *with legs at* (α, β) is the data of line bundles $\mathcal{L}, \mathcal{L}'$ on $S \times_{\mathbf{F}_p} X$ and a diagram

$$\mathrm{Frob}_S^* \mathcal{L} \xrightarrow{i} \mathcal{L}' \xleftarrow{j} \mathcal{L}$$

such that $\mathrm{coker}\, i$ and $\mathrm{coker}\, j$ are supported on the graphs Γ_α and Γ_β, respectively. If $D \subseteq X$ is a divisor such that α, β are disjoint from D, a *level D structure* on a shtuka is a trivialization $h : \mathcal{L}|_{S \times D} \cong \mathcal{O}_{S \times D}$ compatible with Frob_S, in the sense that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Frob}_S^* \mathcal{L}|_{S \times D} & \longrightarrow & \mathcal{L}|_{S \times D} \\ & {\scriptstyle \mathrm{Frob}_S^* h} \searrow \quad \swarrow {\scriptstyle h} & \\ & \mathcal{O}_{S \times D} & \end{array}$$

Let $\mathscr{M}_D^1$ denote the moduli space of rank-1 Drinfeld shtukas with level D structure, let $\underline{\mathrm{Pic}}_D X$ denote the Picard scheme of line bundles on X trivialized over D, and $\mathrm{Pic}_D X = (\underline{\mathrm{Pic}}_D X)(\mathbf{F}_p)$ the group of line bundles on X trivialized over D. There is a commuting diagram:

$$\begin{array}{ccc} \mathscr{M}_D^1 & \xrightarrow{\mathcal{L}} & \underline{\mathrm{Pic}}_D X \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \mathrm{Frob}^* - \mathrm{id}} \\ (X-D)^2 & \longrightarrow & \underline{\mathrm{Pic}}_D X \end{array}$$

where π just sends a shtuka to its legs (α, β). There is a natural action of $\mathrm{Pic}_D X$ on $\mathscr{M}_D^1$, and consequently a natural action of $K^\times \backslash \mathbf{A}^\times = \varprojlim \mathrm{Pic}_D X$ on $\mathscr{M}^1 := \varprojlim \mathscr{M}_D^1$. In this way we obtain an action of $\mathbf{A}^\times$ on a certain moduli space $\mathscr{M}^1$ of shtukas.

Choose a point $p \in X - D$ disjoint from α, β. Then the subgroup J of $\underline{\mathrm{Pic}}_D X$ generated by $\mathcal{O}(p)$ acts on $\mathscr{M}_D^1$ equivariantly over $(X-D)^2$. So there is still a map, which by abuse we also denote $\pi : \mathscr{M}_D^1 / J \to (X-D)^2$.

Let $\underline{\mathbf{Q}}_\ell$ be the constant local system on $\mathscr{M}_D^1/J$. Then $\pi_* \underline{\mathbf{Q}}_\ell$ is a local system on $(X-D)^2$ and corresponds to a representation of $\pi_1(X-D) \times \pi_1(X-D)$. There is a subtlety here, we should actually expect this to be a representation of $\pi_1((X-D)^2)$ and this group is not in general isomorphic to $\pi_1(X-D) \times \pi_1(X-D)$. But in this case, we do in fact get a representation of $\pi_1(X-D) \times \pi_1(X-D)$ because of the way we have set up the Frobenius compatibility (this is the content of Drinfeld's lemma, see [3, Chapter 16] and [1] for more details). We also already had a natural action of $\mathrm{Pic}_D X$, so we have obtained a representation V_ℓ of

$$\pi_1(X-D) \times \pi_1(X-D) \times (\mathrm{Pic}_D X)/J.$$

Theorem 1.2. *We have a decomposition*

$$V_\ell \otimes \overline{\mathbf{Q}}_\ell = \bigoplus_{\chi : \mathrm{Pic}_D X / J \to \overline{\mathbf{Q}}_\ell^\times} \rho_\chi \otimes \rho_\chi^{-1} \otimes \chi$$

where ρ_χ is a character of $\pi_1(X-D)$ compatible with χ.

The desired correspondence is then obtained by assigning χ to ρ_χ. There are several things to prove of course, e.g. that each χ appears exactly once in this decomposition and that the decomposition takes this shape.

