Nodal Decompositions of Graphs

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Abstract

A nodal domain of a function is a maximally connected subset of the domain for which the function does not change sign. Courant's nodal domain theorem gives a bound on the number of nodal domains of eigenfunctions of elliptic operators. In particular, the k^{th} eigenfunction contains no more than k nodal domains. We prove a generalization of Courant's theorem to discrete graphs. Namely, we show that for the k^{th} eigenvalue of a generalized Laplacian of a discrete graph, there exists a set of corresponding eigenvectors such that each eigenvector can be decomposed into at most k nodal domains. In addition, we show this set to be of co-dimension zero with respect to the entire eigenspace.

AMS classification: 05C50

Keywords: Generalized Laplacian; Nodal Domains.

1. Introduction

In the 1923 paper titled "Ein allgemeiner Satzt zur Theorie der Eigenfunktionen selbsadjungierter Differentialausdrücke" [10], as well as in the 1924 text co-authered with David Hilbert "Methoden der mathematischen Physik I" [9], Richard Courant proved a result regarding the zeros of elliptic eigenfunctions, the so-called Courant nodal domain theorem.

Theorem 1.1 (Courant's nodal domain theorem, [10, 9])**.** *Given the self-adjoint second order differential equation $L[u] + \lambda\rho u = 0$, ($\rho \neq 0$), for a domain G with arbitrary homogeneous boundary conditions; if its eigenfunctions are ordered according to increasing eigenvalues, then the nodes of the n^{th} eigenfunction u_n divide the domain into no more than n subdomains. No assumptions are made about the number of independent variables.*

The "nodes" are the nodal set $\{x|u_n(x) = 0\}$ and the "sub-domains" are now referred to as nodal domains. Extensions of Courant's nodal domain theorem are abundant, including to p-Laplacians, Riemannian manifolds, and domains with low regularity assumptions [13, 23, 7, 11, 1]. Most notably, Pleijel's nodal domain theorem is an extension of Theorem 1.1 to vibrating membranes using Faber-Krahn results [24]. Theorem 1.1 is also closely related to the work of Chladni involving the modes of vibration of a rigid surface; the patterns of nodal lines on the surface are referred to as Chladni figures [28, 8]. For further information regarding the importance of Theorem 1.1, the author refers the reader to [2].

Courant's theorem has extensions not only in differential equations, but in graph theory as well. To see the natural extension, we note that many of the matrix representations of graphs, such as the graph Laplacian, have properties that are analogous to continuous elliptic operators.

In fact, there are many connections between spectral graph theory and elliptic partial differential equations. For a thorough and detailed treatment of these connections, we refer the reader to [18].

Nodal decompositions of graphs have applications in graph partitioning, most notably in spectral bisections. The first nodal domain-type theorem for graphs was proved by Davies et al [5], though similar results were previously mentioned, though not fully proved, by Colin de Verdiere in [12] and by Friedman in [17]. A number of results have also been proved by Biyikoglu et al [3, 4] and Gladwell and Zhu [19]. These results will be discussed in greater detail in Section 2. Miroslav Fiedler produced many results related to the nodal domains of eigenvectors corresponding to the algebraic connectivity of a graph [14, 15, 16]. For this reason, such eigenvectors are now referred to as Fielder vectors of the graph. The nodal domains of the Fiedler vector have been used with great success in spectral partitioning [26, 6, 25, 22, 20]. Recently, in [27] it was shown that for every connected graph, there exists a Fiedler vector that produces connected subgraphs in the spectral bisection .

In what follows, we aim to prove a new, direct analogue of Theorem 1.1 for graphs. Namely, we aim to show that for any k, there exists an eigenvector u_k corresponding to the k^{th} eigenvalue that can be decomposed into k or fewer nodal domains. The remainder the paper is as follows. In Section 2, we define the notation of the paper, give basic definitions, and recall existing results in the literature. In Section 3, we give a complete characterization of eigenvectors that take the value zero at some vertex. We will see that the existence of vertices that take the value zero plays a crucial role in the theory of nodal domains on graphs. Finally, we prove a discrete analogue of Theorem 1.1 for graphs, the main result of the paper.

