From the Hecke Category to the Unipotent Locus

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G split semisimple group

$\mathcal{U}$ unipotent locus

$\mathcal{B}$ flag variety

Springer resolution:

$$\tilde{\mathcal{U}} = \{(u, B) \in \mathcal{U} \times \mathcal{B} : u \in B\} \to \mathcal{U}$$

Steinberg variety: $\mathcal{Z} = \tilde{\mathcal{U}} \times_{\mathcal{U}} \tilde{\mathcal{U}}$

Lusztig introduced the *equivariant homology*:

$$\mathrm{H}_*^{\mathrm{BM},G}(\mathcal{Z}, \mathbf{Q}) \simeq \mathbf{Q}[W] \otimes \mathbf{Q}[\mathfrak{t}]$$

where W is the Weyl group and $\mathfrak{t}$ the Cartan.

Coxeter presentation:

$$W = \left\langle s_1, s_2, \ldots, s_r : s_i^2 = 1, \overbrace{s_i s_j s_i \cdots}^{m_{i,j}} = \overbrace{s_j s_i s_j \cdots}^{m_{i,j}} \right\rangle$$

Braid group:

$$Br_W = \left\langle \sigma_1, \sigma_2, \ldots, \sigma_r : \overbrace{\sigma_i \sigma_j \sigma_i \cdots}^{m_{i,j}} = \overbrace{\sigma_j \sigma_i \sigma_j \cdots}^{mi,j} \right\rangle$$

Positive submonoid: $Br_W^+ \subseteq Br_W$

For all $\beta \in Br_W^+$, we'll define $\mathcal{U}(\beta)$ and $\mathcal{Z}(\beta)$.

For the identity $\mathbf{1}$, we'll have $\mathcal{U}(\mathbf{1}) = \tilde{\mathcal{U}}$ and $\mathcal{Z}(\mathbf{1}) = \mathcal{Z}$.

We're interested in

$$\mathbf{A}^{i,j}(\beta) = \mathrm{gr}_j^{\mathbb{W}} \, \mathrm{H}_i^{\mathrm{BM},G}(\mathcal{Z}(\beta), \mathbf{Q})$$

where $\mathbb{W}_{\leq *}$ is the *weight filtration*.

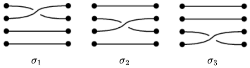
Note $\pm \sum_{i,j} (-1)^i q^{j/2} \dim \mathbf{A}^{i,j}(\beta) = \dfrac{|\mathcal{Z}(\beta)(\mathbf{F}_q)|}{|G(\mathbf{F}_q)|}$.

Goals of talk:

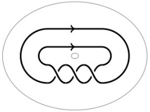
1. Relate $\mathbf{A}^{i,j}(\beta)$ to link homology
2. Relate $\mathcal{U}(\beta) \to \mathcal{U}$ to "full twist" duality in link homology
3. Relate $\mathbf{A}^{i,j}(\beta)$ to representations of DAHAs ("Cherednik algebras")

How do braids relate to topological links?

If $W = S_n$, then $Br_W = Br_n$.



The conjugacy class of $\beta \in Br_n$ is the same information as its *annular closure*.



Embedding into $\mathbf{R}^3$ gives a link.

Ex $\sigma_1^3 \in Br_2$ gives a trefoil, as does $(\sigma_1 \sigma_2)^2 \in Br_3$.

Broué–Michel and Deligne:

$$\mathcal{O} : Br_W^+ \to \left\{ \begin{array}{l} G\text{-varieties over } \mathcal{B} \times \mathcal{B} \\ \text{up to strict isomorphism} \end{array} \right\}$$

It sends $\beta = \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_\ell}$ to

$$p_0 \times p_\ell : \mathcal{O}(\beta) = \{B_0 \xrightarrow{s_1} B_1 \xrightarrow{s_2} \cdots \xrightarrow{s_\ell} B_\ell\} \to \mathcal{B} \times \mathcal{B}$$

where $\xrightarrow{w}$ means relative position w.

