Crystalline and semistable representations

We assume familiarity with de Rham representations. We will mostly follow [2].

The motivation for the discussion of crystalline representations is (very briefly) as follows. Let K be a p-adic field and A/K an abelian variety. Let ℓ be a prime different from p. One has the following very useful criterion for detecting good reduction:

Theorem 0.1 (Néron–Ogg–Shafarevich)**.** *A has good reduction if and only if the (rational) Tate module $V_\ell A$ is unramified.*

We could hope for the same to be true for the p-adic Tate module $V_p A$. Let $\rho : G_K \to \mathrm{GL}(V_p A)$ denote the representation map; then $\det \rho$ is the G_K-representation on $H^{2g}(A, \mathbf{Q}_\ell)$ given by the gth power of the cyclotomic character. The cyclotomic character, however, is not even potentially unramified.

Our goal, then, is to define a reasonable subcategory of p-adic G_K-representations, defined only in terms of Galois representations, such that A has good reduction if and only if $V_p A$ belongs to this subcategory. The correct subcategory is that of the *crystalline* representations, whose definition will arise later in the notes. This fits into Fontaine's period ring formalism; the crystalline representations will be those that are B_{cris}-admissible where B_{cris} is a certain G_K-regular period ring.

We note the following attractive feature of crystalline representations.

Theorem 0.2 (Colmez–Fontaine)**.** *The functor D_{cris} from $\mathrm{Rep}_{G_K}^{\mathrm{cris}}$ to the category of* weakly admissible filtered ϕ-modules over K *is an equivalence of categories.*

We have yet to define all the terms in this theorem, but the point is that we can give a robust description of crystalline representations in terms of (semi)linear algebraic data. We recall that the functor D_{dR} from de Rham representations to Fil_K is *not* fully faithful.

The *semistable* period ring B_{st} and the corresponding category of semistable representations serves a similar role for varieties with semistable reduction.

1 Isocrystals and filtered ϕ-modules

Let k be a perfect field of characteristic p and let $K_0 = W(k)[\frac{1}{p}]$.

Definition 1.1. An *isocrystal* over K_0 is a finite-dimensional K_0-vector space M together with a bijective Frobenius-semilinear endomorphism $\Phi : M \to M$.

We say M is *isotypic* of slope $\lambda \in \mathbf{Q}$ if there is a $W(k)$-lattice $L \subset M$ and integers $r \in \mathbf{Z}, s \in \mathbf{Z}_{>0}$ with $\Phi^s L = p^r L$ and $\lambda = r/s$.

We let $\mathbf{Mod}_{K_0}^{\phi}$ denote the category of isocrystals over K_0.

Remark. Let Γ be a p-divisible group over k. The (covariant) Dieudonné module $\mathbf{D}(\Gamma)$ is a free module over $W(k)$ equipped with a Frobenius-semilinear endomorphism $F : \mathbf{D}(\Gamma) \to \mathbf{D}(\Gamma)$ such that $pD \subseteq FD$. The functor $\Gamma \rightsquigarrow \mathbf{D}(\Gamma) \otimes K_0$ is a functor from the isogeny category of p-divisible groups over k to $\mathbf{Mod}_{K_0}^{\phi}$.

Remark. The category $\mathbf{Mod}_{K_0}^{\phi}$ is abelian.

Theorem 1.2 (Dieudonné–Manin)**.** *If k is algebraically closed, then $\mathbf{Mod}^{\phi}_{K_0}$ is semisimple with one simple object E_λ for each $\lambda \in \mathbf{Q}$. If $\lambda = r/s$ with $r \in \mathbf{Z}$, $s \in \mathbf{Z}_{>0}$ and $(r,s) = 1$, then* $\dim E_\lambda = s$.

Returning to our motivation, let us suppose that K is a p-adic field with residue field k and A is an abelian variety over K with good reduction. Let $\mathcal{A}/\mathcal{O}_K$ be an abelian scheme with generic fiber A. We have a comparison isomorphism

$$H^1_{\mathrm{dR}}(A/K) \cong K \otimes_{K_0} \mathbf{D}(\mathcal{A}_k[p^\infty])_{K_0}$$

(this will hold for any X/K with model $\mathcal{X}/\mathcal{O}_K$ if we replace the Dieudonné crystal by crystalline cohomology). The K-vector space $H^1_{\mathrm{dR}}(A/K)$ is naturally equipped with its Hodge filtration. We are led to introduce the following refined category of isocrystals.

