TESTING k-PLANARITY IS NP-COMPLETE

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ABSTRACT. For all $k \geq 1$, we show that deciding whether a graph is k-planar is NP-complete, extending the well-known fact that deciding 1-planarity is NP-complete. Furthermore, for any $k \geq 1$, we show that, given a graph with local crossing number either at most k or at least $2k$, it is NP-complete to decide whether the local crossing number is at most k or at least $2k$. Finally, we present results regarding the non-existence of drawings that simultaneously approximately minimize both the crossing number and local crossing number of a graph.

1. PRELIMINARIES

A planar drawing of an undirected graph $G = (V, E)$ is an injective mapping from the vertex set V to $\mathbb{R}^2$ paired with an open Jordan curve between u and v for all $(u, v) \in E$, with the condition that each vertex only intersects a curve at its endpoint. Graph drawing is a central area of research in graph theory, with applications to network analysis, bioinformatics, circuit schematics, and software engineering, among other areas. [6]. For an introduction to the field, see [3, 10]. However, in many applications, the corresponding graph is non-planar (e.g., small world networks). Even though the majority of these graphs cannot be drawn in the plane without edge crossings, minimizing the number of crossings that occur (either per edge or in total) is an important task, for both practical and aesthetic reasons in application [15]. This has led to a great deal of recent interest in graph drawings that minimize the number of crossings in some sense (see [11, 16], for example), and has made graph crossings one of the major areas of graph drawing research.

There are a number of competing measures of what exactly constitutes a drawing with few crossings. The total number of pairwise edge crossings of a drawing D is called the crossing number of D, denoted by $cr(D)$. The minimum crossing number over all drawings D of a graph G is called the crossing number of G, denoted by $cr(G)$. Alternatively, it may be that the maximum number of crossings per edge matters more than the total number of edge crossings. A drawing D is k-planar if each edge participates in at most k crossings, and a graph is k-planar if there exists a k-planar drawing of it. The maximum number of crossings per edge of a drawing D is called the local crossing number of D, denoted by $lcr(D)$. The minimum local crossing number over all drawings D of a graph G is called the local crossing number of G, denoted by $lcr(G)$.

Both the crossing number and local crossing number are active areas of research with many open problems, such as the value of these quantities for specific graphs and the existence of approximation algorithms, and many extensions (for instance, see [2, 14]). For a thorough treatment of both of these subjects, see [16]. Despite the importance of the crossing number and local crossing number in producing a quality drawing of a graph, the majority of computational results regarding these two quantities have been algorithmic lower bounds rather than constructions. For instance, deciding if a graph has $cr(G) \leq k$ for a fixed k was shown to be NP-complete in [7]. In [9] and later independently in [12], it was shown that deciding whether a graph is 1-planar is NP-complete (a third proof was later given in [16]). We note that there are a number of linear time planarity testing algorithms, most notably the edge addition method (see [4, 5]).

In this paper, we show that testing k-planarity is NP-complete for all $k \geq 1$. Furthermore, we show that the gap decision problem

> **GAP k-PLANARITY**
> **Input:** A graph $G = (V, E)$ with $lcr(G) \leq k$ or $lcr(G) \geq 2k$.
> **Output:** TRUE if $lcr(G) \leq k$; FALSE otherwise.

is NP-complete.

Our proof proceeds in two simple steps: first we prove that the multigraph variant of gap k-planarity is hard, and then we reduce to the graph case via a technique called edge subdivision. In [12] the authors sketch a way of modifying their 1-planarity proof to give hardness of k-planarity testing in multigraphs, but this approach is quite involved, and filling in the details appears somewhat difficult. Here, we provide a proof of an even stronger result, using a reduction from 3-partition. Our gadget is a simplified version of a technique used in [9] to prove hardness of testing 1-planarity. In doing so, we also provide an alternate proof of the hardness of 1-planarity, one that does not rely on the machinery of K_6 that was needed in [9]. We note that it is fairly straightforward to verify that all of the decision problems considered in this paper are in NP, therefore, it suffices to show hardness.

The remainder of the paper is as follows. In Section 2, we show that the gap k-planarity decision problem is NP-complete. In Section 3, we analyze the sometimes opposing optimization problems of minimizing the crossing number versus minimizing the local crossing number. Through an infinite class of examples, we quantify the inability to simultaneously approximately minimize the crossing number and local crossing number of a drawing.

2. Testing k-planarity is NP-complete

A drawing is said to be normal if each pair of edges cross at most a finite number of times, no three edges cross at a single point, and there are no "touching points." A crossing is an intersection between two edges in which the edges alternate in every sufficiently small neighborhood of the intersection point, whereas in a touching point, the edges are consecutive. In this paper, we will assume all drawings are normal, as touching points can be removed and three edges crossing at a single point can always be perturbed to decrease the crossing number and not increase the local crossing number.