Annular closure $\approx$ pullback along $\Delta : \mathcal{B} \to \mathcal{B} \times \mathcal{B}$

- Shende–Treumann-Zaslow (2013),
- Mellit (2019),
- Casals–Gorsky–Gorsky–Simental (2020)

studied the fibers of $\mathcal{B}(\beta) = \Delta^{-1}(\mathcal{O}(\beta)) \to \mathcal{B}$.

We instead define $\mathcal{U}(\beta)$ by the pullback

$$\begin{CD} \mathcal{U}(\beta) @>>> \mathcal{O}(\beta) \\ @VVV @VVV \\ \mathcal{U} \times \mathcal{B} @>{act}>> \mathcal{B} \times \mathcal{B} \end{CD}$$

where $act(u, B) = (uBu^{-1}, B)$.

$\mathcal{B}(\beta)$ is the fiber of $\mathcal{U}(\beta) \to \mathcal{U}$ above $1 \in \mathcal{U}$.

Define the *Steinberg scheme* of β to be:

$$\mathcal{Z}(\beta) = \mathcal{U}(\mathbf{1}) \times_{\mathcal{U}} \mathcal{U}(\beta)$$

Thus, $\mathcal{U}(\mathbf{1}) = \{(u, B) : uBu^{-1} = B\} = \tilde{\mathcal{U}}$ and $\mathcal{Z}(\mathbf{1}) = \mathcal{Z}$.

Prop (T) $\mathcal{U}(\beta) \to \mathcal{U} \times \mathcal{B}$ is a stratified fiber bundle.

The fibers are paved by algebraic tori.

Ex If $G = \mathrm{SL}_2$, then $\mathcal{U}$ is the quadric cone and $\mathcal{B} = \mathbf{P}^1$.

$$\mathcal{U} \times \mathcal{B} = \overbrace{\{(u, B) : u \in B\}}^{\mathcal{U}(\mathbf{1})} \sqcup \overbrace{\{(u, B) : u \notin B\}}^{\mathcal{U}(\sigma_1)}$$

The map $\mathcal{U}(\sigma_1^2) \to \mathcal{U} \times \mathcal{B}$ is:

- An $\mathbf{A}^1$-bundle over $\mathcal{U}(\mathbf{1})$.
- An $(\mathbf{A}^1 - pt)$-bundle over $\mathcal{U}(\sigma_1)$.

$\mathcal{B}(\sigma_1^2)$ is the pullback of the first bundle to $\{1\} \times \mathcal{B} \simeq \mathbf{P}^1$.

Recall: $\mathcal{Z}(\beta) = \mathcal{U}(\mathbf{1}) \times_{\mathcal{U}} \mathcal{U}(\beta)$ and

$$\mathbf{A}(\beta) = \bigoplus_{i,j} \mathrm{gr}_j^{\mathbb{W}} \mathrm{H}_i^{\mathrm{BM},G}(\mathcal{Z}(\beta))$$

Pull-push along projections from

$$\mathcal{U}(\mathbf{1}) \times_{\mathcal{U}} \mathcal{U}(\mathbf{1}) \times_{\mathcal{U}} \mathcal{U}(\beta)$$

defines an $\mathbf{A}(\mathbf{1})$-action on $\mathbf{A}(\beta)$.

As a ring, $\mathbf{A}(\mathbf{1}) = \mathbf{Q}[W] \ltimes \mathbf{Q}[\mathbf{t}]$.

By Springer theory,

- $\mathrm{Hom}_W(\mathrm{triv}, \mathbf{A}(\beta)) = \bigoplus_{i,j} \mathrm{gr}_j^{\mathbb{W}} \mathrm{H}_i^{\mathrm{BM},G}(\mathcal{U}(\beta))$.
- $\mathrm{Hom}_W(\mathrm{sgn}, \mathbf{A}(\beta)) = \bigoplus_{i,j} \mathrm{gr}_j^{\mathbb{W}} \mathrm{H}_i^{\mathrm{BM},G}(\mathcal{B}(\beta))$.

For $\beta \in Br_n^+$, we can relate $\mathbf{A}(\beta)$ to the link closure $\hat{\beta}$.

