Floer homology for global quotient orbifolds

Miguel Miranda Ribeiro Moreira

Dissertação para obtenção do Grau de Mestre em

Mestrado em Matemática e Aplicações

Orientadores:
Prof. Miguel Abreu
Prof. Leonardo Macarini

Júri

Presidente: Prof. José Mourão
Orientador: Prof. Miguel Abreu
Vogal: Prof. Gustavo Granja

Julho de 2019

Contents

Abstract

In this dissertation we will introduce Floer homology for (compact and Calabi-Yau) global quotient orbifolds, which are obtained as quotients $[X/G]$ of a symplectic manifold X by a finite group G. We study a variation of Floer homology on X defined using Hamiltonian orbits and Floer trajectories which are not 1-periodic but obey a "g-periodicity" condition for a fixed symplectomorphism g, and then we use this variation to define the orbifold Floer complex. We discuss the grading, coherent orientations, the use of Novikov rings and transversality. Transversality is particularly interesting as the orbifold case poses new difficulties. Our main result is an isomorphism between the Floer homology and Chen-Ruan cohomology of an orbifold; this generalizes the well-known isomorphism between Floer and singular homologies in the smooth case.

Keywords – Symplectic geometry, Floer homology, orbifolds, Chen-Ruan cohomology.

Resumo

Nesta dissertação vamos introduzir homologia de Floer para orbivariedades (compactas e Calabi-Yau) que são quocientes globais, obtidas como quocientes $[X/G]$ de uma variedade simplética X por um grupo finito G. Estudamos uma variante de homologia de Floer em X definida usando órbitas Hamiltonianas e trajetórias de Floer que não são 1-periódicas mas obedecem a uma condição de "g-periodicidade" para um simplectomorfismo g fixo. Discutimos a graduação, orientações coerentes, o uso de anéis de Novikov e transversalidade. A transversalidade é particularmente interessante uma vez que o caso das orbivariedades coloca dificuldades novas. O nosso resultado principal é um isomorfismo entre a homologia de Floer e a cohomologia de Chen-Ruan de uma orbivariedade; este generaliza o célebre isomorfismo entre a homologia de Floer e homologia singular no caso suave.

Palavras-chave – Geometria simplética, homologia de Floer, orbivariedades, cohomologia de Chen-Ruan.

Chapter 1

Introduction

1.1 Historical overview of Floer homology

Floer homology was introduced in the '80s by Floer [Flo87, Flo89, FHS95]. Its original purpose was to prove Arnol'd conjecture on the number of fixed points of a Hamiltonian symplectomorphism. Arnol'd conjecture says the following: if H_t is a time dependent Hamiltonian on a compact symplectic manifold X, the number of 1-periodic orbits of the Hamiltonian flow of H_t is at least the sum of the Betti numbers of X (assuming all the 1-periodic orbits are non-degenerate).

Floer's idea was to construct a chain complex generated by the 1-periodic orbits of the Hamiltonian flow whose homology coincided with the singular homology of the manifold. This complex would be something that formally looks like the Morse complex of an action functional $\mathcal{A}_H : \Lambda X \to \mathbb{R}$ defined on the loop space ΛX and whose critical points are precisely the Hamiltonian orbits. However, it turns out that, after this formal similarity, Floer homology gets much different (and more complicated) flavours.

Due to the infinite dimension of ΛX the formal equation for the gradient flow of the action functional $\mathcal{A}_H$ is no longer an ordinary differential equation (as in the finite dimensional Morse case) but a partial differential equation imposed on maps $S^1 \times \mathbb{R} \to X$. Indeed it's a perturbed Cauchy-Riemann equation, which is an elliptic partial differential equation. Thus Floer theory consists largely in the study of the moduli spaces of solutions to these elliptic equations and their compactifications – elliptic regularity, Fredholm theory, Gromov compactness and gluing are some crucial tools in doing so. Another complication of Floer homology is that the indices of the orbits as critical points of $\mathcal{A}_H$ (in the usual Morse sense) are typically infinite. We should remark that both Gromov's earlier introduction of pseudo-holomorphic curves in symplectic manifolds (in the seminal paper [Gro85]) and Conley-Zehnder ideas in the proof of Arnol'd conjecture for $\mathbb{T}^{2n}$ (in [CZ83]) were highly influential in the development of Floer homology.

Floer was able to construct Floer homology, and hence prove Arnol'd conjecture, when the symplectic manifold was monotone (see definition 3.3.8). Essentially this condition was needed to avoid the existence of bubbles ruining the compactness of the moduli spaces, which is needed to prove for instance that the differential satisfies $\partial^2 = 0$. Later Salamon and Hofer extended the proof of Arnol'd conjecture to weakly monotone manifolds (see [HS95]), a case that includes Calabi-Yau manifolds. The general case of Arnol'd conjecture was claimed to be solved by Fukaya-Ono in [FO99]; for this much more sophisticated tools, which are not consensual in the symplectic geometry community, had to be developed. Some key words are multivalued perturbations, Kuranishi structures and virtual transversality.

1.2 Why extending Floer homology to orbifolds?

The goal of this thesis is to give a first approach to Hamiltonian Floer theory for orbifolds. We will construct (under some conditions) Floer homology $HF(\mathcal{X})$ when $\mathcal{X}$ is a compact global quotient orbifold, that is, $\mathcal{X}$ is obtained as the quotient X/G of a manifold X by a finite group G acting on X. Restricting ourselves to global quotients allows us to use well established Floer theory in our construction and not have to do everything from scratch.