In this section, we consider not only drawings of undirected graphs, but multigraphs as well. A multigraph is a graph which is allowed to have multiple edges between vertices, in particular, a triple $G = (V, E, \omega)$ where (V, E) is a simple graph and $\omega : E \to \mathbb{N}^+$ is a function detailing the number of copies $\omega(e)$ of a given edge $e \in E$ in G. All of the previous definitions for k-planarity, crossing number, and local crossing number in a graph are equally applicable for multigraphs.

We will use multigraphs with edges of varying multiplicity to provide a simple proof of hardness of the multigraph version of gap k-planarity

MULTIGRAPH GAP k-PLANARITY
Input: A multigraph $G = (V, E, \omega)$ with $lcr(G) \leq k$ or $lcr(G) \geq 2k$.
Output: TRUE if $lcr(G) \leq k$; FALSE otherwise.

and then use a technique called edge subdivision to show hardness of gap k-planarity testing.

Our proof is via a reduction from the 3-partition problem. The 3-partition decision problem asks whether a multiset A of $3m$ integers can be partitioned into m multisets $A_1, ..., A_m$ such that the sum of the elements in each multiset is the same. Without loss of generality, one may assume that every integer is positive and strictly between a fourth and half of the desired sum B. In addition, we also assume that $B \geq 100$ and $m \geq 4$. One can easily verify that 3-partition remains NP-hard under this mild restriction (for details, see [8]).

Our reduction converts an instance

$$A = \{a_1, ..., a_{3m}\}, \quad \sum_{i=1}^{3m} a_i = Bm, \quad \frac{B}{4} < a_i < \frac{B}{2}, \; i = 1, ..., 3m,$$

of 3-partition into a multigraph $G_A = (V, E, \omega)$ defined as follows:

$$V = \{t, c\} \bigcup \{t_1, ..., t_{3m}\} \bigcup \{c_1, ..., c_{Bm}\} \bigcup \{s_1, ..., s_{3m}\} \bigcup \{\ell_i^j\}_{i=0,1,...,a_j}^{j=1,...,3m},$$

$$E = \left\{(t, t_{3i})(t_{3i}, c_{Bi})(c_{Bi}, c)\right\}_{i=1}^m \bigcup \left\{(t_i, t_{i+1})\right\}_{i=1}^{3m} \bigcup \left\{(c_i, c_{i+1})\right\}_{i=1}^{Bm}$$
$$\bigcup \left\{(t, \ell_0^i), (\ell_0^i, s_i)\right\}_{i=1}^{3m} \bigcup \left\{(s_j, \ell_i^j), (\ell_i^j, c)\right\}_{i=1,...,a_j}^{j=1,...,3m},$$
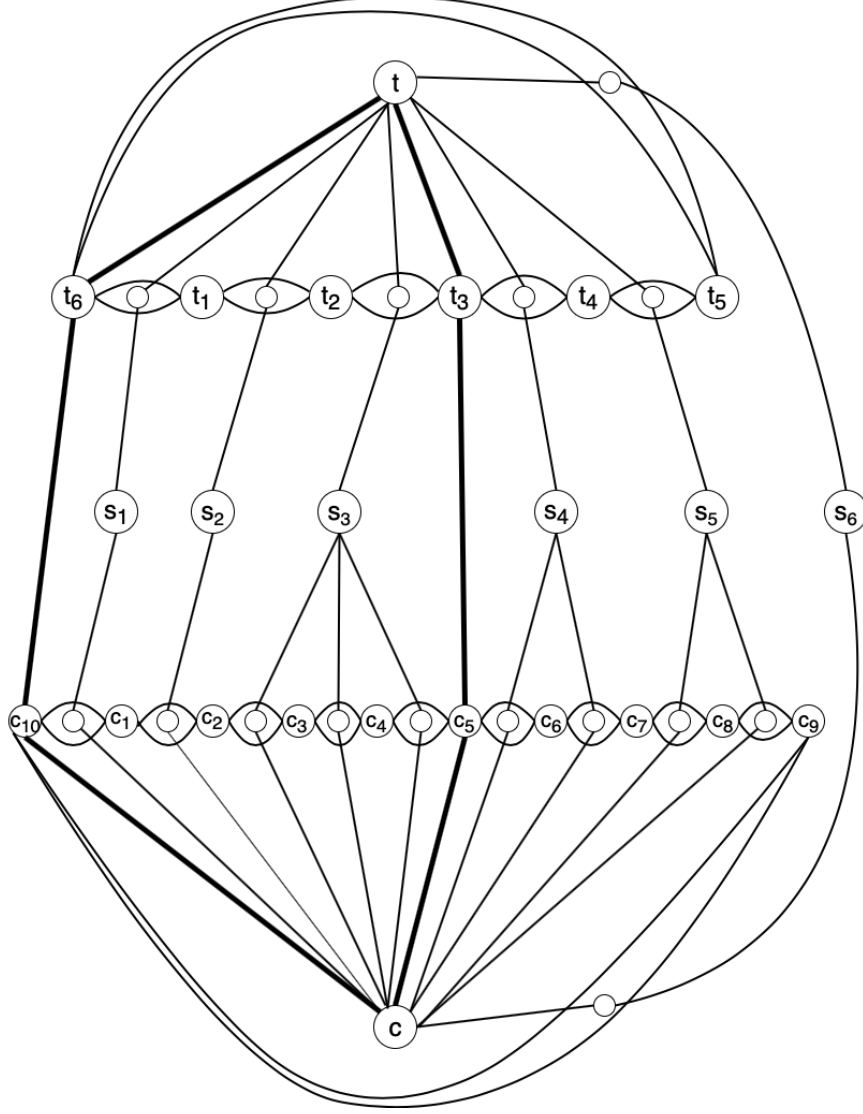
FIGURE 1. A visual example of an embedding of G_A, where $A = \{a_1, a_2, a_3, a_4, a_5, a_6\}$, $a_1 = a_2 = a_6 = 1$, $a_4 = a_5 = 2$, $a_3 = 3$. The edges represent k multi-edges and the bolded edges represent $5Bk$ multi-edges. The paths $P_1 = t\,t_3\,c_5\,c$ and $P_2 = t\,t_6\,c_{10}\,c$ partition the sphere into two regions. The partition of the vertices $s_1, s_2, s_3, s_4, s_5, s_6$ into s_1, s_2, s_3 and s_4, s_5, s_6 by the paths P_1 and P_2 corresponds to the 3-partition into multisets $\{a_1, a_2, a_3\}$ and $\{a_4, a_5, a_6\}$, each with sum equal to five.