2. Notation and Existing Results

We begin by defining the necessary notation. Let the set of connected, undirected, simple graphs $G = (V, E)$ be denoted by $\mathcal{G}$. For a given graph $G \in \mathcal{G}$, let $d(u, v)$ be the distance between the vertices $u, v \in V$, $N(u)$ be the set of vertices $v \in V$ for which $d(u, v) = 1$, and $d(u) := |N(u)|$ be the degree of vertex $u \in V$. For a given graph $G = (V, E)$ and subset $X \subset V$, we denote by $G(X)$ the subgraph of G induced by the vertices X. In addition, if, for a given graph G, there exists a bipartition X, Y of V such that $E \subset X \times Y$, then G is said to be bipartite and may be written as $G = (X, Y, E)$.

The superscript $\cdot^T$ denotes the adjoint with respect to the standard Euclidean ℓ^2 inner product. We define the Lebesgue covering dimension of a set X by $dim(X)$, and define the co-dimension of a subset Y of a vector space X by $codim(Y) = dim(X) - dim(Y)$. In what follows, the vector space X will always be an eigenspace. Let $[n] = \{1, ..., n\}$ and $\mathbf{1}\{\cdot\}$ be the indicator function. For a matrix $A \in \mathbb{R}^{n\times n}$ and some $J \subset [n]$, we denote by $A(J)$ the sub-matrix of A induced by the indices J. In addition, for a given vector $f \in \mathbb{R}^n$, we introduce a partition of $[n]$ based on the sign of the components of f,

$$i_+(f) = \{1 \le i \le n | f(i) > 0\},$$

$$i_-(f) = \{1 \le i \le n | f(i) < 0\},$$

$$i_0(f) = \{1 \le i \le n | f(i) = 0\}.$$

For a given matrix $A \in \mathbb{R}^{n\times n}$ and eigenvalue λ, we denote the corresponding eigenspace by

$$E(\lambda; A) := \{f \in \mathbb{R}^n | Af = \lambda f\}.$$

When it is clear which matrix is being used, we may simply write $E(\lambda)$.

In spectral graph theory literature, there is some debate over which matrix representation of a graph is best, and often the answer changes according to setting and application. In what follows, we take a rather general approach. Namely, we consider the set of generalized Laplacians, defined as follows.

Definition 2.1. *Let $G = (V, E) \in \mathcal{G}$, $|V| = n$. A matrix $M \in \mathbb{R}^{n\times n}$, $M = M^T$, is a generalized Laplacian of G if*

$$\begin{aligned} M(i,j) < 0 \quad &\text{for all} \quad \{v_i, v_j\} \in E, \\ M(i,j) = 0 \quad &\text{for all} \quad \{v_i, v_j\} \notin E, i \neq j. \end{aligned}$$

We note that the degree matrix, the unnormalized and normalized Laplacian matrix, and the negative of the adjacency and signless Laplacian matrix are all generalized Laplacians of G. The random walk Laplacian is not usually in this class, but is similar (in a linear algebraic sense) to the normalized Laplacian.

Any generalized Laplacian can be written in the form

$$M(G) = L + D,$$

where L is a symmetric M-matrix and D is a diagonal matrix. This bears immediate similarity to a continuous Laplacian plus a potential term (for this reason, generalized Laplacian matrices are often called discrete Schrödinger operators).

For an eigenvector f of a generalized Laplacian M of a graph G, the nodal set is defined as

$$\{\{u, v\} \in E | f(u)f(v) \leq 0\} \cup \{u \in V | f(u) = 0\}.$$

In particular, we may refer to $\{\{u, v\} \in E | f(u)f(v) \leq 0\}$ and $\{u \in V | f(u) = 0\}$ as the set of nodal edges and the set of nodal vertices, respectively.