There are several ways why this is reasonably interesting. The first obvious reasons are that orbifolds arise naturally in symplectic geometry – for example as a result of symplectic reduction – and that there are a lot of interesting examples of symplectic orbifolds.

Our particular starting motivation for studying this problem was an observation made in [AMM] by Abreu, Macarini and the author. We showed that one could relate the contact homology of Gorenstein toric contact manifolds to the Chen-Ruan cohomology of certain (toric) fillings. This was done by establishing a combinatorial interpretation for the dimensions of the contact homology groups. However, when the filling is smooth, there is a more direct proof using the work of Oancea-Bourgeois [BO13, BO17] that passes by the symplectic homology (and its positive and/or S^1-equivariant versions) of the filling. Of course that when the filling is an orbifold no such route is possible yet. Another similar looking situation is the recent interpretation of the generalized McKay correspondence using Floer homology by McLean-Ritter in [MR18]. In both cases the results could be proven very directly if we had a construction of orbifold symplectic homology and appropriate adaptations of the tools in [BO13, BO17]. More details about these expected applications will be given in 7.4.

We won't pursue the goal of defining symplectic homology (and its versions) in the orbifold setting in this thesis; we'll restrict ourselves to the compact case. In the smooth compact setting Floer homology is well known to be isomorphic to the singular homology of the manifold (up to a correction of the grading). The above mentioned motivating observations suggested that in the orbifold case we should replace singular homology by Chen-Ruan cohomology. This leads to the main theorem of this thesis; we state it here, omitting some details in the formulation:

Theorem. *Let $\mathcal{X} = [X/G]$ be a Calabi-Yau global quotient orbifold with symplectic form ω. If (H, J) is a pair of Hamiltonian and almost complex structure that allows us to define Floer homology of $\mathcal{X}$ then we have an isomorphism*

$$HF_*(\mathcal{X}, H, J; \Lambda) \cong H^{n-*}_{CR}(\mathcal{X}; \Lambda)$$

between Floer homology and Chen-Ruan cohomology with coefficients in the rational Novikov ring Λ.

This will be theorem 6.0.1. The appearance of Chen-Ruan cohomology seems quite interesting – for instance the fact that the the degree shifting numbers of Chen-Ruan cohomology appear naturally in the Floer construction is quite remarkable for us. The fact that both Chen-Ruan cohomology and Floer homology (more precisely, the Fukaya category) play a role in mirror symmetry may also spark interest in this result.

As far as we know, the main content of this thesis is essentially new. However, there is some work in the direction of an orbifold version of Lagrangian Floer theory and of the Fukaya category. Cho, Hong and others seem to be pursuing this goal for instance in [CH17, CP14]. Our construction was motivated by the orbifold Morse homology defined by Cho-Hong in [CH14], which had precisely this goal in mind. Another work worth mentioning is the definition of a $\mathbb{Z}/2$-invariant Lagrangian Floer homology by Seidel and Smith in [SS10]; contrary to us, they used a Borel type construction.

1.3 Structure of the dissertation

We begin this thesis by laying out some of the foundations of orbifolds in chapter 2. Although after this chapter we'll only consider global quotient orbifolds, we opted to present this introductory chapter in the more general setting. We will define orbifolds, morphisms between orbifolds, cohomology of orbifolds and Chen-Ruan cohomology. We also explain the construction of the Morse complex in [CH14].

The Floer complex of an orbifold $\mathcal{X}$ should be generated by 1-periodic Hamiltonian orbits. When $\mathcal{X} = [X/G]$ is a global quotient, such orbits lift to Hamiltonian orbits $\gamma : [0,1] \to X$ such that $\gamma(1) = g\gamma(0)$. For this reason, in chapter 3 we study a variation of the Hamiltonian Floer complex of a smooth manifold X that replaces 1-periodic orbits by orbits satisfying the condition $\gamma(1) = g\gamma(0)$ (we call this condition g-periodicity) for a fixed symplectomorphism g. This is a reasonably straightforward adaptation of standard Hamiltonian Floer theory and there is nothing essentially new in this chapter (see [DS94,FHS95]), besides giving a bit more detail than what's in the literature at some points. We also mention a way to interpret this in terms of Lagrangian Floer homology in 3.4.

Chapter 4 gives the key definition of Floer homology for global quotient orbifolds. We explain in 4.2 how to define a grading on the Floer homology when $\mathcal{X}$ is Calabi-Yau. It turns out that (as already observed in the Morse case) there is a need to exclude some Hamiltonian orbits from the complex according to how the group acts on their orientation spaces. Because of this we need to give a more detailed account of orientations in 4.3, again only in the Calabi-Yau case.

We discuss the important problem of equivariant transversality in chapter 5, that is, we ask if we can define Floer theory from generic data (H, J). It seems that in general the answer is no because of an obstruction that is explained in 5.1. We give a partial transversality result in section 5.2: we show that we can get transversality for Floer trajectories which aren't fixed by some $g \in G \setminus \{1\}$. This result follows very closely an idea in [KS02] (although our context is slightly different). Given this unsatisfactory situation, in section 5.3 we sketch how to redefine Floer homology of orbifolds in a way that avoids the problem of equivariant transversality.

Our main result, the isomorphism between Floer homology and Chen-Ruan cohomology of a global quotient Calabi-Yau orbifold, is stated and proved in chapter 6. We note that our result is an isomorphism only for some pairs (H, J) (with H a C^2-small autonomous Hamiltonians) because we haven't yet established an invariance result, but it seems to us that such a result should follow from standard Floer theoretical arguments and the ideas in 5.3 to avoid transversality problems again.