$$
\omega(e) = \begin{cases}
k & e \in \left\{(t, \ell_0^i), (\ell_0^i, s_i)\right\}_{i=1}^{3m} \bigcup \left\{(s_j, \ell_i^j), (\ell_i^j, c)\right\}_{i=1,\ldots,a_j}^{j=1,\ldots,3m} \\
2k & e \in \left\{(t_i, t_{i+1})\right\}_{i=1}^{3m} \bigcup \left\{(c_i, c_{i+1})\right\}_{i=1}^{Bm} \\
5Bk & e \in \left\{(t, t_{3i}), (t_{3i}, c_{Bi})(c_{Bi}, c)\right\}_{i=1}^{m}
\end{cases},
$$

where $t_{3m+1} := t_1$, $c_{Bm+1} := c_1$. We give a visual example of this graph in Figure 1.

While the formal definition is somewhat involved, the concept is rather straightforward. A graph drawing can be equivalently thought of as an embedding on the unit sphere $S^2 = \{x \in \mathbb{R}^3 | \|x\| = 1\}$ (via stereographic projection), and we will use this representation throughout the rest of the paper. Qualitatively, a drawing of G_A can be thought of as a partition of the sphere, where t is at the north pole $\vec{n} = (0, 0, 1)$, c is at the south pole $\vec{s} = (0, 0, -1)$, and the m multi-edge paths of the form $P_i = t\,t_{3i}\,c_{Bi}\,c$, $i = 1, \ldots, m$, partition S^2 into m regions. Each star induced by the vertices $s_j, \ell_1^j, \ldots, \ell_{a_j}^j$ corresponds to a number a_j in A, $j = 1, \ldots, 3m$.

The multiset A_i consists of the a_j corresponding to the stars contained in the ith region of S^2. The cycle $C_t = t_1 \, t_2 \ldots t_{3m}$ paired with edges (t, t_{3i}) guarantee that each region has exactly three stars, and the cycle $C_c = c_1 \, c_2 \ldots c_{Bm}$ paired with edges (c, c_{Bi}) guarantee that in each region the number of leaves corresponding to the three stars (and therefore the sum of the corresponding a_js) is exactly B.

We are now prepared to provide a formal proof of our desired result.

Theorem 1. *Multigraph gap k-planarity ($k \geq 1$) is NP-complete.*

We will prove two statements. First, we will show that if A has a 3-partition, then G_A is k-planar. Second, we will show that if A does not have a 3-partition, then $lcr(G_A) \geq 2k$. These two results together complete the proof of Theorem 1.