Definition 1.3. A *filtered ϕ-module* over K is an isocrystal M over K_0 equipped with an exhaustive and separated filtration $\mathrm{Fil}^\bullet$ (indexed by $\mathbf{Z}$) of M_K.

We write MF^{ϕ}_K for the category of filtered ϕ-modules over K.

Remark. The category MF^{ϕ}_K is *not* abelian (we have no right to expect it to be so, since Fil_K is not abelian).

To define the notion of weak admissibility for a filtered ϕ-module, we recall the notions of convex polygons and in particular, the Hodge polygon and Newton polygon.

Definition 1.4. Let $\lambda_0 < \cdots < \lambda_n$ be a given sequence of real numbers and $d_0, \ldots, d_n$ a given sequence of positive integers. The *convex polygon with slopes $\{\lambda_i\}$ and multiplicities* $\{d_i\}$ is constructed recursively as follows.

- Draw the point $v_0 = (0,0)$ in the plane.
- For each $0 \leq i \leq n-1$, draw the line segment of horizontal distance d_i with slope λ_i starting at v_i, and label its endpoint as v_{i+1}.

The resulting object is a finite union of line segments in the plane. Given two convex polygons P and Q we say that $P \geq Q$ if P lies completely on or above Q.

Definition 1.5. Let L denote a field and $M = \bigoplus_{\lambda \in \mathbf{Q}} M_\lambda$ a finite-dimensional $\mathbf{Q}$-graded L-vector space. Let $\lambda_0 < \cdots < \lambda_r$ denote the values for which $M_\lambda \neq 0$ and for $0 \leq i \leq r$ set $d_i = \dim_L M_{\lambda_i}$. The *convex polygon $P(M)$ associated to M* is the convex polygon with slopes $\{\lambda_i\}$ and multiplicities $\{d_i\}$.

More generally if $(M, \mathrm{Fil}^\bullet)$ is a finite-dimensional filtered vector space over L indexed by $\mathbf{Q}$, we set $P(M) \coloneqq P(\mathrm{gr}\, M)$ (note that if the filtration on M is exhaustive and separated then the rightmost point has x-coordinate equal to $\dim M$).

Lemma 1.6. *The y-coordinate of the rightmost point of $P(M)$ coincides with that of $P(\det M)$.*

This is quite useful because it allows us to reduce some arguments to the case of 1-dimensional spaces.

Definition 1.7. Let M denote an object of MF^{ϕ}_K. There are two polygons associated to M:

- the *Hodge polygon* $P_H(M)$, associated to the given filtration on M_K, and
- the *Newton polygon* $P_N(M)$, associated to the slope grading on M.

We let $t_H(M)$ and $t_N(M)$ denote the y-coordinates of the rightmost endpoint of $P_H(M)$ and $P_N(M)$, respectively.

Example 1.8. Let $\mathcal{E}/\mathcal{O}_K$ be an elliptic curve, and $M = \mathbf{D}(\mathcal{E}_k)_{K_0}$. Then $M_K = H^1_{\mathrm{dR}}(E/K)$ is 2-dimensional and has a filtration with $\dim \mathrm{gr}^0 = \dim \mathrm{gr}^1 = 1$, so it has Hodge polygon:

If E has good ordinary reduction, then $P_N(E) = P_H(E)$, whereas if E has good supersingular reduction, then its Newton polygon looks like:

Definition 1.9. Let $M \in \mathrm{MF}_K^{\phi}$. We say that M is *weakly admissible* if the following hold:

1. $P_N(M)$ and $P_H(M)$ have the same right endpoint, and
2. for every subobject M' of M, we have $P_N(M') \geq P_H(M')$.

We let $\mathrm{MF}_K^{\phi,\mathrm{wa}}$ denote the full subcategory of weakly admissible filtered ϕ-modules.

Lemma 1.10. *Let $M \in \mathrm{MF}_K^{\phi}$. Then M is weakly admissible if and only if $t_N(M) = t_H(M)$ and for every subobject M' of M, we have $t_N(M') \geq t_H(M')$.*

Furthermore, M is weakly admissible if and only if $M_{\widehat{K}_0^{\mathrm{ur}}}$ is weakly admissible.