Finally, in chapter 7 we discuss a few possible directions for future work. We speculate about the possibilities of extending our definition to general orbifolds and of defining a product structure on Floer homology of orbifolds. We also discuss in a bit more detail the applications that we could get from extending the tools of symplectic homology to orbifolds.

Chapter 2

Orbifolds

Orbifolds are a generalization of smooth manifolds in which we allow some "not too bad" singularities: more precisely, we allow singularities that locally look like quotients of a smooth manifold by a finite group.

A precise definition and a lot of examples will be given in a while, but first let us digress a little bit about the role of orbifolds in mathematics. Orbifolds have been studied from long ago from an algebraic geometry point of view, and indeed algebraic geometry can provide many examples of orbifolds. They were firstly defined, as analytical objects, by Satake in [Sat56]. In the '70s Thurston studied orbifolds in the context of his geometrization program for 3-manifolds. But it was in the '80s that an immense rise of interest in orbifolds happened, due to work of Dixon, Harvey, Vafa and Witten (see [DHVW85]), who showed that orbifolds could play a role in string theory. Indeed this motivated later the definition of Chen-Ruan cohomology that we'll see, and will play an important role (see 6.0.1), and of Gromov-Witten invariants of orbifolds.

There are several interesting examples of orbifolds appearing naturally. The most obvious source of examples is the study of actions of Lie groups which aren't free but only almost free. A symplectic version of this is that it's very common for a symplectic reduction of a smooth symplectic manifold to be an orbifold; a particular example are toric orbifolds (see [LT97]), which provide a nice large family of symplectic orbifolds. Orbifolds (or their algebraic geometry version – stacks) also arise naturally as moduli spaces (see example 2.1.7). Orbifolds play a crucial role in mirror symmetry: it may happen that the mirror to a smooth variety is an orbifold (see example 2.1.9). Indeed the Chen-Ruan cohomology of an orbifold is very important in mirror symmetry as its product structure encodes information about counting genus 0 holomorphic curves. Finally, considering resolutions of (simple to describe) orbifolds is a good way to get very rich examples of smooth spaces.

2.1 Orbifolds – definitions and examples

2.1.1 Orbifold atlas

We will now give our first definition of orbifold. This definition is in terms of an orbifold atlas and it's the most natural way of capturing the idea that an orbifold is something that locally is a quotient of a manifold by a finite group. This was the original definition of orbifold (called at the time V-manifold) given by Satake in [Sat56]. We start with the definition of orbifold charts, which is slightly technical.

Definition 2.1.1. *Let X be a topological space and $n \geq 1$ an integer. An orbifold chart (of dimension n) on X is a triple $(\widetilde{U}, G, \phi)$ where $\widetilde{U} \subseteq \mathbb{R}^n$ is an open subset, G is a finite group acting smoothly and effectively on $\widetilde{U}$ and $\phi : \widetilde{U} \to X$ is a continuous map inducing a homeomorphism $\widetilde{U}/G$ onto an open subset $U \subseteq X$.*

An embedding of orbifold charts $(\widetilde{U}_1, G_1, \phi_1) \hookrightarrow (\widetilde{U}_2, G_2, \phi_2)$ consists of a smooth embedding $\iota : \widetilde{U}_1 \hookrightarrow \widetilde{U}_2$ such that $\phi_2 \circ \iota = \phi_1$.

An orbifold atlas is a family $\{(\widetilde{U}, G, \phi)\}$ of orbifold charts covering X and compatible in the following sense: given any two charts $(\widetilde{U}_1, G_1, \phi_1)$ and $(\widetilde{U}_2, G_2, \phi_2)$ and a point $x \in U_1 \cap U_2$ there is a third chart $(\widetilde{V}, H, \psi)$ with $x \in \psi(\widetilde{V})$ embedding in $(\widetilde{U}_j, G_j, \phi_j)$ for $j = 1, 2$.

We say that an orbifold atlas refines other if every chart of the former embeds into some chart of the latter. Two orbifold charts are equivalent if they admit a common refinement.

With this we can give the definition of (effective) orbifold.

Definition 2.1.2. *An (effective) orbifold is a Hausdorff and second countable topological space endowed with an equivalence class of orbifold atlas. We generally denote an orbifold by $\mathcal{X}$ and we denote its underlying topological space by $|\mathcal{X}|$. Sometimes we will abuse notation and write $x \in \mathcal{X}$ meaning that $x \in |\mathcal{X}|$.*

The singularities of an orbifold are associated to isotropy of the local group actions (recall that the quotient of a manifold by a free action of a finite group is smooth). So, given $x \in \mathcal{X}$ and an orbifold chart $(\widetilde{U}, G, \phi)$ around x we let $G_x = \{g \in G : g\widetilde{x} = \widetilde{x}\}$ where $\widetilde{x} \in \widetilde{U}$ is some lift of x. It can be shown that, up to conjugacy, this does not depend on the chart we choose. The singular set of $\mathcal{X}$ is the set $\{x \in \mathcal{X} : G_x \neq 1\}$.

2.1.2 Examples

Let's now discuss some examples of orbifolds. The most basic way in which we can get an orbifold is by taking the quotient of a smooth manifold by the action of a finite group. Then it's clear that an usual smooth atlas for the original manifold gives an orbifold atlas on the quotient. These are the examples we're mainly interested in.