Lemma 2. *If A has a 3-partition, then G_A is k-planar.*

Proof. Suppose that A has a 3-partition. We will explicitly describe a k-planar drawing D of G_A. Place t and c at the north and south pole $\vec{n}$ and $\vec{s}$, respectively. Draw the m multi-edge paths P_i from t to c such that they do not cross. Draw the multi-edge cycles C_t and C_c, again, in a non-crossing fashion. Each of the m regions created by the paths P_i corresponds to one of the m multisets in the 3-partition. In each of these regions, place the three star centers s_i, s_j, s_k corresponding to the three elements of the corresponding multiset between C_t and C_c. Place each of the vertices $\ell_0^i, \ell_0^j, \ell_0^k$, in the middle of the $2k$ copies of one of the three multi-edges of the path $t_{3\ell} \, t_{3\ell+1} \, t_{3\ell+2} \, t_{3\ell+3}$. Because our partition of A is a 3-partition, there are a total of B leaves connected to s_i, s_j, s_k. Place each leaf in the middle of the $2k$ copies of one of the B multi-edges of the path $c_{B\ell} \, c_{B\ell+1} \ldots c_{B\ell+B}$. This is a k-planar drawing. An example of this layout is given in Figure 1. $\qquad\square$

Lemma 3. *If A does not have a 3-partition, then $lcr(G_A) \geq 2k$.*

Proof. Suppose that A does not have a 3-partition, but there exists a drawing D of G_A with $lcr(G_A) < 2k$. Without loss of generality, we may assume that t and c are at the north and south pole, respectively. Let $\delta > 0$ be such that there is exactly one vertex and no edge crossings in $B(\vec{n}, \delta)$ and $B(\vec{s}, \delta)$, where $B(x, r) := \{y \in S^2 | \|x - y\| < r\}$. There are m multi-edge paths emanating from t and reaching c, which are non-crossing in $B(\vec{n}, \delta)$ and $B(\vec{s}, \delta)$. We will first look at the clockwise ordering of copies of paths P_i in $B(\vec{n}, \delta)$.

It may be the case that copies of P_i interlace with copies of other paths in $B(\vec{n}, \delta)$. The copies of P_i partition $B(\vec{n}, \delta)$ into $5Bk$ regions, each of which contains some number of copies of other paths. No two copies of P_i can partition these non-P_i paths into two regions containing at least $6(2k-1)+1$ non-P_i paths each, as this would contradict $(2k-1)$-planarity because P_i is of length three. The number of non-P_i paths $(5Bk(m-1))$ is more than three times $6(2k-1)+1$, so one of the original $5Bk$ regions must contain at least $6(2k-1)+1$ non-P_i paths, and, therefore, all but at most $6(2k-1)$ non-P_i paths. Let us denote this region by R^*. Each of the $5Bk - 1$ regions which are at least $6(2k-1)+1$ regions away from R^* cannot contain any non-P_i paths. By removing at most $24(2k-1)$ copies of each P_i, $i = 1, ..., k$, we now have a drawing in which no copies of paths P_i, P_j interlace in $B(\vec{n}, \delta)$ or $B(\vec{s}, \delta)$, with at least $5Bk - 24(2k-1) > (5B - 48)k$ copies of each P_i remaining.

This creates a local clockwise ordering of the paths both at $B(\vec{n}, \delta)$ and $B(\vec{s}, \delta)$. If these two orderings are not the same, then two multi-edge paths must cross, a contradiction, as each path is three multi-edges long, and $(5B - 48)k > 3(2k-1)$. Next, we observe that the ordering in $B(\vec{n}, \delta)$ and $B(\vec{s}, \delta)$ must be the natural ordering (or the reversal of it), namely, the paths must be ordered $P_1, ..., P_m$. Suppose to the contrary, that the ordering is such that there exists an i such that P_i and P_{i+1} are not adjacent. Then there is a multi-edge cycle of length five consisting of $t \, t_{3i} \, t_{3i+1} \, t_{3i+2} \, t_{3i+3}$, which $(5B - 48)k > 5(2k-1)$ edge-disjoint copies of some multi-edge path P_j must cross, a contradiction. See Figure 2 for a visual example.

Next, for every vertex t_{3i} (and, similarly for c_{Bi}), we investigate the local structure of the at least $(5B - 48)k$ copies of edges (t, t_{3i}) and (t_{3i}, c_{Bi}). Let $\delta^* > 0$ be such that the δ^* neighborhood of the location of t_{3i} (denoted $N(t_{3i}, \delta^*)$) has no edge crossings and only one vertex. Using the same argument as above, we can remove some relatively small number of copies of (t, t_{3i}) and (t_{3i}, c_{Bi}) and be left with edges which do not locally interlace.