Theorem 1.11. *The category $\mathrm{MF}_K^{\phi,\mathrm{wa}}$ is abelian (even though MF_K^{ϕ} is not!).*

2 Crystalline representations

As we have seen, for a variety X/K with good reduction, the crystalline cohomology of $\mathcal{X}_k$ has the structure of a filtered ϕ-module. However, there is no Frobenius structure on B_{dR} so the functor D_{dR} is insufficient to study good reduction phenomena. To remedy this, we will construct a ring $B_{\mathrm{cris}} \subseteq B_{\mathrm{dR}}$ with the following properties:

1. B_{cris} is a G_K-stable $A_{\mathrm{inf}}[\frac{1}{p}]$-subalgebra of B_{dR} containing the element $t = \log([\varepsilon])$.
2. B_{cris} is G_K-regular with $B_{\mathrm{cris}}^{G_K} = K_0$.
3. The natural map $K \otimes_{K_0} B_{\mathrm{cris}} \to B_{\mathrm{dR}}$ is injective and induces an isomorphism of associated graded algebras (where the LHS has the subspace filtration).
4. There is an injective G_K-equivariant map $\phi : B_{\mathrm{cris}} \to B_{\mathrm{cris}}$ extending the Frobenius map on A_{inf}, and $\phi(t) = pt$.
5. We have an identification $(\mathrm{Fil}^0 B_{\mathrm{cris}})^{\phi=1} = \mathbf{Q}_p$. That is, the Frobenius fixed points in $B_{\mathrm{cris}} \cap B_{\mathrm{dR}}^+$ can be identified with $\mathbf{Q}_p$.

Before we proceed with the construction, let us see what such properties will give us.

Properties (1) and (2) tell us that we can apply the period ring formalism, so that we will have a functor $V \rightsquigarrow D_{\mathrm{cris}}(V)$ that lands in the category of finite-dimensional K_0-vector spaces, and we obtain a category $\mathrm{Rep}_{G_K}^{\mathrm{cris}}$ of B_{cris}-admissible representations, called *crystalline representations.*

Via the injection $K \otimes_{K_0} B_{\mathrm{cris}} \hookrightarrow B_{\mathrm{dR}}$ and the Frobenius structure on B_{cris}, we see that D_{cris} factors through MF_K^{ϕ}. Henceforth we consider D_{cris} as a functor $\mathrm{Rep}_{G_K}^{\mathrm{cris}} \to \mathrm{MF}_K^{\phi}$.

Proposition 2.1. *Suppose k is algebraically closed. Then the crystalline characters of G_K are exactly the Tate twists.*

Proof. Let $\psi : G_K \to \mathbf{Q}_p^{\times}$ be a crystalline character acting on 1-dimensional $\mathbf{Q}_p$-vector space V. After a Tate twist, we may assume that ψ has Hodge–Tate weight 0. It follows from Sen–Tate theory that $\psi(G_K)$ is finite and hence ψ has open kernel and is trivialized over a finite extension L/K. Then we have

$$\begin{aligned} D_{\mathrm{cris}}(\psi) &= (B_{\mathrm{cris}} \otimes_{\mathbf{Q}_p} V)^{G_K} \\ &= ((B_{\mathrm{cris}} \otimes_{\mathbf{Q}_p} V)^{G_L})^{\mathrm{Gal}(L/K)} \\ &= (K_0 \otimes_{\mathbf{Q}_p} V)^{\mathrm{Gal}(L/K)} \\ &= K_0 \otimes_{\mathbf{Q}_p} V^{\mathrm{Gal}(L/K)} \end{aligned}$$

so we have

$$\dim_{K_0} D_{\mathrm{cris}}(\psi) = \dim_{\mathbf{Q}_p} V^{\mathrm{Gal}(L/K)} \leq \dim_{\mathbf{Q}_p} V = \dim_{K_0} D_{\mathrm{cris}}(\psi)$$

so we conclude that $V^{\mathrm{Gal}(L/K)} = V$ and thus ψ is trivial.