The comparison between the continuous and discrete case is not without its difficulties. In particular, for the continuous case, it is well known that $\{x|u_n(x) = 0\}$ is of co-dimension one [7]. However, for a finite graph, the nodal set is some positive proportion of the edges and may include vertices as well. It is the latter property that leads to problems. In fact, the existence of eigenvectors with nodal vertices is the major barrier to an exact graph analogue to Courant's theorem. For this reason, in the graph theory literature, there is a key distinction between nodal domains that do include nodal vertices, and nodal domains that do not.

Definition 2.2. *A strong (resp. weak) nodal domain of a graph G with respect to an eigenvector f is a maximally connected subgraph H satisfying $f(u)f(v) > 0$ (resp. $f(u)f(v) \geq 0$) for all $u, v \in V_H$. The number of strong (resp. weak) nodal domains of a graph G with respect to f is denoted by $\mathcal{G}(f)$ (resp. $\mathcal{W}(f)$).*

When the set of nodal vertices $\{u|f(u) = 0\}$ is empty, the definition of a weak and strong nodal domain is equivalent, and $\mathcal{G}(f) = \mathcal{W}(f)$.

In spite of the complications introduced by nodal vertices, certain results are still possible. Most notably, Biyikoglu et al proved the following.

Theorem 2.3 (Biyikoglu et al, [3, 4])**.** *Let M be a generalized Laplacian of a connected undirected graph with n vertices. Then any eigenvector f_k corresponding to the k^{th} eigenvalue λ_k with multiplicity r has at most k weak nodal domains and $k + r - 1$ strong nodal domains, namely,*

$$\mathcal{W}(f_k) \leq k \qquad \text{and} \qquad \mathcal{G}(f_k) \leq k + r - 1.$$

This analogue of Courant's theorem is exact, save for when nodal vertices exist. The bounds $\mathcal{W}(f_k) \leq k$ and $\mathcal{G}(f_k) \leq k + r - 1$ have been shown to be tight for numerous examples, including the star graph and Peterson's graph. In addition to Theorem 2.3, the following corollary for disconnected graphs was proven in [3, 4].

Corollary 2.4. *Let M be a generalized Laplacian of a graph G with c connected components. Then any eigenfunction f_k corresponding to eigenvalue λ_k with multiplicity r satisfies*

$$\mathcal{W}(f_k) \leq k + c - 1 \qquad \text{and} \qquad \mathcal{G}(f_k) \leq k + r - 1.$$

The most notable extensions of Theorem 2.3 to date are from Gladwell and Zhu.

Theorem 2.5 (Gladwell-Zhu, [19])**.** *There exist orthogonal eigenfunctions f_k of $M(G)$ such that $\mathcal{G}(f_k) \leq k$ for $k = 1, ..., n$.*

Corollary 2.6. *Suppose that λ_k is an eigenvalue with multiplicity r and eigenspace $E(\lambda_k)$. Then there exists a basis $\{f_k, ..., f_{k+r-1}\}$ of $E(\lambda_k)$ such that $\mathcal{G}(f_k) \leq k$ for all $j = k, ..., k + r - 1$.*

While Theorem 2.5 and Corollary 2.6 are strong extensions, examples exist where the set of eigenvectors $f_k \in E(\lambda_k)$ satisfying $\mathcal{G}(f_k) \leq k$ may be of positive co-dimension. We supply one such example for illustration.

Example 2.7. *Consider the graph Laplacian of the star graph S_n on n vertices. We can quickly verify that $\lambda_2 = 1$ and has multiplicity $n - 2$. The corresponding eigenspace $E(\lambda_2) = \{f | f(v_0) = 0, \langle f, 1\rangle = 0\} \approx \mathbb{R}^{n-2}$ has a zero valuated vertex at the centroid v_0. However, for a Fiedler vector to satisfy $\mathcal{G}(f) \leq 2$, precisely $n - 3$ of the vertices v_i, $i > 0$, must equal zero. Therefore, this is a set of co-dimension $n - 3$ with respect to $E(\lambda_k)$.*

Rather than dealing with strong or weak domains, we concern ourselves with nodal decompositions of graphs. In the continuous case, the nodal domains make up a decomposition of the domain minus a set of positive co-dimension, which can be adjoined to the partition in an arbitrary way. In contrast, for the discrete case, the results of weak and strong nodal domain theorems do not give any information regarding nodal decompositions of graphs. In some ways, for graphs, nodal decompositions rather than weak and strong nodal domains are the most natural analogue. We have the following definition.