The copies of (t, t_{3i}) partition $N(t_{3i}, \delta^*)$ into $(5B - 48)k$ regions, each of which contains some number of copies of (t_{3i}, c_{Bi}). No two copies of (t, t_{3i}) can partition the copies of (t_{3i}, c_{Bi}) into two regions containing
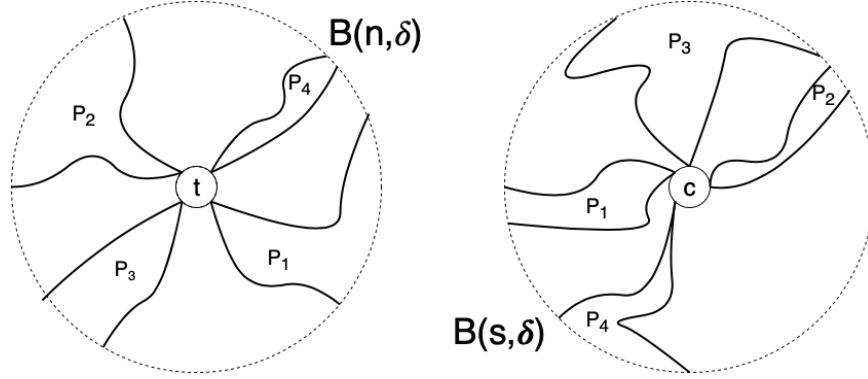
FIGURE 2. A vertical view of $B(\vec{n}, \delta)$ and $B(\vec{s}, \delta)$. The clockwise ordering P_1, P_3, P_2, P_4 is the same for both $B(\vec{n}, \delta)$ and $B(\vec{s}, \delta)$. However, this cannot correspond to an embedding D with $lcr(D) < 2k$, as either P_3 or P_4 must cross the cycle $t\, t_3\, t_4\, t_5\, t_6$ to reach c.

FIGURE 3. Three copies of P_1, labeled so that the ordering agrees at each vertex.

at least $2(2k-1)+1$ copies of (t_{3i}, c_{Bi}) each, as this would contradict $(2k-1)$-planarity. The number of copies of (t_{3i}, c_{Bi}) $((5B-48)k)$ is more than three times $2(2k-1)+1$, so one of the original $(5B-48)k$ regions must contain at least $2(2k-1)+1$ copies of (t_{3i}, c_{Bi}), and, therefore, all but at most $2(2k-1)$ copies of (t_{3i}, c_{Bi}). By removing at most $2(2k-1)$ copies of (t, t_{3i}) and (t_{3i}, c_{Bi}) each, the local ordering in this neighborhood is such that copies of these two edges do not interlace. Therefore, by removing a total of at most $4(2k-1)$ copies of each path P_i, $i = 1, ..., k$, we now have edges which are locally well-ordered, and at least $(5B-48)k - 4(2k-1) > (5B-56)k$ copies of each P_i remain.

We can now formally define each copy of the path, using the ordering of edges in $B(\vec{n}, \delta)$ and labeling paths so that the ordering of copies of (t, t_{3i}) matches the ordering of (t_{3i}, c_{Bi}) locally (and the same for (t_{3i}, c_{Bi}) and (c_{Bi}, c) locally). See Figure 3 for an illustration.

We can now define a partition of S^2 based on the m non-overlapping regions enclosed by the middle edge-disjoint copies of P_i and P_{i+1} (we define middle based on the ordering of $B(\vec{n}, \delta)$). These middle edge-disjoint copies cannot cross any copy of any other path, as there are at least $(5B-56)k/2 - 1 > 6(2k-1) - 2$ edge-disjoint paths separating them. Let R_i be the region defined by the middle copies of P_i and P_{i+1}.

In addition, R_i must fully contain a large number of copies of P_i and P_{i+1}. In particular, the middle copy of P_i cannot cross any copy of P_i which is at least $6(2k-1)$ copies away in the initial ordering. Therefore, R_i fully contains at least $(5B-56)k/2 - 1 - 6(2k-1) > 2Bk$ copies of P_i and P_{i+1} each. R_i must fully contain the paths $t_{3i}\, t_{3i+1}\, t_{3i+2}\, t_{3i+3}$, otherwise there would be $2Bk > 3(2k-1)$ edge-disjoint paths to cross. The same argument holds for $c_{Bi}\, c_{Bi+1} \dots c_{Bi+B}$ by noting that $2Bk > B(2k-1)$.

We now consider the locations of the vertices s_j. Each vertex s_j must be separated from t by all multi-edge copies of C_t, otherwise there would be a 5-cycle $t\, t_{3i}\, t_{3i+1}\, t_{3i+2}\, t_{3i+3}$ separating $a_j > Bk/4 > 5(2k-1)$ edge-disjoint paths from s_j to c. Each region has at most three vertices s_j, otherwise one of the multi-edges in the path $t_{3i}\, t_{3i+1}\, t_{3i+2}\, t_{3i+3}$ would be crossed at least $2k$ times. Therefore each region has exactly three vertices s_j.