HOMFLY–PT–Khovanov–Rozansky:

$$\mathbf{P} : \{\textit{links}\}/\textit{isotopy} \to \mathbf{Z}[a^{\pm 1}, t^{\pm 1}](q^{1/2})$$

Implicit in Galashin–Lam: For all $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell} \in Br_n^+$,

$$[a^{\ell+n-1}]\mathbf{P}(\hat{\beta}) \propto \sum_{i,j} t^i q^{j/2} \dim \mathrm{gr}_j^{\mathbb{W}} \mathrm{H}_i^{\mathrm{BM},G}(\mathcal{B}(\beta))$$

where $[-]$ means "coefficient of $(-)$."

Thm (T) For any W and $\beta \in Br_W^+$:

$$\mathbf{P}(\hat{\beta}) \propto \sum_{i,j,k} t^{-i} q^{j/2} (q^{1/2} a^2 t)^{-k} \dim \mathrm{Hom}_W(\Lambda^k(\mathbf{t}), \mathbf{A}^{i,j}(\beta))$$

Ex Recall that for $G = \mathrm{SL}_2$, we described $\mathcal{U}(\sigma^2)$.

Using $|G(\mathbf{F}_q)| = q^3 - q$, we compute:

$$\frac{|\mathcal{U}(\sigma^2)(\mathbf{F}_q)|}{|G(\mathbf{F}_q)|} = \frac{1 - q + q^2}{1 - q}$$

$$\frac{|\mathcal{Z}(\sigma^2)(\mathbf{F}_q)|}{|G(\mathbf{F}_q)|} = \frac{1 + q^2}{1 - q} = \frac{1 - q + q^2 + q}{1 - q}$$

These reflect $\mathbf{P}(\widehat{\sigma^2}) = \dfrac{q^{-1/2}}{1-q}(a(1 - q + q^2 t^2) + a^3 q t^3)$.

Beyond SL_2 and SL_3, not all irreps of W are in $\Lambda^*(\mathbf{t})$.

For all W, there's a $\otimes$-triangulated category $\mathbf{H}_W$, the *Hecke category*, and a $\otimes$-trace functor:

$$\mathrm{HHH} : \mathbf{H}_W \to \mathbf{grVect}_3$$

Rouquier: $\mathcal{R} : Br_W \to \left\{ \begin{array}{l} \text{objects of } \mathbf{H}_W \text{ up to strict} \\ \text{isomorphism} \end{array} \right\}$

Webster–Williamson: $\mathbf{P}(\hat{\beta}) \propto \dim \mathrm{HHH}(\mathcal{R}(\beta))$

Thm (T) For all W, we can factor HHH as

$$\mathbf{H}_W \xrightarrow{\mathrm{AH}} \mathbf{grMod}_2(\mathbf{A}(1)) \xrightarrow{\mathrm{Hom}_W(\Lambda^*(\mathbf{t}), -)} \mathbf{grVect}_3$$

such that $\mathbf{A}(\beta) \propto \mathrm{AH}(\mathcal{R}(\beta))$ for positive β.

AH should match Gorsky–Hogancamp–Wedrich *et al*.

Use split form G_0 over $\mathbf{F}_q$ and $G = G_0 \otimes \bar{\mathbf{F}}_q$.

$\mathbf{H}_W = \mathrm{K}^b(\mathrm{C}_{\mathcal{B}\times\mathcal{B}})$, where

$$\mathrm{C}_{\mathcal{B}\times\mathcal{B}} = \left\{ j_{w,!*}\bar{\mathbf{Q}}_\ell\langle i\rangle : \begin{array}{l} w \in W \\ i \in \mathbf{Z} \end{array} \right\} \subseteq \mathrm{D}^b_{mix,G}(\mathcal{B}_0 \times \mathcal{B}_0)$$

and j_w is the inclusion $\{B \xrightarrow{w} B'\} \subseteq \mathcal{B}_0 \times \mathcal{B}_0$.