Definition 2.1.3. *Let X be a manifold and G be a finite group acting on X. Then we denote by $\mathcal{X} = [X/G]$ the orbifold with underlying topological space X/G and orbifold atlas induced by a smooth atlas of X. We call such orbifolds global quotient orbifolds.*

More generally, suppose G is a compact Lie group acting on X; we say that the action of G on X is almost free if the isotropy groups G_x are finite for every $x \in X$. Then we can give an orbifold structure to the quotient X/G; this is true by the slice theorem, which roughly speaking says that near $x \in X$ the quotient X/G looks like $(G \times T_xS)/G_x$ where $T_xS = T_xX/T_x(G \cdot x)$. We still denote by $[X/G]$ the resulting orbifold. This is the most natural way to describe orbifolds, and indeed every orbifold is of this form.

Theorem 2.1.4. *Every orbifold can be obtained as the quotient $[X/G]$ of a smooth manifold X by a compact Lie group G acting on X almost freely.*

Proof. See [ALR07, Corollary 1.24]. ☐

We now give some very concrete examples of orbifolds. This is mainly intended to show the richness of the world of orbifolds and to mention a few places where they appear.

Example 2.1.5. Let $\mathbb{T}^4 = \mathbb{R}^4/\mathbb{Z}^4$ be the four dimensional torus and consider the involution $\tau : \mathbb{T}^4 \to \mathbb{T}^4$ given by

$$\tau(x_1, x_2, x_3, x_4) = (-x_1, -x_2, -x_3, -x_4).$$

Then $\langle\tau\rangle \cong \mathbb{Z}/2$ acts on $\mathbb{T}^4$ and the resulting quotient $\mathbb{T}^4/(\mathbb{Z}/2)$ is called the Kummer surface. This orbifold has 16 singular points, of the form $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$ with $\varepsilon_j \in \{0, 1/2\}$.

Example 2.1.6. Let X be a smooth manifold. The symmetric product of X is X^n/S_n where S_n acts on X^n by permuting the coordinates. This space is the configuration space of n undistinguishable points in X. Singularities come from elements of X^n having some coordinates repeated.

Example 2.1.7. The moduli spaces $\mathcal{M}_{g,n}$ and their Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$, for $2g-2+n > 0$, have an orbifold structure. The simplest interesting example is $\mathcal{M}_{1,1}$, which is the moduli space of elliptic curves, that is, holomorphic structures on the torus. An elliptic curve takes the form $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ for some $\tau \in H = \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$ and the elliptic curves corresponding to $\tau, \tau' \in H$ are isomorphic if and only if there is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2;\mathbb{Z}) \text{ such that } \tau' = \frac{a\tau + b}{c\tau + d}.$$

This action is not effective as $-I$ acts trivially, so we can replace $SL(2;\mathbb{Z})$ by $PSL(2;\mathbb{Z}) = SL(2;\mathbb{Z})/\langle -I\rangle$ to get an effective orbifld $H/PSL(2;\mathbb{Z})$. There are two singular points: i with isotropy group $\mathbb{Z}/2$ and $e^{\frac{2\pi i}{3}}$ with isotropy group $\mathbb{Z}/3$.

Example 2.1.8. Let $a_1, \dots, a_n \in \mathbb{Z}$ be coprime integers and consider the action of S^1 on

$$S^{2n-1} = \left\{ (z_1, \dots, z_n) \in \mathbb{C}^n : \sum_{j=1}^{n} |z_j|^2 = 1 \right\}$$

given by

$$e^{2\pi i t} \cdot (z_1, \dots, z_n) = \left(e^{2\pi i a_1 t} z_1, \dots, e^{2\pi i a_n t} z_n\right).$$

This action is almost free and the resulting quotient is an orbifold, called a weighted projective space and denoted $\mathbb{C}P^n(a_1, \dots, a_n)$. For example at $(1, 0, \dots, 0)$ the isotropy group of the action is $\mathbb{Z}/a_1$. It can be shown that weighted projective spaces are not global quotients.

Example 2.1.9. Let

$$Y = \left\{ [z_0 : \dots : z_4] : z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 + \phi z_0 z_1 z_2 z_3 z_4 \right\} \subseteq \mathbb{C}P^4$$

be a quintic hypersurface in $\mathbb{C}P^4$ with $\phi \in \mathbb{C}$ a generic constant. Let $\zeta = e^{\frac{2\pi i}{5}}$ and consider automorphisms $e_1, e_2, e_3 : Y \to Y$ given by

$$\begin{aligned} e_1(z_0, z_1, z_2, z_3, z_4) &= (z_0, \zeta z_1, z_2, z_3, \zeta^{-1} z_4) \\ e_2(z_0, z_1, z_2, z_3, z_4) &= (z_0, z_1, \zeta z_2, z_3, \zeta^{-1} z_4) \\ e_3(z_0, z_1, z_2, z_3, z_4) &= (z_0, z_1, z_2, \zeta z_3, \zeta^{-1} z_4). \end{aligned}$$

Then e_1, e_2, e_3 generate a group isomorphic to $(\mathbb{Z}/5)^3$ acting on Y. The quotient orbifold is called the mirror quintic and it really is the mirror of a quintic (see [LS14]).

Example 2.1.10. A big family of orbifolds is given by toric varieties. Toric orbifolds are classified by simplicial fans (see [CLS11, Theorem 3.1.19]). If one wants to consider toric orbifolds with a symplectic structure, a classification in terms of certain labelled polytopes is given in [LT97].

2.1.3 Orbifolds as Lie groupoids

A more conceptual way of thinking of orbifolds is in terms of (proper étale) Lie groupoids. This approach was pioneered in [MP97] and it has some advantages: it allows us to deal with non-effective orbifolds, it makes some concepts associated to orbifolds much more natural looking, such as the orbifold classifying space or orbifold morphisms, and it avoids the local treatment of orbifolds. This also makes the definition of orbifold more related to the definition of stacks, its algebraic counterpart.