The vertices s_j are also separated from c by all copies of C_c. Suppose this is not the case. If no copy of the multi-edge cycle C_c separates s_j from c, then there are $4k$ edge-disjoint multi-edge cycles separating s_j from t, a contradiction. Then s_j must be separated from c by two copies of some edge (c_i, c_{i+1}), a contradiction to the $a_j > Bk/4 > 2(2k-1)$ edge-disjoint paths from s_j to c. Therefore, C_c separates s_j from c.

Because A does not have a 3-partition, one of these regions, say R_i, must have three vertices $s_{j_1}, s_{j_2}, s_{j_3}$ such that the sum of their leaves exceeds B. However, there are only B multi-edges in the path $c_{Bi} \, c_{Bi+1} \dots c_{Bi+B}$, so one such multi-edge must have more than one path $s_j \, \ell_p^j \, c$ crossing it, a contradiction. The proof is complete. $\qquad\square$

Although it is not necessary for the proof of Theorem 1, one can also verify the stronger statement that $lcr(G_A) = k$ if and only if A has a 3-partition, otherwise $lcr(G_A) = 2k$.

From here, we reduce from multigraph gap $2k$-planarity testing to gap k-planarity testing in a straight-forward way. Given a multigraph $G = (V, E, \omega)$, we define the edge subdivision of G to be the undirected graph $G_* = (V_*, E_*)$ constructed by subdividing each edge of G into two edges with a new vertex between them (i.e. replacing $e = (u, v) \in E$ by $(u, x_e), (x_e, v) \in E_*$, where $x_e \in V_*$ is a unique vertex for each copy of e). The key property of this edge subdivision is the following.

Lemma 4. *Let G_* be the edge subdivision of the multigraph G. Then*

$$lcr(G_*) = \left\lceil \frac{lcr(G)}{2} \right\rceil.$$

Proof. If G_* is k-planar, then taking any k-planar drawing of G_* and reversing the subdivision operation gives a drawing of G which is clearly $2k$-planar. Conversely, given a $2k$-planar embedding of G, we can obtain from it a k-planar embedding of G_* by placing the x_e vertices "in the middle" of the crossings on e so that each segment has at most k crossings, after possibly perturbing the drawing of G slightly so that no three edges cross at a single point. Therefore,

$$lcr(G) \le 2lcr(G_*) \quad \text{and} \quad lcr(G_*) \le \left\lceil \frac{lcr(G)}{2} \right\rceil,$$

which, by integrality of $lcr(G_*)$, implies that $lcr(G_*) = \left\lceil \frac{lcr(G)}{2} \right\rceil$. $\qquad\square$

From here, the following two theorems immediately follow.

Theorem 5. *Gap k-planarity is NP-complete for all $k \ge 1$.*

Theorem 6. *Deciding whether a graph is k-planar is NP-complete for all $k \ge 1$.*

3. Crossing number vs local crossing number

In this section, we consider the problem of approximately minimizing both the crossing number and local crossing number of an embedding. In particular, we define

$$r(G) := \min_D \frac{cr(D)}{cr(G)} \frac{lcr(D)}{lcr(G)},$$

where the minimum is taken over all drawings D of a non-planar graph G. If $r(G)$ is small, then it means that there exists a drawing of G which simultaneously approximately minimizes both the total number of crossings and the maximum number of crossings per edge. However, if $r(G)$ is large, then these two minimization problems are clearly incompatible. We will prove the following result.

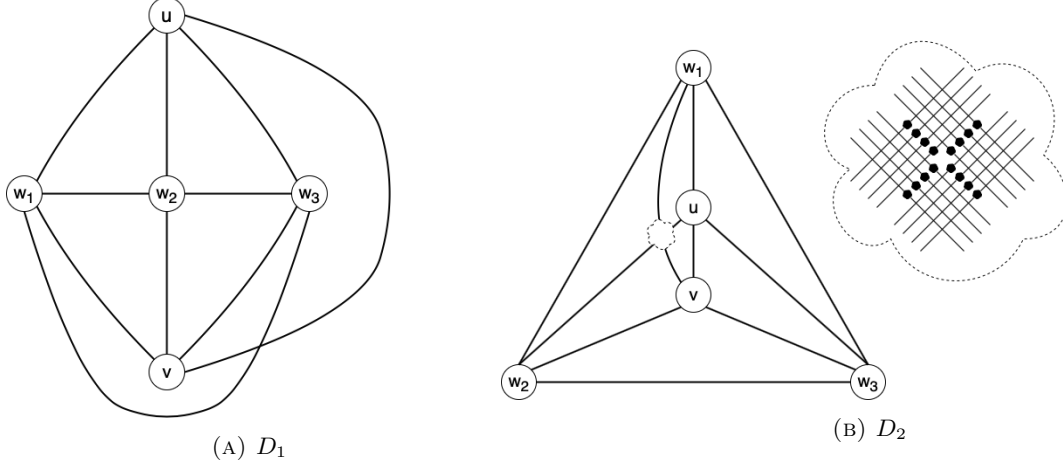**Theorem 7.** *Let $\mathcal{G}_n^{\neg p}$ be the set of non-planar graphs of order n. Then*

$$cn^{1/2} \le \max_{G \in \mathcal{G}_n^{\neg p}} r(G) \le Cn$$

for all $n \ge 5$, for fixed constants c, C.