Definition 2.8. *A nodal decomposition of a graph G with respect to an eigenvector f is a partition of the vertex set V,*

$$\{V_i\}_{i=1}^{s}, \quad \bigcup_{i=1}^{s} V_i = V, \quad V_i \cap V_j = \emptyset \text{ for all } i \neq j,$$

such that the subgraphs $G(V_i)$, $i = 1, .., s$, are the strong nodal domains of some vector g satisfying

$$g(v) = \begin{cases} +1 \text{ or } -1 & \text{if } f(v) = 0 \\ f(v) & \text{if } f(v) \neq 0 \end{cases}.$$

The minimum number s for which a nodal decomposition exists is denoted by $\mathcal{D}(f)$.

If, in a decomposition, a nodal vertex is treated like a positively or negatively valuated vertex, then we will say the vertex is positively or negatively "charged." In what follows, we will prove that the k^{th} eigenvalue of a generalized Laplacian has a set of corresponding eigenvectors $U \subset E(\lambda_k)$ such that U has co-dimension zero with respect to $E(\lambda_k)$ and every $f_k \in U$ satisfies $\mathcal{D}(f_k) \leq k$. This result is stated explicitly in Theorem 3.6.

3. Main Result

In this section, we prove the main result of the paper. First, we give a complete characterization of the nodal vertices of an eigenvalue. We must make a clear distinction between the nodal vertices of an eigenvalue and of an eigenvector. The nodal vertices of an eigenvector are the indices for which the eigenvector takes the value zero. The nodal vertices of an eigenvalue, on the other hand, are the indices for which every corresponding eigenvector takes the value zero. When an eigenvalue is simple, these two definitions are equivalent. For a repeated eigenvalue, there always exists an eigenvector with a nodal vertex, though the eigenvalue itself may have no nodal vertices.

We can use nodal vertices to define an equivalence relation on an eigenspace. Namely, if λ is an eigenvalue of a generalized Laplacian M of the graph $G = (V, E) \in \mathcal{G}$, $V = [n]$, then we have an equivalence relation on $E(\lambda)$

$$f \sim g \text{ if and only if } i_0(f) = i_0(g).$$

Let us formally define the nodal vertices of an eigenvalue

$$i_0(\lambda) := \bigcap_{f \in E(\lambda)} i_0(f),$$

and denote the equivalence class induced by $i_0(\lambda)$ by $[i_0(\lambda)]$. This equivalence class is non-empty. In fact, we can show that $E(\lambda) \backslash [i_0(\lambda)]$ has positive co-dimension.

Theorem 3.1. *Let λ be an eigenvalue of a generalized Laplacian M of the graph $G = (V, E) \in \mathcal{G}$ with corresponding eigenspace $E(\lambda)$. Then*

$$codim(E(\lambda) \backslash [i_0(\lambda)]) > 0.$$

Proof. We will first prove that $[i_0(\lambda)]$ is non-empty. It suffices to show that for any given $f_1, f_2 \in E(\lambda)$, there exists some $f_3 \in E(\lambda)$ such that $i_0(f_3) = i_0(f_1) \cap i_0(f_2)$. One such example is given by

$$f_3 = f_1 + \alpha f_2, \text{ where } \alpha > \frac{\max_{u \in [n]} |f_1(u)|}{\min_{u \in ([n] \backslash i_0(f_2))} |f_2(u)|}.$$

By induction, we see that $[i_0(\lambda)]$ is non-empty. Let $|i_0(\lambda)| = r$. Let us define the space

$$W = \{g \in \mathbb{R}^{n-r} | g = f([n] \backslash i_0(\lambda)) \text{ for some } f \in E(\lambda)\}.$$

W is a subspace of $\mathbb{R}^{n-r}$ and $dim(W) = dim(E(\lambda))$. Now let us define the set

$$U_j = \{g \in W | j \in i_0(g)\}.$$

We note that U_j is a subspace of W, but there exists some element $g^* \in W$, $i_0(g^*) = \emptyset$. Therefore $dim(U_j) < dim(W) = dim(E(\lambda))$. The proof is complete. □

Based on Theorem 3.1, we may safely restrict our attention to the equivalence class $[i_0(\lambda)]$. We briefly recall the concepts of matrix reducibility and articulation points.