Similarly:

C_G	(unipotent character sheaves on G_0)
C'_G	(summands of the Grothendieck sheaf on G_0)
$\mathrm{C}_\mathcal{U}$	(summands of the Springer sheaf on $\mathcal{U}_0$)

$\mathcal{B} \times \mathcal{B} \xleftarrow{act} G \times \mathcal{B} \xrightarrow{pr} G \xleftarrow{i} \mathcal{U}$ induces:

$$\theta : \mathbf{H}_W \xrightarrow{pr_* act^*} \mathrm{K}^b(\mathrm{C}_G) \to \mathrm{K}^b(\mathrm{C}'_G) \xrightarrow{i^*} \mathrm{K}^b(\mathrm{C}_\mathcal{U})$$

Realization functors:

$$\rho : \mathrm{K}^b(\mathrm{C}_{(-)}) \to \mathrm{D}^b_G(-)$$

AH (up to shifts) is:

$$\mathbf{H}_W \xrightarrow{\rho\theta} \mathrm{D}^b_G(\mathcal{U}) \xrightarrow{\mathrm{gr}^{\mathbb{W}}_* \mathrm{Hom}^*(-, \mathcal{S}pr)^\vee} \mathbf{grMod}_2(\mathbf{A}(\mathbf{1}))$$

HHH is:

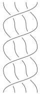
$$\mathbf{H}_W \xrightarrow{pr_* \, act^*} \mathrm{K}^b(\mathrm{C}_G) \xrightarrow{\mathrm{gr}^{\mathbb{W}}_* \mathrm{H}^*(G, (-) \otimes \bar{\mathbf{F}}_q)} \mathrm{K}^b(\mathbf{grVect}_2)$$

To show HHH factors through AH, need work of Rider:

$$\mathrm{Hom}_{\mathrm{D}^b_G(\mathcal{U})}(\rho(K), \rho(L)) \simeq \bigoplus_n \mathrm{Hom}_{\mathrm{K}^b(\mathrm{C}_{\mathcal{U}})}(K, L\langle n\rangle[-n]_{\mathrm{K}^b})$$

To show $\mathrm{AH}(\mathcal{R}(\beta)) = \mathbf{A}(\beta)$, need $\rho(\mathcal{R}(\beta)) = j_{\beta,!}\bar{\mathbf{Q}}_\ell$.

The *full twist* is a canonical central element $\pi \in Br^+_W$.



Gorsky–Hogancamp–Mellit–Nakagane, refining Kálmán:

For $\beta \in Br_n$ of length ℓ, the following match up to a power of t:

- $[a^{\ell-n+1}]\mathbf{P}(\widehat{\beta})$ (“bottom a-degree”)
- $[a^{\ell+n-1}]\mathbf{P}(\widehat{\beta\pi})$ (“top a-degree”)

Ex In Br_2, the full twist is $\pi = \sigma^2$.

$$[a^{-1}]\mathbf{P}(\hat{\mathbf{1}}) = \frac{q^{1/2}}{1-q}, \qquad [a^3]\mathbf{P}(\widehat{\sigma^2}) = \frac{q^{1/2}t^3}{1-q}$$

Cor (T) For positive β:

$$\begin{aligned} \mathrm{gr}_*^{\mathbb{W}} \mathrm{H}_*^{\mathrm{BM},G}(\mathcal{U}(\beta)) &\simeq \mathrm{Hom}_W(\mathrm{triv}, \mathbf{A}(\beta)) \\ &\simeq \mathrm{Hom}_W(\mathrm{sgn}, \mathbf{A}(\beta\pi)) \\ &\simeq \mathrm{gr}_*^{\mathbb{W}} \mathrm{H}_*^{\mathrm{BM},G}(\mathcal{B}(\beta\pi)) \end{aligned}$$

Conj (T) There's a $\mathbb{W}_{\leq *}$-preserving homeomorphism:

$$[\mathcal{U}(\beta)/G] \approx [\mathcal{B}(\beta\pi)/G]$$

Ex For $G = \mathrm{SL}_2$:

$$\mathcal{U}(\mathbf{1}) = T^\vee \mathbf{P}^1, \qquad \mathcal{B}(\sigma^2) = (\mathbf{P}^1)^2 - \mathbf{P}^1$$

Here both stack quotients are $[pt/(\{\pm 1\} \times \mathbf{G}_m)]$.