Recall that a Lie groupoid $\mathcal{G}$ consists of a manifold of objects G_0 and a manifold of arrows G_1 with the following structure maps: the target and source maps $s, t : G_1 \to G_0$, unit map $u : G_0 \to G_1$, inverse map $i : G_1 \to G_1$ and composition map $m : G_1 \,{}_s\!\times_t G_1 \to G_1$. All these maps are required to be smooth and to obey the "obvious" properties: composition has to be associative and has to interact in the expected way with the unit and inverse maps. Moreover we require s, t to be submersions – this is needed so that $G_1 \,{}_s\!\times_t G_1$ is a manifold. We sometimes write $g : x \to y$ meaning that $g \in G_1$ and $s(g) = x, t(g) = y$.

Definition 2.1.11. *A Lie groupoid $\mathcal{G}$ is said to be proper if the map $(s, t) : G_1 \to G_0 \times G_0$ is proper.*

The Lie groupoid $\mathcal{G}$ is said to be étale if s and t are local diffeomorphisms; in this case we can define the dimension of $\mathcal{G}$ to be $\dim \mathcal{G} = \dim G_0 = \dim G_1$.

An orbifold groupoid is an étale and proper Lie groupoid. Associated to it we have an underlying topological space: its orbit space $|\mathcal{G}| = G_0/\!\sim$ where $\sim$ is the equivalence relation defined by $x \sim y$ if and only if there is $g \in G_1$ such that $x = s(g)$ and $y = t(g)$.

Note that in an étale proper Lie groupoid the isotropy groups $G_x = s^{-1}(x) \cap t^{-1}(x)$ (that is, the arrows from an object $x \in G_0$ to itself) are finite since they are discrete and compact because $\mathcal{G}$ is étale and proper, respectively.

Before we proceed, let's sketch how this is related to the previous definition of orbifold in terms of orbifold atlases. First, global quotients $[X/G]$ can be represented by a Lie groupoid $\mathcal{G} = G \ltimes X$ with

$$G_0 = X \ , \ G_1 = G \times X \ , \ s(g, x) = x \text{ and } t(g, x) = g \cdot x.$$

Its orbit space $|\mathcal{G}|$ is clearly X/G. In general, if we have an orbifold atlas $\{(\widetilde{U}_\alpha, G_\alpha, \phi_\alpha)\}$ we can construct an orbifold groupoid by gluing $G_\alpha \ltimes \widetilde{U}_\alpha$ is a way along the lines of [ALR07, Example 1.33].

Conversely, given an orbifold groupoid we can construct an atlas as follows: given $x \in G_0$ the isotropy group G_x acts on a neighborhood $\widetilde{U}_x$ of x, as we will now explain. Given $g \in G_x$ let $\sigma : \widetilde{U}_x \to V_g$ be a local inverse to s such that $\sigma(x) = g$ where $x \in \widetilde{U}_x \subseteq G_0$ and $g \in V_g \subseteq G_1$ are open neighbourhoods; also, assume that t maps V_g to $\widetilde{U}_x$. Then we get a map $g = t \circ \sigma : \widetilde{U}_x \to \widetilde{U}_x$ that is the action of g on $\widetilde{U}_x$. With this we construct an orbifold atlas $\{(\widetilde{U}_x, G_x, \phi_x)\}_{x \in G_0}$ where $\phi_x : \widetilde{U}_x \hookrightarrow G_0 \to |\mathcal{G}|$. We'll use repeatedly this idea that an arrow $g : x \to y$ extends to a diffeomorphism $g : \widetilde{U}_x \to \widetilde{U}_y$ from an open neighbourhood of x to an open neighbourhood of y mapping x to y.

Of course different orbifold groupoids may lead to the same orbifold, just in the same way as different atlases may be equivalent (if they admit a common refinement). For groupoids, this notion is captured in the definition of Morita equivalence.

Definition 2.1.12. *An equivalence of Lie groupoids is a homomorphism $\phi : \mathcal{H} \to \mathcal{G}$ such that the map*

$$t\pi_1 : G_1 \,{}_s\!\times_\phi H_0 \to G_0$$

sending a pair $(g, y) \in G_1 \times H_0$ such that $s(g) = \phi(y)$ to $t(g)$ is a surjective submersion and the diagram

$$\begin{array}{ccc} H_1 & \xrightarrow{\phi} & G_1 \\ \downarrow{\scriptstyle (s,t)} & & \downarrow{\scriptstyle (s,t)} \\ H_0 \times H_0 & \xrightarrow{\phi\times\phi} & G_1 \times G_1 \end{array}$$

is a fibered product in the category of manifolds.

A Morita equivalence of orbifold groupoids $\mathcal{H}$ and $\mathcal{G}$ is a diagram of the form

$$\mathcal{H} \longleftarrow \mathcal{K} \longrightarrow \mathcal{G}$$

where both morphisms are equivalences of Lie groupoids.

The first condition implies that an equivalence ϕ is an essentially surjective functor, that is, every object in $\mathcal{G}$ is isomorphic to some object in the image of ϕ. The second condition implies that ϕ is full and faithful, that is, the map $\phi : G_1(x, y) \to H_1(\phi(x), \phi(y))$ is a bijection for any $x, y \in G_0$. Indeed this is what one gets by asking that $t\pi_1$ is surjective and and that the diagram is a fibered product in the category of sets, respectively. So this strange looking definition is just an enhancement of usual equivalence of categories to this smooth setting. It should be noted that if $\phi : \mathcal{H} \to \mathcal{G}$ is an equivalence then the induced continuous map on the orbit spaces $|\phi| : |\mathcal{H}| \to |\mathcal{G}|$ is a homeomorphism.