To prove the upper bound, we make use of the well-known crossing lemma.

Theorem 8 (Crossing Lemma, [1, 13])**.** *Let $G = (V, E)$ be a graph with $|E| \ge \lambda |V|$. Then*

$$cr(G) \ge (\lambda^{-2} - 3\lambda^{-3}) \frac{|E|^3}{|V|^2}.$$

(A) D_1

(B) D_2

FIGURE 4. A visual demonstration of the two drawings D_1 and D_2 of the graph G.

Let us temporarily restrict ourselves to graphs satisfying $|E| \geq \frac{9}{2}|V|$. By Theorem 8,

$$\frac{4}{243}\frac{|E|^3}{|V|^2} \leq cr(D) \leq \frac{|E|}{2}lcr(D).$$

In addition, any drawing D satisfying $cr(D) = cr(G)$ must also satisfy $lcr(D) < |E|$, and therefore $r(G) < |E|$. If this is not the case, then there are two edges which cross each other more than once. Removing these crossings locally decreases the crossing number, a contradiction. This produces a natural upper bound of

$$r(G) < \frac{|E|}{\frac{2}{|E|}cr(D)} = \frac{|E|^2}{2cr(D)} \leq \frac{|E|^2}{2\left(\frac{4}{243}\frac{|E|^3}{|V|^2}\right)} = \frac{243|V|^2}{8|E|}.$$

Combining our two bounds of $|E|$ and $\frac{243|V|^2}{8|E|}$, we obtain the upper estimate of Theorem 7.

To produce the lower bound, we give an infinite class of examples, based on adaptive edge subdivision of a multigraph version of K_5 (the smallest non-planar graph). In particular, let $G = (V, E)$, with

$$V = \{u, v, w_1, w_2, w_3\} \bigcup \{x_{1,2}^i, x_{2,3}^i, x_{1,3}^i\}_{i=1}^{k^4} \bigcup \{x_{u,1}^{i,j}, x_{u,2}^{i,j}, x_{u,3}^{i,j}\}_{i=1,\dots,k^3}^{j=1,\dots,k-1} \bigcup \{x_{v,1}^{i,j}, x_{v,2}^{i,j}, x_{v,3}^{i,j}\}_{i=1,\dots,k^3}^{j=1,\dots,k-1},$$

$$E = \left\{(u, x_{u,\ell}^{i,1}), (x_{u,\ell}^{i,1}, x_{u,\ell}^{i,2}), \dots, (x_{u,\ell}^{i,k-1}, w_\ell)\right\}_{i=1,\dots,k^3}^{\ell=1,2,3} \bigcup \left\{(v, x_{v,\ell}^{i,1}), (x_{v,\ell}^{i,1}, x_{v,\ell}^{i,2}), \dots, (x_{v,\ell}^{i,k-1}, w_\ell)\right\}_{i=1,\dots,k^3}^{\ell=1,2,3}$$

$$\bigcup \left\{(w_1, x_{1,2}^i), (x_{1,2}^i, w_2), (w_2, x_{2,3}^i), (x_{2,3}^i, w_3), (w_3, x_{1,3}^i), (x_{1,3}^i, w_1)\right\}_{i=1}^{k^4} \bigcup (u, v),$$

for some natural number k. This graph can be thought of as a multigraph of K_5 on the vertices u, v, w_1, w_2, w_3, where edges $(u, w_1), (u, w_2), (u, w_3), (v, w_1), (v, w_2), (v, w_3)$ have multiplicity k^3; $(w_1, w_2), (w_2, w_3), (w_3, w_1)$ have multiplicity k^4; and (u, v) has only one edge. Each copy of the edge (w_i, w_j) is replaced by a standard edge subdivision $(w_i, x_{i,j}^\ell), (x_{i,j}^\ell, w_j)$, $\ell = 1, \dots, k^4$, but each copy of the edges of the form (u, w_i) (and (v, w_i), resp.) is replaced by a path of length k given by $(u, x_{u,i}^{\ell,1}), (x_{u,i}^{\ell,1}, x_{u,i}^{\ell,2}), \dots, (x_{u,i}^{\ell,k-1}, w_i)$, $\ell = 1, \dots, k^3$. To refer to the path subdivision of an edge e in K_5, we will simply write P_e.