To finish the proof, it will suffice to compute $D_{\mathrm{cris}}(\mathbf{Q}_p(r))$. By definition, it is the space $B_{\mathrm{cris}}[-r]$ on which G_K acts by $\chi_{\mathrm{cycl}}^{-r}$, and we know this space is at most 1-dimensional. On the other hand, we know that G_K acts on t by the cyclotomic character, so we see that $D_{\mathrm{cris}}(\mathbf{Q}_p(r)) = K_0 t^{-r}$. This is the filtered isocrystal of dimension 1, slope $-r$, and filtration supported in degree $-r$. $\square$

Proposition 2.2. *The functor D_{cris} has the following properties:*

1. *The map $K \otimes_{K_0} D_{\mathrm{cris}}(V) \to D_{\mathrm{dR}}(V)$ is an isomorphism in Fil_K and V is de Rham.*

2. *For $V \in \mathrm{Rep}_{G_K}^{\mathrm{cris}}$, the comparison isomorphism*

$$\alpha : B_{\mathrm{cris}} \otimes_{K_0} D_{\mathrm{cris}}(V) \xrightarrow{\sim} B_{\mathrm{cris}} \otimes_{\mathbf{Q}_p} V$$

   is Frobenius-equivariant, G_K-equivariant, and the base change α_K is an isomorphism of filtered K-vector spaces.

3. *D_{cris} is fully faithful.*

Proof. The map $K \otimes_{K_0} D_{\text{cris}}(V) \to D_{\text{dR}}(V)$ is injective since $K \otimes_{K_0} B_{\text{cris}} \to B_{\text{dR}}$ is injective. So we have

$$\begin{aligned}\dim_{K_0} D_{\text{cris}}(V) &= \dim_K(K \otimes_{K_0} D_{\text{cris}}(V)) \\ &\le \dim_K D_{\text{dR}}(V) \\ &\le \dim_{\mathbf{Q}_p} V \\ &= \dim_{K_0} D_{\text{cris}}(V)\end{aligned}$$

so we conclude that all the inequalities are equalities and in particular, $K \otimes_{K_0} D_{\text{cris}}(V) \to D_{\text{dR}}(V)$ is an isomorphism and V is de Rham.

The second statement follows from the first as follows. We only need to check that α_K is a filtered isomorphism, so we need to check that $\operatorname{gr} \alpha_K$ is an isomorphism. By property (3) and the second statement we can write α_K as

$$(K \otimes_{K_0} B_{\text{cris}}) \otimes_K D_{\text{dR}}(V) \xrightarrow{\sim} (K \otimes_{K_0} B_{\text{cris}}) \otimes_{\mathbf{Q}_p} V$$

whose associated graded is then

$$\operatorname{gr}(B_{\text{dR}} \otimes_K D_{\text{dR}}(V)) \to \operatorname{gr}(B_{\text{dR}} \otimes_{\mathbf{Q}_p} V).$$

So the statement reduces to the corresponding statement for B_{dR}.

For fully faithfulness, it is enough to define a functor $V_{\text{cris}} : \mathrm{MF}_K^{\phi} \to \mathrm{Rep}_{G_K}$ such that

$$V_{\text{cris}}(D_{\text{cris}}(V)) = V$$

for all $V \in \mathrm{Rep}_{G_K}^{\text{cris}}$. This is provided by the Frobenius structure via property (5). We set

$$V_{\text{cris}}(M) = \mathrm{Fil}^0(B_{\text{cris}} \otimes_{K_0} M)^{\phi=1}.$$

Then we compute for V crystalline:

$$\begin{aligned}V_{\text{cris}}(D_{\text{cris}}(V)) &= \mathrm{Fil}^0(B_{\text{cris}} \otimes_{K_0} D_{\text{cris}}(V))^{\phi=1} \\ &= \mathrm{Fil}^0(B_{\text{cris}} \otimes_{\mathbf{Q}_p} V)^{\phi=1} \\ &= \mathrm{Fil}^0(B_{\text{cris}})^{\phi=1} \otimes_{\mathbf{Q}_p} V \\ &= V.\end{aligned}$$

□

Proposition 2.3. *D_{cris} factors through $\mathrm{MF}_K^{\phi,\text{wa}}$.*

Proof. By Lemma 1.10 we may assume that k is algebraically closed.

We need to show that for any subobject M' of M, we have $t_N(M') \ge t_H(M')$ with equality when $M = M'$. We will reduce to the case that $\dim M' = 1$. Indeed, let $m' = \dim M'$. Since D_{cris} is compatible with tensors, quotients, etc. the desired statement for $\wedge^{m'} V$ and subobject $\det M' \subseteq \wedge^{m'} M$ implies the statement for V and subobject $M' \subseteq M$.

