σ-conjugacy classes and the Kottwitz map

Let $F/\mathbf{Q}_p$ a finite extension, $\Gamma = \mathrm{Gal}(\overline{F}/F)$, $L \subseteq \overline{F}$ the maximal unramified extension of F. Let v denote the valuation on L, normalized so that $v(\pi_F) = 1$. To an algebraic group G/F we assign the abelian group $B(G)$ of σ-conjugacy classes of elements in $G(L)$. This is naturally a pointed set and can be interpreted as $H^1(\langle\sigma\rangle, G(L))$.

The goal of this note is to define a natural map

$$B(G) \longrightarrow X^*(Z(\widehat{G})^\Gamma)$$

which will be an isomorphism when G is a torus. We will focus on the case of tori, and then generalize. This is an exposition of (a subset of) Kottwitz's paper "Isocrystals with additional structure."

1 The Kottwitz map

1.1 Tori

We begin with the case of a torus T/F. Then we are looking to define a map

$$B(T) \longrightarrow X^*(\widehat{T}^\Gamma) \cong X_*(T)_\Gamma.$$

In fact we will define the isomorphism in the opposite direction.

Proposition 1.1. *There is a natural isomorphism of functors $A : X_*(-)_\Gamma \to B(-)$ from the category of tori over F to the category of abelian groups (unique up to negation).*

We normalize so that the resulting map $\mathrm{End}(\mathbf{G}_m) \to B(\mathbf{G}_m)$ sends 1 to 1.

Let us, for the moment, suppose we know that such an F exists. We will be able to compute an explicit formula for what it should be, and then check that this indeed works.

Lemma 1.2. *The functor $X_*(-)$ is pro-represented by the pro-torus*

$$\mathbf{T}_F := \varprojlim_{E/F} R_{E/F}\mathbf{G}_m$$

where the limit runs over finite Galois extensions E/F and the transition maps are norms.

Proof. For each E/F finite Galois let

$$\mu_E : \mathbf{G}_{m,E} \to (R_{E/F}\mathbf{G}_m)_E \cong \prod_{\Gamma_{E/F}} \mathbf{G}_{m,E}$$

be the map $a \mapsto (a, 1, 1, \dots)$ where the first slot corresponds to $1 \in \Gamma_{E/F}$. Let $\mu \in X_*(T)$ with field of definition K, so $\mu : \mathbf{G}_{m,K} \to T_K$. The key claim is that there is a unique map $f_\mu : R_{K/F}\mathbf{G}_m \to T$ such that $\mu = f_\mu \circ \mu_K$. This is equivalent to saying that there is a unique Galois-equivariant map

$$f_{\mu,K} : \prod_{\Gamma_{K/F}} \mathbf{G}_{m,K} \to T_K$$

which is μ when restricted to the first component. But $\Gamma_{K/F}$ acts on $\prod \mathbf{G}_{m,K}$ by $\sigma(x_g)_g = (x_{g\sigma})_g$, so the desired map is given explicitly as

$$f_{\mu,K}((x_g)_g) = \prod_g g^{-1}\mu(x_g).$$

Compatibility with the transition maps follows from the above stated uniqueness and the compatibility of the maps μ_E with the transition maps. □

Proposition 1.3. *Let T/F a torus and E/F a Galois extension splitting T. Let $E_0 = E \cap L$. Then for any $\mu \in X_*(T)$ we have*

$$A_T(\mu) = N_{E/E_0}\mu(\pi_E) \in T(E_0)$$

where π_E is any uniformizer of E.