2 Equicharacteristic shtukas

Drinfeld used the same setup to give a proof of the Langlands correspondence for GL_2 over K, using Drinfeld shtukas of rank 2 instead of rank 1. The rest of the proof is similar, but there are additional technicalities in the geometry of the moduli space $\mathscr{M}_D^2$. Laurent Lafforgue pushed these techniques further to prove the Langlands correspondence for GL_n over function fields. Vincent Lafforgue gave a modified proof which applied to all reductive groups G. In this situation, it becomes necessary to allow for a more general definition of shtuka which allows for higher rank bundles and arbitrarily many legs. We will still focus on the case of GL_n, but will allow for these more general shtukas, of which Drinfeld shtukas are a special case.

Definition 2.1. Let $S/\mathbf{F}_p$ a scheme, $x_1, \ldots, x_m \in X(S)$, and $U = S \times_{\mathbf{F}_p} X - \bigcup_i \Gamma_{x_i}$. A *shtuka with legs at* $x_1, \ldots, x_m$ is a vector bundle $\mathcal{E}$ on $S \times_{\mathbf{F}_p} X$ with an isomorphism $\varphi : \mathrm{Frob}_S^* \mathcal{E}|_U \to \mathcal{E}|_U$.

The local version of this is naturally obtained from this definition: choose a closed point $x \in X(\mathbf{F}_p)$ and let $\widehat{X}$ be the formal completion of X along x. Then $\widehat{X} \cong \operatorname{Spf} \mathbf{F}_p[\![T]\!]$ is an adic space.

Definition 2.2. Let $S/\mathbf{F}_p$ an adic space, $x_1, \ldots, x_m \in \widehat{X}(S)$, and $U = S \times_{\mathbf{F}_p} \widehat{X} - \bigcup_i \Gamma_{x_i}$. A *local shtuka with legs at* $x_1, \ldots, x_m$ is a vector bundle $\mathcal{E}$ on $S \times_{\mathbf{F}_p} \widehat{X}$ with an isomorphism $\varphi : \mathrm{Frob}_S^* \mathcal{E}|_U \to \mathcal{E}|_U$ that is meromorphic along $\bigcup_i \Gamma_{x_i}$.

Remark. Let's recall what it means to be meromorphic along a Cartier divisor. Let Y be a uniform analytic adic space. On small enough open subsets, we can assume $Y = \operatorname{Spa}(R, R^+)$ and the Cartier divisor is determined by a non-zerodivisor $f \in R$. Let Z be the support of the divisor and $U = X \setminus Z$ the complement. There is a canonical map $R[f^{-1}] \hookrightarrow H^0(U, \mathcal{O}_U)$ which is not in general an isomorphism ([3, §5.3]). An element $g \in H^0(U, \mathcal{O}_U)$ is called *meromorphic* if it comes from $R[f^{-1}]$.

It then makes sense to ask for a map of vector bundles to be meromorphic along a Cartier divisor; after choosing trivializations locally over some small enough affinoid open V, such a map is given by an element of $\mathrm{Mat}_n(H^0(V, \mathcal{O}_V))$ and we can ask for the entries to be meromorphic.

In the case that $S = \operatorname{Spa} C$, where $C/\mathbf{F}_p$ is a nonarchimedean algebraically closed field, we have

$$S \times_{\mathbf{F}_p} \operatorname{Spa} \mathbf{F}_p[\![T]\!] = \operatorname{Spa} C \times_{\operatorname{Spa} \mathcal{O}_C} \operatorname{Spa} \mathcal{O}_C[\![T]\!] = \bigcup_n \operatorname{Spa} C \left\langle \frac{T}{\varpi^{1/n}} \right\rangle$$

is the open unit disk $\mathbf{D}_C$ over C. Here $C\langle T/\varpi^{1/n}\rangle$ is $A[1/\varpi]$ where A is the (ϖ, T)-adic completion of $\mathcal{O}_C[T/\varpi^{1/n}]$. A leg is given by a map $\mathbf{F}_p[\![T]\!] \to C$, which is exactly specified by an element of $C^{\circ\circ} = \mathfrak{m}_{\mathcal{O}_C} = \mathbf{D}_C(\operatorname{Spa} C)$. A local shtuka over S with legs at $x_1, \ldots, x_m \in \mathbf{D}_C(\operatorname{Spa} C)$ is therefore just a vector bundle $\mathcal{E}$ on the open unit disk $\mathbf{D}_C$ with a Frobenius linear endomorphism defined away from the geometric points $x_1, \ldots, x_m$, and meromorphic at these points. Here by "Frobenius linear" we have to be careful that Frobenius acts by pth powers on C but trivially on the free variable T.