Definition 3.2. *A matrix $A \in \mathbb{R}^{n \times n}$ is reducible if there exists a permutation matrix $\pi \in \mathbb{R}^{n \times n}$ such that*

$$\pi A \pi^T = \begin{pmatrix} A_{1,1} & A_{1,2} \\ 0 & A_{2,2} \end{pmatrix}.$$

Definition 3.3. *A matrix $A \in \mathbb{R}^{n \times n}$ has degree of reducibility r if there exists a permutation matrix $\pi \in \mathbb{R}^{n \times n}$ such that*

$$\pi A \pi^T = \begin{pmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,r+1} \\ 0 & A_{2,2} & \cdots & A_{2,r+1} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & A_{r+1,r+1} \end{pmatrix},$$

with $A_{i,i}$ irreducible, $1 \leq i \leq r+1$.

Definition 3.4. *Let $G = (V, E) \in \mathcal{G}$ be a graph. If the removal of vertex $v \in V$ results in a disconnected graph, we say vertex v is an articulation point (or cut vertex) of G. In addition, if the removal of a subset $V_0 \subset V$ results in a disconnected graph, we say that the set of vertices V_0 is an articulation set of G.*

It is well known that a graph G is connected if and only if its corresponding generalized Laplacian matrix $M(G)$ is irreducible. Let us write M in block notation, $i_0(\lambda) = \{1, ..., r\}$, $[n] \backslash i_0(\lambda) = \{r+1, ..., n\}$,

$$M = \begin{pmatrix} N & -A \\ -A^T & \widetilde{M} \end{pmatrix},$$

$N \in \mathbb{R}^{r \times r}$, $\widetilde{M} \in \mathbb{R}^{(n-r) \times (n-r)}$, $A \in \mathbb{R}^{r \times (n-r)}$, and also write vectors $f \in \mathbb{R}^n$ in a similar fashion $f = (f_0^T, \widetilde{f}^T)^T$, $f_0 \in \mathbb{R}^r$, $\widetilde{f} \in \mathbb{R}^{n-r}$. Even further, suppose the matrices N and $\widetilde{M}$ have degree of reducibility p and q, respectively. Then we may write the matrices N, $\widetilde{M}$, and A in the form

$$N = \begin{pmatrix} N_1 & 0 & \cdots & 0 \\ 0 & N_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & N_p \end{pmatrix}, \widetilde{M} = \begin{pmatrix} M_1 & 0 & \cdots & 0 \\ 0 & M_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & M_q \end{pmatrix}, A = \begin{pmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,q} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,q} \\ \vdots & \vdots & \ddots & \vdots \\ A_{p,1} & A_{p,2} & \cdots & A_{p,q} \end{pmatrix},$$

where N_i, $i = 1, ..., p$, and M_j, $j = 1, ..., q$, are irreducible. Here, some $A_{i,j}$ may be zero. Again, we may write $\widetilde{f}$ in a similar fashion $\widetilde{f} = (f_1^T, f_2^T, ..., f_q^T)^T$. We may naturally associate each submatrix N_i and M_j with a corresponding vertex subset V_i and $\widetilde{V}_j$ in V, respectively. This structure induces a natural bipartite graph $H = (X, Y, E_H)$, $X = \{x_1, ..., x_p\}$, $Y = \{y_1, ..., y_q\}$, with $\{x_i, y_j\} \in E_H$ if and only if $A_{i,j} \not\equiv 0$.