Informally, one can think of an orbifold groupoid as extra structure on the underlying topological space in analogy to an atlas being extra structure on a manifold/orbifold. In this analogy, equivalence of groupoids can be thought as a refinement of atlases, and thus Morita equivalence is analogous to the existence of a common refinement.

Definition 2.1.13. *An orbifold is a Morita equivalence class of orbifold groupoids. If $\mathcal{G}$ is some groupoid in this class, we say that it presents the corresponding orbifold.*

Remark 2.1.14. Some care is in order with this definition since the notion of étale groupoid is not stable under Morita equivalence. For instance if G is a compact non-finite Lie group acting almost freely on X then $\mathcal{G} = G \ltimes X$ is not étale but it's Morita equivalent to an étale groupoid. Even if $\mathcal{G}$ is not étale but is Morita equivalent to an orbifold groupoid we will say that it presents the corresponding orbifold.

2.2 Orbifold morphisms, orbibundles and the orbifold loop space

Giving a correct definition of morphisms between orbifolds that has good properties is something trickier than it might seem. Unfortunately, with the "straightforward" definition of a smooth map between orbifolds as a map between the underlying topological spaces that lifts in an appropriate manner to charts, which was originally given by Satake [Sat56], the pullback of orbibundles may be not an orbibundle. Nevertheless we state this definition here.

Definition 2.2.1. *Let $\mathcal{X}$, $\mathcal{Y}$ be effective orbifolds with underlyng topological spaces X, Y. A smooth morphism is a continuous map $f : X \to Y$ together with (an equivalence class of) lifts $\widetilde{f}_\alpha : \widetilde{U}_\alpha \to \widetilde{V}_\alpha$, where $\{(\widetilde{U}_\alpha, G_\alpha, \phi_\alpha)\}$ and $\{(\widetilde{V}_\alpha, H_\alpha, \psi_\alpha)\}$ are orbifold atlases of $\mathcal{X}$ and $\mathcal{Y}$, respectively, such that $f(\phi(\widetilde{U}_\alpha)) \subseteq \psi(\widetilde{V}_\alpha)$ and $\psi_\alpha \circ \widetilde{f}_\alpha = f_{|\phi(\widetilde{U}_\alpha)} \circ \phi_\alpha$.*

As we mentioned, this definition is not quite good enough. Two better suited versions are the notions of good map by Chen-Ruan [CR04] and of strong map by Moerdijk-Pronk [MP97], which turn out to be equivalent. We give the latter, which is formulated in the groupoid language (unlike good maps, which are defined using charts). We give the definition using a categorical language.

Definition 2.2.2. *Let* OGpoid *be the category of orbifold groupoids (which is a full subcategory of Lie groupoids) and let* OGpoid' *be the category obtained by identifying two morphisms in* OGpoid *if they are related by a natural transformation. Let* $\Sigma \subseteq \mathrm{Mor}(\mathrm{OGpoid}')$ *be the collection of equivalences of Lie groupoids. Then the orbifold category* Orb *is defined as the localization*

$$\mathrm{Orb} = \mathrm{OGpoid}'[\Sigma^{-1}].$$

An orbifold morphism is a morphism in this category. Every such morphism can be represented (in a non-unique way) by a diagram

$$\mathcal{G} \xleftarrow{\epsilon} \mathcal{K} \xrightarrow{\phi} \mathcal{H}$$

where ϵ *is an equivalence of Lie groupoids and* ϕ *is a homomorphism of Lie groupoids.*

It can be shown that orbifold morphisms are smooth orbifold maps in the "local lifts" sense. This definition is definitely not very practical: to understant orbifold morphisms from $\mathcal{G} \to \mathcal{H}$ we have to consider every equivalence $\epsilon : \mathcal{K} \to \mathcal{G}$ (in our analogy, we have to consider every possible refinement of a cover). It's impossible to compute anything (except very simple examples) directly. To understand the loop space of an orbifold, which will play a role later, we need a brief introduction to orbibundles.

2.2.1 Orbibundles

The notion of orbibundle generalizes vector bundles for smooth manifolds.

Definition 2.2.3. *Let* $\mathcal{G}$ *be an orbifold groupoid. A vector bundle over an orbifold presented by* $\mathcal{G}$ *consists in a vector bundle* $\pi : E \to G_0$ *together with a linear action of* $\mathcal{G}$*, that is, for every* $g : x \to y$ *we get a linear map between the fibers* $E_x \to E_y$*.*

The total space of the vector bundle is the orbifold presented by the groupoid $\mathcal{E} = \mathcal{G} \ltimes E$ *defined by*

$$\mathcal{E}_0 = E,\ \mathcal{E}_1 = G_1\ {}_s\!\times_p E,\ s(g,e) = e \text{ and } t(g,e) = g(e).$$

We have a natural projection map $\pi : \mathcal{E} \to \mathcal{G}$*. This map is said to be a (vector) orbibundle.*

Following a similar idea define principal orbibundles. We note that an orbibundle is not a vector bundle. Indeed the fiber at $x \in G_0$ is E_x/G_x, which is a quotient of a vector space by a finite group. If G_x acts trivially on E_x for every $x \in G_0$ we call $\mathcal{E}$ a honest bundle.

Note that orbibundles over a global quotient orbifold $[X/G]$ are precisely the same as G-equivariant vector bundles over X, which we discuss in appendix B.1.