Proof. By enlarging E, we can assume there is a map $f : R_{E/F}\mathbf{G}_m \to T$ such that $f \circ \mu_E = \mu$. By functoriality, we have a diagram

$$\begin{array}{ccc} \mathbf{Z}[\Gamma_{E/F}] & \longrightarrow & B(R_{E/F}\mathbf{G}_m) \\ \downarrow & & \downarrow{\scriptstyle f} \\ X_*(T) & \longrightarrow & B(T) \end{array}$$

We have an equality

$$f(N_{E/E_0}\mu_E(\pi_E)) = N_{E/E_0}f(\mu_E(\pi_E)) = N_{E/E_0}\mu(\pi_E)$$

(since f is defined over F, it is Galois-equivariant on E-points). So we are reduced to the case $T = R_{E/F}\mathbf{G}_m$. In this case, we have a commuting diagram

$$\begin{array}{ccc} \mathbf{Z}[\Gamma_{E/F}] & \longrightarrow & B(T) \\ \downarrow & & \downarrow{\scriptstyle N_{E/F}} \\ \mathbf{Z} & \longrightarrow & B(\mathbf{G}_m) \end{array}$$

and the right-hand downward arrow is an isomorphism. We have another commutative diagram as follows:

$$\begin{array}{ccc} T(E) & \xrightarrow{N_{E/E_0}} & T(E_0) \\ {\scriptstyle N_{E/F}}\downarrow & & \downarrow{\scriptstyle N_{E/F}} \\ \mathbf{G}_m(E) & \xrightarrow[N_{E/E_0}]{} & \mathbf{G}_m(E_0) \end{array}$$

(there are two different kinds of norm maps here, one $T \to \mathbf{G}_m$ denoted $N_{E/F}$ and one $E \to E_0$ denoted N_{E/E_0}). The key observation is that $N_{E/F}\mu_E(\pi_E) = \pi_E$, so

$$N_{E/F}(N_{E/E_0}\mu_E(\pi_E)) = N_{E/E_0}(\pi_E) = u\pi_F$$

for a p-adic unit u. So $N_{E/E_0}\mu_E(\pi_E)$ is the element of $B(T)$ that maps to the class of π_F in $B(\mathbf{G}_m)$, so we're done. □

Corollary 1.4. *Suppose E is unramified (so T is also unramified). Then $A_T(\mu) = \mu(\pi_F)$.*

So we now understand the map $X_*(-)_\Gamma \to B(-)$ completely explicitly for tori. It is easy to check that this is a natural transformation of functors. We also note that if $\mu = \mathrm{id} \in \mathrm{End}(\mathbf{G}_m)$ then we have $A_{\mathbf{G}_m}(\mathrm{id}) = 1 \in B(\mathbf{G}_m) \cong \mathbf{Z}$. It follows that $A_{\mathbf{G}_m}$ is bijective. It follows that A_T is bijective for any $T = R_{E/F}\mathbf{G}_m$ and consequently for any product of such tori.

Lemma 1.5. *For any torus T, the map $A_T : X_*(T)_\Gamma \to B(T)$ is surjective.*

Proof. There is some torus T' of the form $R_{E/F}\mathbf{G}_m$ which surjects onto T. Since $A_{T'}$ is an isomorphism and $B(T') \to B(T)$ is surjective, we get the desired conclusion via commutativity of the diagram

$$\begin{array}{ccc} X_*(T')_\Gamma & \longrightarrow & X(T)_\Gamma \\ \downarrow & & \downarrow \\ B(T') & \longrightarrow & B(T). \end{array}$$

□

Proposition 1.6. *For any torus T, the map $A_T : X(T)_\Gamma \to B(T)$ is bijective.*

Proof. Choose T' as in Lemma 1.5 and let T'' be the kernel of $T' \twoheadrightarrow T$. Then we have $A_{T''} : X_*(T'')_\Gamma \to B(T'')$ is surjective. Applying the four-lemma to the diagram

$$\begin{array}{ccccccc} X_*(T'')_\Gamma & \longrightarrow & X_*(T')_\Gamma & \longrightarrow & X(T)_\Gamma & \longrightarrow & 1 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ B(T'') & \longrightarrow & B(T') & \longrightarrow & B(T) & \longrightarrow & 1 \end{array}$$