A mixed characteristic shtuka will replace the smooth curve X with $X = \operatorname{Spec} \mathbf{Z}$, and the closed point x will be given by a prime number p. Then $\widehat{X} = \operatorname{Spf} \mathbf{Z}_p$. We would like to define local shtukas by making these substitutions everywhere in Definition 2.2. However, we immediately run into the problem that there is no suitable analog of $S \times_{\mathbf{F}_p} \widehat{X}$. There is no natural 0-dimensional object to replace $\mathbf{F}_p$ over which $\operatorname{Spf} \mathbf{Z}_p$ lives, and of course there is also the issue that S and $\operatorname{Spf} \mathbf{Z}_p$ have different characteristics. Once we have a suitable replacement for this product space, we will have to take some care to give an appropriate definition of legs.

The next section is dedicated to solving these two issues.

3 Mixed-characteristic shtukas

3.1 The product space $S \dot{\times} \operatorname{Spa} \mathbf{Z}_p$

The intuition is that given R of characteristic p, the tensor product "$R \otimes \mathbf{Z}_p$" should be a universal ring admitting a map from R and $\mathbf{Z}_p$. As we have already discussed, there is no nontrivial such object in the literal sense. However, $W(R)$ has most of the desired properties. It admits a map from $\mathbf{Z}_p$ (as topological rings) and admits a map from R (as topological multiplicative monoids).

If (R, R^+) is a Tate Huber pair with $\varpi \in R$ a pseudouniformizer, we should then expect "$\operatorname{Spa} R^+ \times \operatorname{Spa} \mathbf{Z}_p$" to be $\operatorname{Spa} W(R^+)$. So it is sensible to define $\operatorname{Spa}(R, R^+) \times \operatorname{Spa} \mathbf{Z}_p$ to be the subset of $\operatorname{Spa} W(R^+)$ where $[\varpi] \neq 0$. For technical reasons that we will soon see, it is best to restrict this definition to perfectoid Huber pairs.

Definition 3.1. Let (R, R^+) be a perfectoid Huber pair in characteristic p and $S = \operatorname{Spa}(R, R^+)$. We define the product space

$$S \dot{\times} \operatorname{Spa} \mathbf{Z}_p := \{[\varpi] \neq 0\} \subseteq \operatorname{Spa} W(R^+).$$

Since $\mathcal{O}$ and $\mathcal{O}^+$ are always sheaves on perfectoid spaces, this extends to a definition of $S \dot{\times} \operatorname{Spa} \mathbf{Z}_p$ for any $S \in \operatorname{Perf}$.

Proposition 3.2. *Let $S \in \operatorname{Perf}$.*

1. *The space $S \dot{\times} \operatorname{Spa} \mathbf{Z}_p$ is a uniform analytic adic space.*
2. *There is a canonical isomorphism of diamonds $(S \dot{\times} \operatorname{Spa} \mathbf{Z}_p)^\diamond \simeq S \times \operatorname{Spd} \mathbf{Z}_p$.*

Before discussing the proof of this result, let us observe that it implies that $\operatorname{Spd} \mathbf{Z}_p$ is an *absolute diamond* in the sense that $S \times \operatorname{Spd} \mathbf{Z}_p$ is a diamond for any $S \in \operatorname{Perf}$, even though $\operatorname{Spd} \mathbf{Z}_p$ is not itself a diamond. In fact, $\mathcal{D} \times \operatorname{Spd} \mathbf{Z}_p$ is a diamond for any diamond $\mathcal{D}$; this follows from the previous fact by choosing a quasi-pro-étale perfectoid cover $S \to \mathcal{D}$.