Suppose we are concerned with the k^{th} eigenvalue λ_k of M with multiplicity m,

$$\lambda_1 \leq ... < \lambda_k = ... = \lambda_{k+m-1} < \lambda_{k+m} \leq ... \leq \lambda_n.$$

We note that λ_k and $\widetilde{f}$ are an eigenpair of $\widetilde{M}$, and $\widetilde{f}$ is in the null space of A. In addition, the multiplicity of λ_k with respect to $\widetilde{M}$ is at least m. In fact, we can fully characterize the eigenspace $E(\lambda_k)$. Indeed, this space is precisely the set of all f such that $f_j \in E(\lambda_k; M_j)$, $j = 1, ..., q$, and

$$\sum_{j=1}^{q} A_{i,j} f_j = 0$$

for all $i = 1, ..., p$. Let us denote the multiplicity and index of λ_k with respect to M_j by m_j and k_j, respectively. Then we immediately have that the multiplicity and index of λ_k with respect to $\widetilde{M}$ is given by

$$\widetilde{m} = \sum_{j=1}^{q} m_j \text{ and } \widetilde{k} = [\sum_{j=1}^{q} k_j] - q + 1.$$

We recall the well-known eigenvalue interlacing theorem.

Theorem 3.5 (Eigenvalue interlacing theorem, [21])**.** *Let S be a real $n \times m$ matrix ($n > m$) such that $S^T S = I$, and let A be a symmetric $n \times n$ matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$. Define $B = S^T AS$ and let B have eigenvalues $\mu_1 \leq \mu_2 \leq ... \leq \mu_m$. Then the eigenvalues of B interlace those of A, namely*

$$\lambda_i \leq \mu_i \leq \lambda_{n-m+i}.$$

By eigenvalue interlacing,

$$\lambda_k = \lambda_{k+m-1}(M) < \lambda_{k+m}(M) \leq \lambda_{k+m}(\widetilde{M}),$$

which implies that

$$\widetilde{k} \leq k + m - \widetilde{m}.$$

For each eigenspace $E(\lambda_k; M_j)$, we can construct a basis $\{\widetilde{f}_i^{(1)}, ..., \widetilde{f}_i^{(m_i)}\} \subset \mathbb{R}^{|\widetilde{V}_j|}$ satisfying $i_0(\widetilde{f}_j^{(\sigma)}) = \emptyset$ for all $j = 1, ..., q$, $\sigma = 1, ..., m_j$. Let $f_j^{(\sigma)} \in \mathbb{R}^n$ be the extension of $\widetilde{f}_j^{(\sigma)}$ to $\mathbb{R}^n$, namely,

$$f_j^{(\sigma)}(v) = \widetilde{f}_j^{(\sigma)}(v) \quad \text{for all } v \in \widetilde{V}_j, \quad f_j^{(\sigma)}(v) = 0 \text{ otherwise}.$$

For every $f \in E(\lambda_k; M)$, we can define a mapping from f to its corresponding representation in the basis $\{f_j^{(\sigma)}\}_{j=1,...,q}^{\sigma=1,...,m_j}$,

$$f = \sum_{j=1}^{q} \sum_{\sigma=1}^{m_j} \alpha_j^{(\sigma)} f_j^{(\sigma)}.$$

Indeed, in this new basis, $E(\lambda, M)$ is the subspace of $\mathbb{R}^{\widetilde{m}} = \{\{\alpha_j^{(\sigma)}\}_{j=1,...,q}^{\sigma=1,...,m_j} \mid -\infty < \alpha_j^{(\sigma)} < \infty\}$ satisfying

$$\sum_{j=1}^{q} \sum_{\sigma=1}^{m_j} \alpha_j^{(\sigma)} [A_{ij} f_j^{(\sigma)}] = 0 \quad \text{for all } i = 1, ..., p.$$

Furthermore, because λ_k has multiplicity m with respect to M, $E(\lambda; M)$ can be represented by a system of $\ell = \widetilde{m} - m$ homogeneous linear equations in the variables $\{\alpha_j^{(\sigma)}\}_{j=1,...,q}^{\sigma=1,...,m_j}$,

$$h_i(\{\alpha_j^{(\sigma)}\}_{j=1,...,q}^{\sigma=1,...,m_j}) = \sum_{j=1}^{q} \sum_{\sigma=1}^{m_j} c_{ij}^{(\sigma)} \alpha_j^{(\sigma)} = 0 \qquad i = 1, ..., \ell,$$

for some set of constants $c_{ij}^{(\sigma)} \in \mathbb{R}$, $i = 1, ..., \ell$, $j = 1, ..., q$, $\sigma = 1, ..., m_j$.