yields the desired result. □

1.2 Groups with simply connected derived subgroup

We now want to extend this to reductive groups. For the sake of simplicity, we will assume that G_{der} is simply connected (which will suffice in many applications) but what follows can be done for a general connected reductive group. Define the torus $T := G/G_{\mathrm{der}}$. Then we have canonical maps

$$B(G) \longrightarrow B(T) \xrightarrow{\sim} X_*(T)_\Gamma \xrightarrow{\sim} X^*(\widehat{T}^\Gamma)$$

so it will suffice to see that $\widehat{T} = Z(\widehat{G})$. This is a general fact about reductive G with simply connected G_{der} (since the dual of a simply connected group is centerless).

In fact, this map is surjective and we can identify a natural subset of $B(G)$ which bijects onto $X^*(Z(\widehat{G})^\Gamma)$.

2 σ-conjugacy classes and the slope morphism

2.1 The slope morphism

We again begin by working in the category of tori over F. We show that there is a functorial homomorphism $B(-) \to X_*(-)^\Gamma_{\mathbf{Q}}$ for tori. This is easier; indeed an element $b \in T(L)$ defines an element of $\mathrm{Hom}(X^*(T)^\Gamma_{\mathbf{Q}}, \mathbf{Q}) \cong X_*(T)^\Gamma_{\mathbf{Q}}$ via

$$\lambda \mapsto v(\lambda(b)).$$

Since we are only looking at Galois equivariant characters, changing b to a σ-conjugate will change $\lambda(b)$ to a σ-conjugate, which does not change the valuation. So this is well-defined as a map from $B(T)$.

We want a generalization to connected reductive groups (at least with simply connected derived subgroup). Let $\mathbf{D}$ denote the pro-torus over F with $X^*(\mathbf{D}) = \mathbf{Q}$. Explicitly,

$$\mathbf{D} = \varprojlim_{n \in \mathbf{N}} \mathbf{G}_m$$

where the transition maps are the power maps. Then $X_*(T)^\Gamma_{\mathbf{Q}} = \mathrm{Hom}(\mathbf{D}, T)$. The correct generalization is then to show that $\mathbf{D}$ represents $B(-)$ in the category of connected reductive groups over F.

Let G be a reductive group, and let ρ be a faithful representation on an L-vector space V. There is also a natural map from $B(G)$ to isocrystal structures on V (a σ-semilinear endomorphism on V) where $g \in G(L)$ is assigned to the map $\Phi = g(1 \otimes \sigma) : V \otimes L \to V \otimes_\sigma L$. This gives rise to a slope decomposition on V, which is equivalent to a $\mathbf{Q}$-grading and hence an action of $\mathbf{D}$. So we get a corresponding map $\alpha_{\rho,g} \in \mathrm{Hom}(\mathbf{D}, \mathrm{GL}(V))$.

Thus, the choice of $g \in G(L)$ determines a map of Tannakian categories $\mathrm{Rep}(G) \to \mathrm{Rep}(\mathbf{D})$ and hence an element $\nu_g \in \mathrm{Hom}(\mathbf{D}, G)$. The assignment $g \mapsto \nu_g$ gives a functorial map $\nu : B(G) \to \mathrm{Hom}(\mathbf{D}, G)$.

2.2 Basic elements

Definition 2.1. We say that $g \in G(L)$ is *basic* if $\nu_g : \mathbf{D} \to G$ factors through $Z(G)$. We write $B(G)_b$ for the basic σ-conjugacy classes.

Now suppose G_{der} is simply connected and let $T = G/G_{\mathrm{der}}$ as before.

Proposition 2.2. *The map $B(G)_b \to B(T)$ is bijective. In particular, the restriction*

$$B(G)_b \to X_*(Z(\widehat{G})^\Gamma)$$

is bijective.

This identifies a natural subset of $B(G)$ which bijects onto the image of the Kottwitz map; we do not include the proof here.