Example 2.2.4. The tangent space $T\mathcal{G}$ to an orbifold can be defined as an orbibundle. Take $E = TG_0$ and consider the following $\mathcal{G}$-action: for each $g : x \to y$ we extend to a diffeomorphism $g : U_x \to U_y$ from a neighbourhood of x to a neighbourhood of y. Taking the differential we get a map $(dg)_x : T_xG_0 \to T_yG_0$.

Orbibundles can be pulled-back by orbifold morphisms. Indeed if $\epsilon : \mathcal{K} \to \mathcal{G}$ is an equivalence, then there is a correspondence between orbibundles over $\mathcal{K}$ and orbibundles over $\mathcal{G}$, which we denote by ϵ_*. Thus, the pull-back of a vector bundle $\mathcal{E}$ over $\mathcal{H}$ by an orbifold morphism represented by $\mathcal{G} \xleftarrow{\epsilon} \mathcal{K} \xrightarrow{\phi} \mathcal{H}$ is defined to be $\epsilon_*\phi^*\mathcal{E}$. The same applies for principal bundles.

2.2.2 Orbifold loop space

We can describe the loop space of a global quotient, that is, classify orbifold morphisms $S^1 \to \mathcal{X}$ where $\mathcal{X} = [X/G]$. This is done in [ALR07, Example 2.48] and here we sketh the idea. The projection map $X \to \mathcal{X}$ is a principal G-orbibundle over $\mathcal{X}$; thus, if we have an orbifold morphism $S^1 \to \mathcal{X}$ we can pull-back via this morphism and we get a G-bundle $\pi : E \to S^1$ with a G-equivariant map $\varphi : E \to X$.

Conversely, given the bundle E and the map φ, the quotient by G induces the map $S^1 \to \mathcal{X}$. So there is a correspondence between loops in $\mathcal{X}$ and such data: this is made precise in [ALR07, Corollary 2.46].

Since $E \to S^1$ is a (possibly disconnected) covering space we can lift the loop id : $S^1 \to S^1$ to a path $\sigma : [0,1] \to E$ such that $\sigma(1) = g\sigma(0)$ for some $g \in G$. Composing with φ we get a path $\gamma = \varphi \circ \sigma : [0,1] \to X$ such that $\gamma(1) = g\gamma(0)$. If we choose a different lift $\sigma' = h\sigma$ for some $h \in G$ we would get the pair $(h\gamma, hgh^{-1})$ instead of (γ, g), so these pairs should correspond to the same loop.

$$
\begin{tikzcd}
{[0,1]} \arrow[r, "\sigma"] \arrow[rr, bend left, "\gamma"] \arrow[dr] & E \arrow[r, "\varphi"] \arrow[d, "\pi"] & X \arrow[d] \\
 & S^1 \arrow[r] & \mathcal{X}
\end{tikzcd}
$$

Proposition 2.2.5. *The orbifold loop space $\Omega\mathcal{X}$ of a global quotient orbifold $\mathcal{X} = [X/G]$ is the (infinite dimensional) orbifold presented by the orbifold groupoid*

$$\Omega\mathcal{X} = G \ltimes \{(\gamma, g) \in C^\infty([0,1], X) \times G : \gamma(1) = g\gamma(0)\}$$

where G acts by $h \cdot (\gamma, g) = (h\gamma, hgh^{-1})$.

A particular case of this is when we take $X = pt$ to be a point. Then the loop space of $[pt/G]$ is the set of conjugacy classes of G. This shows a very important phenomenon: we might have different orbifold morphisms with the same topological realization. In other words, the forgetful functor Orb $\to$ Top sending an orbifold to its underlying topological space is not faithful.

We now take the opportunity to introduce the inertia orbifold. This is the set of orbifold loops $|\gamma| : S^1 \to \mathcal{G}$ such that the topological realization $S^1 \to |\mathcal{G}|$ is constant. If $|\gamma|$ is constant equal to $x \in |\mathcal{G}|$ then γ factors through $S^1 \to [pt/G_x] \hookrightarrow \mathcal{G}$. Thus the set of constant loops is

$$|\Lambda\mathcal{G}| = \{(x, (g)_{G_x}) : x \in |\mathcal{G}| \text{ and } g \in G_x\}$$

where $(g)_{G_x}$ denotes the conjugacy class of g in G_x. This set can be given an orbifold structure as follows:

Definition 2.2.6. *Let $\mathcal{G}$ be an orbifold groupoid. Consider the set of arrows*

$$S_{\mathcal{G}} = \{g \in G_1 : s(g) = t(g)\}$$

and the obvious $\mathcal{G}$-action on $S_{\mathcal{G}}$. Then the inertia orbifold $\Lambda\mathcal{G}$ of $\mathcal{G}$ is the orbifold presented by the orbifold groupoid

$$\mathcal{G} \ltimes S_{\mathcal{G}}.$$

Once again, Morita equivalence of groupoids induces a Morita equivalence of the inertia groupoids $\mathcal{G} \ltimes S_{\mathcal{G}}$, so the inertia orbifold is well defined.

A particular case is that of global quotients, that is, $\mathcal{G} = G \ltimes X$ (or more generally when G is a Lie group acting almost freely on X). In this case, the inertia orbifold can be described as follows:

Proposition 2.2.7. *Let $\mathcal{G} = G \ltimes X$ where G is a compact Lie group acting almost freely on X. Then $\Lambda\mathcal{G}$ is presented by the groupoid*

$$\bigsqcup_{(g)} C(g) \ltimes X^g$$

where $C(g) = \{h \in G : hg = gh\}$ is the centralizer of g, X^g is the fixed point set of g and the disjoint union runs over the conjugacy classes of G.