Thus we are reduced to the case of $M' \subseteq M$ such that $\dim M' = 1$. Again by Tate twisting, we may suppose that $t_H(M') = 0$.

If $M' = M$, then by the above example, M is the trivial representation so we are done.

If $M' \neq M$, let $x \in M'$ be a K_0-basis for M'. We can write $\phi(x) = \lambda x$ for some $\lambda \in K_0$; since k is algebraically closed we may assume that $\lambda = p^n$ for some n (by scaling x). We also know $x \in (\mathrm{Fil}^0(B_{\mathrm{dR}}) \otimes_{\mathbf{Q}_p} V) \setminus (\mathrm{Fil}^1(B_{\mathrm{dR}}) \otimes_{\mathbf{Q}_p} V)$. Let $\{v_1, \ldots, v_n\}$ a $\mathbf{Q}_p$-basis of V, and write

$$x = \sum b_i \otimes v_i$$

for $b_i \in B_{\mathrm{cris}}$. Then $\phi(b_i) = p^n b_i$ for all i, and there is at least one value i_0 for which $b_{i_0} \notin \mathrm{Fil}^1(B_{\mathrm{dR}}) \otimes_{\mathbf{Q}_p} V$.

Suppose (towards a contradiction) that $n < 0$. Letting $b' = b_{i_0}/t^n$ we see that $\phi(b') = b'$ and $b' \in \mathrm{Fil}^{-n}(B_{\mathrm{cris}}) \subseteq \mathrm{Fil}^0(B_{\mathrm{cris}})$ but $b' \notin \mathrm{Fil}^{-n+1}(B_{\mathrm{cris}})$. Property (5) tells us that $b' \in \mathbf{Q}_p$. But $\mathbf{Q}_p \cap \mathrm{Fil}^1(B_{\mathrm{cris}}) = 0$ so $b' = 0$, so $b_{i_0} = 0$, a contradiction. So $n \geq 0$, as desired. □

Recall from the introduction that in fact the following optimal statement is true! But we will not prove it here.

Theorem 2.4 (Colmez–Fontaine). *$D_{\mathrm{cris}} : \mathrm{Rep}^{\mathrm{cris}}_{G_K} \to \mathrm{MF}^{\phi,\mathrm{wa}}_K$ is an equivalence of categories.*

3 Semistable representations

For this we take a brief aside into the theory of ℓ-adic Galois representations.

Definition 3.1. An ℓ-adic G_K-representation is called *semistable* if the inertia subgroup acts unipotently.

Theorem 3.2 (Grothendieck). *If $K/\mathbf{Q}_p$ is finite then any ℓ-adic G_K-representation is potentially semistable.*

(In fact the theorem is somewhat more general in terms of which fields K are allowed.)

For any p-adic field K we can see that the tame inertia can be decomposed as $I_K^t = \prod_{q \neq p} \mathbf{Z}_q(1)$. Any ℓ-adic Galois representation (ρ, V) is also potentially tame, and after passing to a further finite extension it will act through the pro-ℓ quotient $I^t_{K,\ell} \cong \mathbf{Z}_\ell(1)$.

Hence we will assume we have a semistable representation V whose inertial action factors through $\mathbf{Z}_\ell(1)$. Since the inertia acts unipotently we see that for all $g \in I^t_{K,\ell}$ we have $\rho(g) = \exp(c(g)N)$ where $N \in \mathrm{Hom}(V, V(-1))$ is nilpotent and $c(g)$ is the scalar corresponding to g under the identification $I^t_{K,\ell} \cong \mathbf{Z}_\ell(1)$. We call such N the *monodromy operator* associated to V. Observe that if V is unramified we have $N = 0$.

The most naively stated monodromy theorem for p-adic Galois representations will of course be false. We will define a suitable analogous notion of semistability for p-adic Galois representations such that we obtain an analogous theorem for the class of de Rham representations. Varieties over K with semistable reduction will produce semistable representations in this setup. With some care we see that their p-adic étale cohomology produces the following structure.

Definition 3.3. A *filtered* (ϕ, N)-module over K is a filtered ϕ-module M together with a K_0-linear "monodromy operator" $N : M \to M$ such that $N\phi = p\phi N$. We write $\mathrm{MF}^{\phi,N}_K$ for the resulting category.