Proof of Proposition 3.2(2). Suppose we know $X = S \dot{\times} \operatorname{Spa} \mathbf{Z}_p$ exists as an analytic adic space. We want to show that $X^\diamond = S \times \operatorname{Spd} \mathbf{Z}_p$. We can assume $X = \operatorname{Spa}(R, R^+)$ is affinoid. Let $T = \operatorname{Spa}(A, A^+) \in \operatorname{Perf}$. By definition $X^\diamond(T)$ consists of untilts $T^\sharp = \operatorname{Spa}(A^\sharp, A^{\sharp+})$ together with a map $T^\sharp \to S \dot{\times} \operatorname{Spa} \mathbf{Z}_p$. Such a map is equivalent to a map $W(R^+) \to A^{\sharp+}$ such that $[\varpi]$ is invertible in $A^\sharp$. We claim that there is a bijection

$$\left\{ \begin{matrix} \text{maps } W(R^+) \to A^{\sharp+} \\ \text{where } [\varpi] \text{ invertible in } A^\sharp \end{matrix} \right\} \simeq \left\{ \begin{matrix} \text{maps } R^+ \to A^+ \\ \text{where } \varpi \text{ invertible in } A \end{matrix} \right\}$$

Given a map $W(R^+) \to A^{\sharp+}$ on the LHS, we see that $[\varpi]$ is a pseudouniformizer of $A^\sharp$ and reducing modulo $(p, [\varpi])$ gives a map $R^+/\varpi \to A^{\sharp+}/(p, [\varpi])$. Taking an inverse limit over Frobenius gives a map $R^+ \to A^{\sharp\flat+} = A^+$. In the other direction, given $R^+ \to A^+$, we obtain $W(R^+) \to W(A^+) \to A^{\sharp+}$ via the θ map.

So to give a map $W(R^+) \to A^{\sharp+}$ with $[\varpi]$ invertible is equivalent to giving a map $R^+ \to A^+$ extending to a map of Huber pairs $(R, R^+) \to (A, A^+)$. It follows that

$$X^\diamond(T) \simeq \{(T^\sharp, T \to S)\} \simeq (S \times \operatorname{Spd} \mathbf{Z}_p)(T).$$

□

We will not give the full proof that $S \dot{\times} \operatorname{Spa} \mathbf{Z}_p$ is an adic space, however it is worth mentioning the analogy with the equicharacteristic situation. Suppose that $S = \operatorname{Spa} C$ where $C/\mathbf{F}_p$ is nonarchimedean and algebraically closed. We saw earlier that

$$\operatorname{Spa} C \times_{\mathbf{F}_p} \operatorname{Spa} \mathbf{F}_p[\![T]\!] = \mathbf{D}_C = \bigcup_n \operatorname{Spa} C\left\langle \frac{T}{\varpi^{1/n}} \right\rangle$$

The RHS makes it clear that this is an analytic adic space. In the mixed characteristic situation, we have $\mathbf{F}_p[\![T]\!]$ replaced by $\mathbf{Z}_p$, and the corresponding decomposition is

$$\operatorname{Spa} C \dot{\times} \operatorname{Spa} \mathbf{Z}_p = \bigcup_n \operatorname{Spa}(R_n, R_n^+)$$

where $R_n^+ = \mathcal{O}_C \left\langle \frac{p}{[\varpi^{1/p^n}]} \right\rangle$, where the Tate algebra brackets here denote $[\varpi]$-adic completion of the polynomial algebra $W(\mathcal{O}_C)\left[\frac{p}{[\varpi^{1/p^n}]}\right]$, and $R_n = R_n^+[1/[\varpi]]$. To show that the resulting space is an analytic adic space, one shows that each R_n is sousperfectoid (see [3, Proposition 11.2.1]).

This is the mixed characteristic version of "open unit disk" though it is not literally an open unit disk in any real sense.

3.2 The correct notion of legs

A naïve definition of legs would be maps $S \to \operatorname{Spa} \mathbf{Z}_p$. However, as we have already discussed, every $S \in \mathrm{Perf}$ is fibered uniquely over $\operatorname{Spa} \mathbf{Z}_p$ and so this cannot be the correct analog. We might try to define the graph of a leg directly, i.e. sections $S \to S \dot{\times} \operatorname{Spa} \mathbf{Z}_p$, but this doesn't even make sense as there is no natural map $S \dot{\times} \operatorname{Spa} \mathbf{Z}_p \to S$. The correct replacement will be maps $S \to \operatorname{Spd} \mathbf{Z}_p$, which can equivalently be viewed as sections $S \to (S \dot{\times} \operatorname{Spa} \mathbf{Z}_p)^{\diamondsuit}$. By definition, these are exactly untilts $S^\sharp$ of S!