We can reduce to:

Conj (T) For all $B_0, B_1 \in \mathcal{B}$, there's a $\mathbb{W}_{\leq *}$-preserving homeomorphism:

$$\{u \in \mathcal{U} : uB_0u^{-1} = B_1\} \approx \{B \in \mathcal{B} : B_0 \xrightarrow{w_0} B \xrightarrow{w_0} B_1\}$$

Above, $w_0 \in W$ is the longest element.

Kawanaka matched their $\mathbf{F}_q$-point counts in 1975.

Ex Suppose $G = \mathrm{SL}_3$ and $B_0 \xrightarrow{w_0} B_1$.

The varieties are $\mathbf{G}_m \times X_1$ and $\mathbf{G}_m \times X_3$, where:

$$X_d = \{(x, y, z) \in \mathbf{A}^3 : xyz = (x-1)^d\}$$

Homeomorphic but not isomorphic.

A braid $\beta \in Br_W^+$ is *periodic* of slope $\frac{m}{n}$ iff $\beta^n = \pi^m$.

For such β, we can compute

$$[\mathbf{A}(\beta)]_q = \sum_{i,j} (-1)^i q^{j/2} \operatorname{gr}_j^{\mathbb{W}} \mathrm{H}_i^{\mathrm{BM},G}(\mathcal{Z}(\beta))$$

fairly explicitly.

Uses a q-deformation that Jones used to compute $\mathbf{P}|_{t=-1}$ for torus knots.

Thm (T) $[\mathbf{A}(\beta)]_q$ is the graded character of a virtual module over the *rational DAHA*

$$\mathbf{D}_{m/n}^{rat} = \frac{\mathbf{Q}[W] \ltimes (\mathbf{Q}[\mathfrak{t}] \otimes \mathbf{Q}[\mathfrak{t}^\vee])}{[x,y] - \langle x,y\rangle - \frac{m}{n} \sum_{\alpha\in\Phi^+} \langle x,\alpha^\vee\rangle\langle\alpha,y\rangle s_\alpha}$$

For *cuspidal* slopes $\frac{m}{n}$, it is *simple spherical* or almost so.

Oblomkov–Yun, inspired by Varagnolo–Vasserot, construct $\mathbf{D}_{m/n}^{rat}$-actions on modules

$$\operatorname{gr}_*^{\mathbb{P}} \mathrm{H}_{\mathbf{G}_m}^*(\mathcal{M}_{m/n})|_{\epsilon=1}$$

where $\mathcal{M}_{m/n}$ is a homogeneous *parabolic Hitchin fiber*.

Here, $\mathrm{H}_{\mathbf{G}_m}^*(pt) = \mathbf{Q}[\epsilon]$ and $\mathbb{P}$ is a *perverse filtration*.

Conj (T) The $\mathbf{A}(\mathbf{1})$-action on $\mathbf{A}(\beta)$ lifts to a *graded AHA* action on:

$$\mathrm{H}_*^{\mathrm{BM},\,G\times\mathbf{G}_m}(\mathcal{Z}(\beta))$$

For periodic β of slope $\frac{m}{n}$, induces a $\mathbf{D}_{m/n}^{rat}$-action on:

$$\operatorname{gr}_*^{\mathbb{W}} \mathrm{H}_*^{\mathrm{BM},\,G\times\mathbf{G}_m}(\mathcal{Z}(\beta))|_{\epsilon=1}$$

If β is periodic and $Br_W \to W$ sends $\beta \mapsto w$, then

$$C(w) \subseteq W$$

is a complex reflection group.

In OY, the $\mathbf{D}^{rat}_{m/n}$-action commutes with a $Br_{C(w)}$-action.

For slopes $\frac{1}{n}$, Broué–Michel conjectured

$$Br_{C(w)} = C(\beta).$$

They showed $C(\beta)^+$ acts on the étale site of the *Deligne–Lusztig variety* $X(\beta)$.

$X(\beta)$ is the pullback of $O(\beta)$ along the graph of Frobenius.

Is there a $C(\beta)^+$-action on $\mathrm{gr}^{\mathbb{W}}_* \mathrm{H}^{\mathrm{BM}, G \times \mathbf{G}_m}_*(\mathcal{Z}(\beta))|_{\epsilon=1}$?

Thank you for listening.