To define the notion of semistability, we will construct a period ring B_{st} with the following properties:

1. There are G_K-stable inclusions $B_{\mathrm{cris}} \hookrightarrow B_{\mathrm{st}} \hookrightarrow B_{\mathrm{dR}}$.
2. B_{st} is G_K-regular with $B_{\mathrm{st}}^{G_K} \cong K_0$.
3. The natural map $K \otimes_{K_0} B_{\mathrm{st}} \to B_{\mathrm{dR}}$ is injective (and thus induces an isomorphism on associated gradeds).
4. There is an injective map $\phi : B_{\mathrm{st}} \to B_{\mathrm{st}}$ extending the corresponding map on B_{cris}.
5. There is an element $Y \in B_{\mathrm{st}}$ such that $\phi(Y) = pY$, and the map $B_{\mathrm{cris}}[X] \to B_{\mathrm{st}}$ of B_{cris}-algebras sending $X \mapsto Y$ is an isomorphism.

From this structure we can construct $N : B_{\mathrm{st}} \to B_{\mathrm{st}}$ given by $N = -\frac{d}{dY}$. By construction, $N\phi = p\phi N$. Consequently the resulting functor D_{st} has natural target $\mathrm{MF}_K^{\phi,N}$. We also note that $B_{\mathrm{st}}^{N=0} = B_{\mathrm{cris}}$. The ring B_{st} gives rise to the category $\mathrm{Rep}_{G_K}^{\mathrm{st}}$ of *semistable* representations and a functor $D_{\mathrm{st}} : \mathrm{Rep}_{G_K}^{\mathrm{st}} \to \mathrm{MF}_K^{\phi,N}$.

Theorem 3.4. *For G_K-representations we have the sequence of implications*

$$\text{crystalline} \implies \text{semistable} \implies \text{de Rham} \implies \text{Hodge–Tate.}$$

Proof. We only have to prove the first two arrows. Let V be a crystalline representation; then we have

$$\begin{aligned}
\dim_{K_0} D_{\mathrm{cris}}(V) &= \dim_{K_0} D_{\mathrm{st}}(V)^{N=0} \\
&\leq \dim_{K_0} D_{\mathrm{st}}(V) \\
&\leq \dim_{\mathbf{Q}_p} V \\
&= \dim_{K_0} D_{\mathrm{cris}}(V)
\end{aligned}$$

so we conclude that all inequalities are equalities; in particular $N = 0$ on $D_{\mathrm{st}}(V)$ and V is semistable.

The proof that semistable implies de Rham proceeds exactly as in Proposition 2.2. $\square$

Similarly to the crystalline case we get a linear algebra classification of semistable representations.

Theorem 3.5. *The functor D_{st} is fully faithful with essential image $\mathrm{MF}_K^{\phi,N,\mathrm{wa}}$.*

By the proof of Theorem 3.4 we can assert that the essential image of the crystalline representations are those filtered (ϕ, N)-modules for which $N = 0$.

Most importantly, we can state a deep theorem which serves as a p-adic analogue of the unipotent monodromy theorem, known as *Fontaine's semistability conjecture.*

Theorem 3.6 (Berger, André–Kedlaya–Mebkhout)**.** *A representation is de Rham if and only if it is potentially semistable.*

4 Crystalline and semistable period rings

Having deduced several formal consequences of the existence of period rings B_{cris} and B_{st} with certain properties, it remains to actually construct such rings. This aspect of the theory is the most technical; we will have to proceed in several steps.

We let $I \subseteq A_{\text{inf}}$ denote the ideal $\ker\theta$, which is principal and generated by any choice of distinguished element ξ. Recall that B_{dR}^+ was defined to be the I-adic completion of $A_{\text{inf}}[\frac{1}{p}]$. We define the subring A_{cris} of B_{dR}^+ via

$$A_{\text{cris}} := \left\{ \sum_{n \geq 0} a_n \frac{\xi^n}{n!} \,\middle|\, a_n \in A_{\text{inf}}, a_n \to 0 \ p\text{-adically} \right\}$$