Given an untilt $S^\sharp$, we have to describe what we mean by the "graph" of $S^\sharp$, in analogy with Definition 2.2. It will be enough to describe this in the affinoid situation, so consider $S = \operatorname{Spa}(R, R^+)$ and suppose $S^\sharp = \operatorname{Spa}(R^\sharp, R^{\sharp+})$. We have a canonically defined map $\theta : W(R^+) \to R^{\sharp+}$. The image of $[\varpi]$ in $R^\sharp$ is $\varpi^\sharp$ which is invertible, so the resulting map $S^\sharp = \operatorname{Spa}(R^\sharp, R^{\sharp+}) \to \operatorname{Spa} W(R^+)$ factors through $S \dot{\times} \operatorname{Spa} \mathbf{Z}_p$. In other words, we have a diagram:

$$\begin{array}{ccc} \operatorname{Spa} R^{\sharp+} & \longrightarrow & \operatorname{Spa} W(R^+) \\ \uparrow & & \uparrow \\ S^\sharp & \dashrightarrow & S \dot{\times} \operatorname{Spa} \mathbf{Z}_p \end{array}$$

On the left, $S^\sharp$ sits inside $\operatorname{Spa} R^{\sharp+}$ as the locus where $\varpi \neq 0$ and on the right, $S \dot{\times} \operatorname{Spa} \mathbf{Z}_p$ sits inside $\operatorname{Spa} W(R^+)$ as the locus where $[\varpi] \neq 0$. So the dashed arrow actually exists as a well-defined map of adic spaces. We now state without proof (from [3, Proposition 11.3.1]).

Proposition 3.3. *Given $S^\sharp \in (\operatorname{Spd} \mathbf{Z}_p)(S)$, the resulting map $S^\sharp \to S \dot{\times} \operatorname{Spa} \mathbf{Z}_p$ is the inclusion of a closed Cartier divisor.*

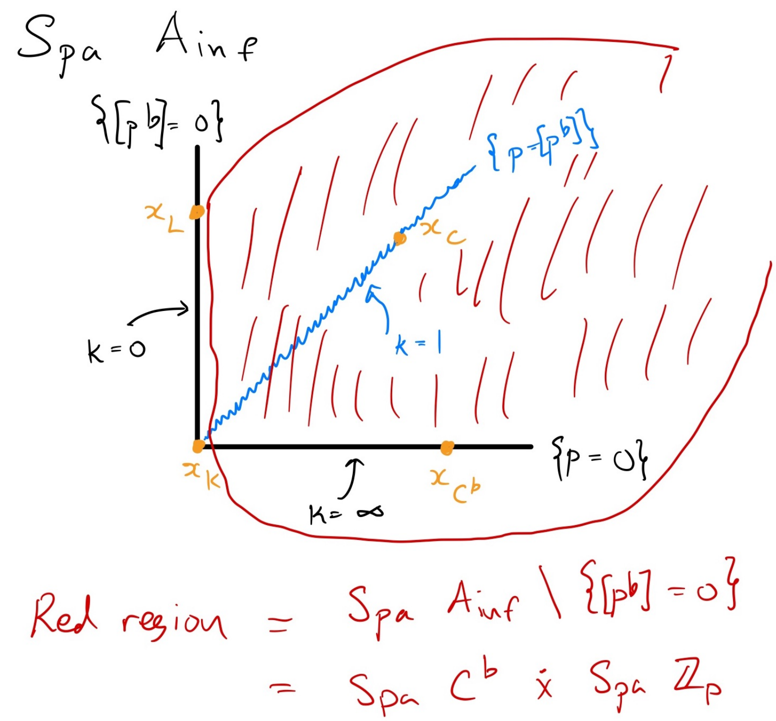
The intuition should be clear. The element ξ cuts out $\operatorname{Spa} R^{\sharp+}$ in $\operatorname{Spa} W(R^+)$, and ξ is a non-zerodivisor generating a closed ideal in $W(R^+)$. However, the subtleties in the definition of closed Cartier divisor make this quite technical to check, see [3, Proposition 11.3.1] for details.

Given $x \in (\operatorname{Spd} \mathbf{Z}_p)(S)$ corresponding to untilt $S^\sharp$, the graph of x will refer to the resulting map $\Gamma_x : S^\sharp \to S \dot{\times} \operatorname{Spa} \mathbf{Z}_p$. We now have all the ingredients we need to define shtukas in mixed characteristic.