We first consider two different drawings of G. Let D_1 be the drawing in which the cycles $C_w := P_{(w_1,w_2)}P_{(w_2,w_3)}P_{(w_1,w_3)}$ separates u and v and the only edge crossings are the single edge (u, v) crossing all copies of $P_{(w_1,w_3)}$. In this case we have $lcr(D_1) = cr(D_1) = k^4$. Alternatively, let D_2 be the drawing in which u and v are on the same side of every copy of the cycle C_w, and the only edge crossings are all the copies of $P_{(u,w_2)}$ crossing all the copies of $P_{(v,w_1)}$. In this case, due to the subdivision into paths of length k, we have $lcr(D_2) = k^2$ and $cr(D_2) = k^6$. See Figure 4 for a visual representation of these two drawings.

We will show that no drawing D can produce a significantly better approximation to both crossing number and local crossing number than the two drawings described above. In particular, we will show that for all drawings D of G, $lcr(D)\,cr(D) \geq \tilde{c}k^8$ for some $\tilde{c}$.

Suppose D is such that at least k^3 edge disjoint copies of C_w do not separate u and v. If this is not the case, then $lcr(D) \, cr(D) \geq (k^4 - k^3)^2$. Let S_u be the "star" created by $P_{(u,w_1)}$, $P_{(u,w_2)}$, and $P_{(u,w_3)}$ (with S_v defined similarly). We have k^3 edge disjoint copies of S_u, S_v, and C_w, with u and v on the same side of every copy of C_w. Given one copy each of S_u, S_v, and C_w, with u and v on the same side of C_w, one of the three subgraphs must cross. This implies that there must be a total of at least k^6 crossings. As there are only $12k^4$ edges, the local crossing number is at least $k^2/12$. Noting that G has $6(k-1)k^3 + 3k^4 + 5$ vertices completes the proof of Theorem 7.

Acknowledgements

References

[1] Miklós Ajtai, Vašek Chvátal, Monroe M Newborn, and Endre Szemerédi. Crossing-free subgraphs. *North-Holland Mathematics Studies*, 60(C):9–12, 1982.

[2] John Asplund, Thao Do, Arran Hamm, and Vishesh Jain. On the k-planar local crossing number. *Discrete Mathematics*, 342(4):927–933, 2019.

[3] Giuseppe Di Battista, Peter Eades, Roberto Tamassia, and Ioannis G Tollis. *Graph drawing: algorithms for the visualization of graphs*. Prentice Hall PTR, 1998.

[4] John M Boyer and Wendy J Myrvold. On the cutting edge: simplified O(n) planarity by edge addition. *J. Graph Algorithms Appl.*, 8(2):241–273, 2004.

[5] Hubert De Fraysseix, Patrice Ossona De Mendez, and Pierre Rosenstiehl. Trémaux trees and planarity. *International Journal of Foundations of Computer Science*, 17(05):1017–1029, 2006.

[6] Giuseppe Di Battista, Peter Eades, Roberto Tamassia, and Ioannis G Tollis. Algorithms for drawing graphs: an annotated bibliography. *Computational Geometry*, 4(5):235–282, 1994.

[7] Michael R Garey and David S Johnson. Crossing number is NP-complete. *SIAM Journal on Algebraic Discrete Methods*, 4(3):312–316, 1983.

[8] Michael R Garey and David S Johnson. *Computers and intractability*, volume 29. wh freeman New York, 2002.

[9] Alexander Grigoriev and Hans L Bodlaender. Algorithms for graphs embeddable with few crossings per edge. *Algorithmica*, 49(1):1–11, 2007.

[10] Michael Kaufmann and Dorothea Wagner. *Drawing graphs: methods and models*, volume 2025. Springer, 2003.

[11] Stephen G Kobourov, Giuseppe Liotta, and Fabrizio Montecchiani. An annotated bibliography on 1-planarity. *Computer Science Review*, 25:49–67, 2017.

[12] Vladimir P Korzhik and Bojan Mohar. Minimal obstructions for 1-immersions and hardness of 1-planarity testing. *Journal of Graph Theory*, 72(1):30–71, 2013.

[13] Frank Thomson Leighton. New lower bound techniques for VLSI. *Mathematical systems theory*, 17(1):47–70, 1984.

[14] János Pach, László A Székely, Csaba D Tóth, and Géza Tóth. Note on k-planar crossing numbers. *Computational Geometry*, 68:2–6, 2018.

[15] Helen C. Purchase. Effective information visualisation: a study of graph drawing aesthetics and algorithms. *Interacting with computers*, 13(2):147–162, 2000.

[16] Marcus Schaefer. *Crossing numbers of graphs*. CRC Press, 2018.

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