Now that we have sufficiently detailed the structure of eigenvalues with nodal vertices, we are prepared to prove the main result of the paper.

Theorem 3.6. *Let $G = (V, E)$ be a connected graph and M be an associated generalized Laplacian. Then for any eigenvalue λ_k, there exists a corresponding eigenvector f_k such that $\mathcal{D}(f_k) \leq k$. Furthermore, the set of $f_k \in E(\lambda_k)$ with $\mathcal{D}(f_k) \leq k$ has co-dimension zero.*

Proof. By Theorem 3.1, we may restrict ourselves to $[i_0(\lambda_k)]$. If $i_0(\lambda_k) = \emptyset$, then $\mathcal{W}(f) = \mathcal{G}(f) = \mathcal{D}(f)$ for all $f \in E(\lambda_k)$ and, using Theorem 2.3, we have $\mathcal{D}(f) \leq k$. Therefore, in what follows, we may suppose that $i_0(\lambda_k)$ is non-empty.

Let $H = (X, Y, E_H)$, $|X| = p$, $|Y| = q$, be the bipartite graph induced by the nodal vertices $i_0(\lambda_k)$. Because $G \in \mathcal{G}$, H is connected. Suppose without loss of generality that the irreducible components of N and $\widetilde{M}$ are ordered such that in the corresponding graph H, for all $i > 1$, $d(x_i, x_j) = 2$ for some $j < i$, and the vertices of Y, $y_1, ..., y_q$, are ordered such that

$$\min_{x_i \in N(y_{j_1})} i \leq \min_{x_i \in N(y_{j_2})} i \quad \text{if and only if} \quad j_1 \leq j_2.$$

Let m and $\widetilde{m}$ be the multiplicity of the eigenvalue λ_k with respect to M and $\widetilde{M}$, respectively, and $\ell = \widetilde{m} - m$. Let $\widetilde{k}$ be the index of λ_k with respect to $\widetilde{M}$. By Corollary 2.4, for any $\widetilde{f}$ with $\widetilde{M}\widetilde{f} = \lambda_k \widetilde{f}$, $i_0(\widetilde{f}) = \emptyset$, we have

$$\mathcal{W}(\widetilde{f}) = \mathcal{G}(\widetilde{f}) \leq \widetilde{k} + q - 1,$$

giving

$$\mathcal{W}(\widetilde{f}) = \mathcal{G}(\widetilde{f}) \leq k + m - \widetilde{m} + q - 1 = k + (q - \ell - 1).$$

For the remainder of the proof, we will consider vectors $f \in E(\lambda; M)$ in the basis $\{f_j^{(\sigma)}\}_{j=1,...,q}^{\sigma=1,...,m_j}$ and work within the space $\mathbb{R}^{\widetilde{m}} = \{\{\alpha_j^{(\sigma)}\}_{j=1,...,q}^{\sigma=1,...,m_j} | -\infty < \alpha_j^{(\sigma)} < \infty\}$. Let us define

$$j_i, \sigma_i := \operatorname*{argmax}_{j,\sigma} \; \big(j + \frac{\sigma}{m_j}\big) \mathbf{1}\{c_{ij}^{(\sigma)} \neq 0\}, \quad i = 1, ..., \ell,$$

and without loss of generality suppose $(j_{i_1}, \sigma_{i_1}) = (j_{i_2}, \sigma_{i_2})$ if and only if $i_1 = i_2$. If this is not the case, then by manipulation of the equations $\{h_i\}_{i=1}^{\ell}$ this can be achieved in a manner similar to reducing a matrix to row echelon form.