Definition 3.4. Let $S \in \mathrm{Perf}$, $x_1, \ldots, x_m \in (\operatorname{Spd} \mathbf{Z}_p)(S)$, and $U = S \dot{\times} \operatorname{Spa} \mathbf{Z}_p - \bigcup_i \Gamma_{x_i}$. A *local shtuka with legs at* $x_1, \ldots, x_m$ is a vector bundle $\mathcal{E}$ on $S \dot{\times} \operatorname{Spa} \mathbf{Z}_p$ with an isomorphism $\varphi : \mathrm{Frob}_S^* \mathcal{E}|_U \to \mathcal{E}|_U$ that is meromorphic along $\bigcup_i \Gamma_{x_i}$.

Let $C/\mathbf{Q}_p$ be a complete nonarchimedean algebraically closed field. We will consider shtukas over $\operatorname{Spa} C^\flat$ with one leg at C. We can choose $\pi = p$ to be the pseudouniformizer of C and $p^\flat = (p, p^{1/p}, p^{1/p^2}, \cdots)$ to be that of its tilt. The untilt C of $C^\flat$ corresponds to an element $\xi = p - [p^\flat] \in A_{\mathrm{inf}} := W(\mathcal{O}_{C^\flat})$, primitive of degree 1. We will describe the data of a shtuka over $S = \operatorname{Spa} C^\flat$ with one leg at C. In this case $S \dot{\times} \operatorname{Spa} \mathbf{Z}_p$ is $\operatorname{Spa} A_{\mathrm{inf}} \setminus \{[p^\flat] = 0\}$ and a shtuka is a vector bundle on this space, Frobenius equivariant away from $\operatorname{Spa} C \hookrightarrow S \dot{\times} \operatorname{Spa} \mathbf{Z}_p$, i.e. away from the closed Cartier divisor determined by ξ. On the picture on the next page, this is the geometric point x_C.

We will need it later, so let us discuss the shape of $\operatorname{Spa} A_{\mathrm{inf}}$ in some more detail. There is the Frobenius endomorphism φ on $\operatorname{Spa} A_{\mathrm{inf}}$ coming from the Frobenius on $\mathcal{O}_{C^\flat}$. Let k be the residue field of $\mathcal{O}_{C^\flat}$, and $L = W(k)[1/p]$. There are points $x_k, x_L, x_C, x_{C^\flat}$ labeled on the figure below (subscript indicates the residue field of the point).



Similarly to the case of $\operatorname{Spa} \mathcal{O}_C[\![T]\!]$, we can look at $\mathcal{Y} = \operatorname{Spa} A_{\mathrm{inf}} \setminus \{x_k\}$, this is the complement of the non-analytic point. There is again a map $\kappa : \mathcal{Y} \to [0, \infty]$ given by

$$\kappa(x) = \log \frac{|[p^\flat](\widetilde{x})|}{|p(\widetilde{x})|}$$

where $\widetilde{x}$ is a maximal rank-1 generalization of x. We see that $\kappa(x_L) = 0$, $\kappa(x_C) = 1$, $\kappa(x_{C^\flat}) = \infty$. We also see that we have the identity $\kappa \circ \varphi = p\kappa$.

There is a notion from integral p-adic Hodge theory closely related to that of shtukas over $\operatorname{Spa} C^\flat$ with one leg at C, namely that of a *Breuil–Kisin–Fargues module*. This is a finite free A_{inf}-module M together with an isomorphism $\varphi_M : (F^*M)[\xi^{-1}] \to M[\xi^{-1}]$. What we have already seen is that a Breuil–Kisin–Fargues module canonically defines a shtuka over $\operatorname{Spa} C^\flat$ with one leg at C.

Theorem 3.5. *The natural functor from Breuil–Kisin–Fargues modules to shtukas over* $\operatorname{Spa} C^\flat$ *with one leg at* C *is an equivalence of categories.*

The nontrivial content of this theorem is that any shtuka, a priori defined only over the locus $\{[p^\flat] \neq 0\}$, can be extended over all of $\operatorname{Spa} A_{\text{inf}}$.

Observe that the definition of a Breuil–Kisin–Fargues module bears a striking resemblance to the notion of isocrystal over a perfect field of characteristic p. This suggests that shtukas over $\operatorname{Spa} C^\flat$ with one leg at C will be closely related to p-divisible groups.

References

[1] V. Drinfeld. Langlands' conjecture for GL(2) over functional fields.

[2] D. Hansen. Notes on diamonds, http://www.davidrenshawhansen.com/diamondnotes.pdf.

[3] P. Scholze and J. Weinstein. Berkeley lectures on p-adic geometry, 2014.