Alternatively, this can be characterized as the p-adic completion of the divided power envelope of A_{inf} with respect to I. We claim that the natural Frobenius on $A_{\text{inf}}[\frac{1}{p}]$ extends to A_{cris}. To verify this, we note that $\phi(\xi) \equiv \xi^p \mod pA_{\text{inf}}$ so we can write $\phi(\xi) = \xi^p + pa$ for some $a \in A_{\text{inf}}$. We can rewrite this in the form

$$\phi(\xi) = p \cdot \left(a + (p-1)! \cdot \frac{\xi^p}{p!} \right) \qquad \Longrightarrow \qquad \phi\left(\frac{\xi^n}{n!}\right) = \frac{p^n}{n!} \cdot \left(a + (p-1)! \cdot \frac{\xi^p}{p!} \right)^n$$

so we conclude that A_{cris} has a natural Frobenius map extending that on A_{inf}.

Definition 4.1. We set $B_{\text{cris}}^+ := A_{\text{cris}}[\frac{1}{p}] \subseteq B_{\text{dR}}^+$.

Lemma 4.2. *The element $t \in B_{\text{dR}}^+$ belongs to A_{cris} and $\phi(t) = pt$.*

Proof sketch. That $t \in A_{\text{cris}}$ follows from the fact that $[\varepsilon] - 1 = a'\xi$ for some $a' \in A_{\text{inf}}$; we can take $a_n = (n-1)!(-a')^n$.

Since $t = \log([\varepsilon])$ with $[\varepsilon] \in A_{\text{inf}}$ a Teichmüller lift, we expect $\phi(t) = pt$. To make this rigorous requires slightly more care working with the power series. $\square$

Definition 4.3. We set $B_{\text{cris}} := B_{\text{cris}}^+[\frac{1}{t}]$.

Verifying properties (1)–(5) of B_{cris} is quite tricky and involved; we omit the proofs (some partial proofs appear in [2], with references to the literature). We note that B_{cris} is not a field.

To construct B_{st} we consider an element

$$Y = \log[p^\flat] := \sum_{n \geq 1} \frac{(1 - [p^\flat]/p)^{n-1}}{n}.$$

This is a well-defined element of B_{dR}^+. Again, it takes some additional technique to conclude that this Y is transcendental over B_{cris}. We *impose* that $\phi(Y) = pY$.

Let us examine how G_K acts on Y. For each $g \in G_K$ we have

$$g[p^\flat] = [p^\flat] \cdot [\varepsilon]^{c(g)}$$

for some $c(g) \in \mathbf{Z}_p$. So we have

$$gY = Y + c(g)t.$$

Remark. This is not the only valid choice of Y, i.e. B_{st} is not canonically defined as a subring of B_{dR}.

5 Example: the Tate curve

Let $E := \mathbf{C}_K^\times/p^{\mathbf{Z}}$ denote the (rigid analytic) Tate curve with parameter p. Then E has an algebraization over K with bad semistable reduction.

The p^n-power torsion of E is given by

$$E[p^n](\overline{K}) = \{\zeta_{p^n}^a (p^{1/p^n})^b\}_{a,b\in\mathbf{Z}/p^n\mathbf{Z}}$$

for some choices of ζ_{p^n}, p^{1/p^n}. So we see that

$$T_pE \cong \mathbf{Z}_p\langle e, f\rangle$$

where e corresponds to the system of p-power roots of unity for ε and f corresponds to a system of p-power roots of p for $p^\flat$. In this basis we have that the G_K-representation $V = T_pE$ is given by

$$g \mapsto \begin{bmatrix} \chi_{\mathrm{cycl}}(g) & c(g) \\ & 1 \end{bmatrix}.$$

Using this explicit description, we see that the elements of $B_{\mathrm{st}} \otimes_{\mathbf{Q}_p} V$ given by

$$t^{-1} \otimes e, \qquad -Yt^{-1} \otimes e + 1 \otimes f$$

are Galois invariant elements. So $D_{\mathrm{st}}(V)$ is spanned by these, and we see that V is semistable but not crystalline.

It can be checked in the same way that any parameter $q \in \mathcal{O}_K$ with $|q| < 1$ yields a semistable G_K-representation.

Remark. Historically, Fontaine constructed B_{st} from B_{cris} by adjoining the necessary periods coming from Tate curves. So we have done this a bit backwards.

References

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[2] O. Brinon and B. Conrad. CMI Summer School Notes on p-adic Hodge theory.

[3] J.-M. Fontaine and Y. Ouyang. Theory of p-adic Galois representations.