Let us fix the values of $\{\alpha_{j_i}^{(\sigma_i)}\}_{i=1}^{\ell}$ as a function of the values of other variables, namely,

$$\alpha_{j_i}^{(\sigma_i)} = -\frac{1}{c_{ij_i}^{(\sigma_i)}} \sum_{\alpha_j^{(\sigma)} \neq \alpha_{j_i}^{(\sigma_i)}} c_{ij}^{(\sigma)} \alpha_j^{(\sigma)}, \quad i = 1, ..., \ell.$$

We now have $E(\lambda_k; M) = \{\{\alpha_j^{(\sigma)}\}_{j=1,...,q}^{\sigma=1,...,m_j} \backslash \{\alpha_{j_i}^{(\sigma_i)}\}_{i=1,...,\ell} | -\infty < \alpha_j^{(\sigma)} < \infty\}$. Let us define

$$W := \{y_j \in Y | \; j = j_i \; \text{ for some } i\}.$$

We will now inductively create vectors $f \in E(\lambda_k; M)$ with $\mathcal{D}(f) \leq k$.

For any $f \in E(\lambda_k; M)$, $i_0(\tilde{f}) = \emptyset$, $\tilde{f}$ already has at most $k + (q - \ell - 1)$ strong nodal domains. We aim to produce a set of decompositions that decreases the number of nodal domains by at least $q - \ell - 1$. We will do this inductively, by traversing the elements of $Y \backslash W$, $|Y \backslash W| \geq q - \ell$, in chronological order.

Consider the first element $\hat{y}_1 \in Y \backslash W$, $\hat{y}_1 \in N(x_{\hat{i}_1})$, $\hat{y}_1 \notin N(x_i)$ for all $i < \hat{i}_1$. Let $\widetilde{V}_{\hat{j}_1}$ be the vertex set corresponding to $\hat{y}_1$, and $v \in \widetilde{V}_{\hat{j}_1} \cap N(V_{\hat{i}_1})$. Without loss of generality, suppose every element in the basis of $E(\lambda_k; M_{\hat{j}_1})$ takes a positive value at v. We define $\alpha_{\hat{j}_1}^{(\sigma)} > 0$ for all $\sigma = 1, ..., m_{\hat{j}_1}$. We give all the vertices in $V_{\hat{i}_1}$ a positive charge.

Now, let us consider the t^{th} element $\hat{y}_t \in Y \backslash W$, $\hat{y}_t \in N(x_{\hat{i}_t})$, $\hat{y}_t \notin N(x_i)$ for all $i < \hat{i}_t$. Let $\widetilde{V}_{\hat{j}_t}$ be the vertex set corresponding to $\hat{y}_t$, and $v \in \widetilde{V}_{\hat{j}_t} \cap N(V_{\hat{i}_t})$. Without loss of generality, suppose every element in the basis of $E(\lambda_k; M_{\hat{j}_t})$ takes a positive value at v. If $V_{\hat{i}_t}$ already has a charge, let us define $\alpha_{\hat{j}_t}^{(\sigma)}$, $\sigma = 1, ..., m_{\hat{j}_t}$, to be the same sign as the charge of $V_{\hat{i}_t}$. If $V_{\hat{i}_t}$ has not been assigned a charge, we choose the charge to be the same sign as an attached non-nodal vertex that has already been given a valuation. By our ordering of Y, such a non-nodal vertex must exist. In this case, again, we define $\alpha_{\hat{j}_t}^{(\sigma)}$, $\sigma = 1, ..., m_{\hat{j}_t}$, to be the same sign as the charge of $V_{\hat{i}_t}$. We have decreased the number of nodal domains in our decomposition by at least one.

Recalling that $|Y \backslash W| \geq q - \ell$, we have constructed a set of eigenvectors f with nodal decompositions that reduce the number of domains by at least $q - \ell - 1$, as desired. Because we have only restricted the signs of certain elements, the set of eigenvectors which we have restricted ourselves to is of co-dimension zero with respect to $E(\lambda_k; M)$. The proof is complete. □

Acknowledgement

The author would like to thank Louisa Thomas for improving the style of the presentation.

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