Proof. In this case $S_{\mathcal{G}} = \{(g,x) : gx = x\} = \bigsqcup_{g\in G} X^g$, so

$$\mathcal{G} \ltimes S_{\mathcal{G}} = G \ltimes \left(\bigsqcup_{g\in G} X^g \right)$$

where $h \in G$ acts on $\bigsqcup_{g\in G} X^g$ by sending $x \in X^g$ to $hx \in X^{hgh^{-1}}$. The groupoid $\bigsqcup_{(g)} C(g) \ltimes X^g$ embeds in an obvious way in $\mathcal{G} \ltimes S_{\mathcal{G}}$ and this embedding is an equivalence. Note that there are no arrows between X^g and $X^{g'}$ if g, g' are in different conjugacy classes, that the arrows from X^g to X^g are of the form (x,g) for some $x \in X^g$ and $g \in C(g)$, and that if $x \in X^{g'}$ and $g = hg'h^{-1}$ then x is isomorphic (as an object in the category $\mathcal{G} \ltimes S_{\mathcal{G}}$) to $hx \in X^g$. □

2.3 Cohomology of orbifolds

We will now give an appropriate definition of cohomology for orbifolds. Unless we take coefficients in $\mathbb{Q}$ or $\mathbb{R}$ we don't expect the cohomology of the orbifold to be simply the cohomology of the underlying topological space $|\mathcal{G}|$ as this doesn't retain any information about the orbifold structure. For instance when $\mathcal{G} = G \ltimes X$ it's reasonable to expect that $H^*(\mathcal{G}) = H^*_G(X)$ is equivariant cohomology (see appendix B). This is done by constructing a "homotopically correct" version of the underlying topological space.

Definition 2.3.1. *Let $\mathcal{G}$ be an orbifold groupoid (or, more generally, a category). Let*

$$G_n = \{(g_1, \dots, g_n) \in G_1^n : s(g_i) = t(g_{i+1}) \text{ for } i = 1, \dots n-1\}$$

denote the set of n composable arrows. These sets form a simplicial set, called the nerve of $\mathcal{G}$, with face operators $d_i : G_n \to G_{n-1}$ given by

$$d_i(g_1, \dots, g_n) = \begin{cases} (g_2, \dots, g_n) & \text{if } i = 1 \\ (g_1, \dots, g_{n-1}) & \text{if } i = n \\ (g_1, \dots, g_i g_{i+1} \dots, g_n) & \text{otherwise} \end{cases} .$$

The classifying space of $\mathcal{G}$ is the the geometric realization of its nerve

$$B\mathcal{G} = \left(\bigsqcup_{n\geq 0} G_n \times \Delta^n \right) \Big/ (d_i(g), x) \sim (g, \delta^i(x))$$

where Δ^n is the n-simplex and $\delta^i : \Delta^{n-1} \to \Delta^n$ is the inclusion of the i-th facet.

When $\mathcal{G}$ is the groupoid with one object and morphism group G, that is, $\mathcal{G} = G \ltimes pt$, this recovers the usual construction of the classifying space of G (see appendix B). When $\mathcal{G} = G \ltimes X$ the classifying space of $\mathcal{G}$ is the Borel construction

$$B\mathcal{G} = EG \times_G X.$$

It can be shown that Morita equivalent groupoids have classifying spaces which are weakly homotopy equivalent. Thus, any homotopy invariant of $B\mathcal{G}$ gives rise to an invariant of the orbifold represented by $\mathcal{G}$. In particular we can define cohomology.

Definition 2.3.2. *Let $\mathcal{G}$ be a groupoid presenting an orbifold. We define its cohomology with coefficients in any ring R to be the singular cohomology of $B\mathcal{G}$:*

$$H^*(\mathcal{G}; R) = H^*(B\mathcal{G}; R).$$

In particular if $\mathcal{G} = G \ltimes X$ then

$$H^*(G \ltimes X) = H^*(X \times_G EG) = H^*_G(X).$$

Proposition 2.3.3. *With rational coefficients the cohomology of an orbifold is the singular cohomology of its underlying topological space:*

$$H^*(\mathcal{G}; \mathbb{Q}) \cong H^*(|\mathcal{G}|; \mathbb{Q})$$

Proof. There is a canonical map $B\mathcal{G} \to |\mathcal{G}|$ with fiber BG_x at $x \in |\mathcal{G}|$. Since G_x are finite, the fibers BG_x have trivial rational cohomology. By the Vietoris-Begle theorem the induced homomorphism in (rational) cohomology is an isomorphism. □

Note that in particular this implies proposition B.0.4. Rational cohomology of orbifolds satisfies Poincaré duality.

Proposition 2.3.4 (Poincaré duality)**.** *Let $\mathcal{G}$ present a compact and oriented orbifold of dimension n. Then*

$$\dim H^*(\mathcal{G}, \mathbb{Q}) = \dim H^{n-*}(\mathcal{G}, \mathbb{Q}).$$

Proof. It's enough to show that $X = |\mathcal{G}|$ is a $\mathbb{Q}$-homology manifold, i.e. to prove that for every $x \in X$ there is an open neighborhood $U \subseteq X$ of x such that

$$H^*(U, U \setminus \{x\}; \mathbb{Q}) \cong \begin{cases} \mathbb{Q} & \text{if } * = n \\ 0 & \text{otherwise.} \end{cases}$$

But by choosing a coordinate chart around x we have

$$H^*(U, U \setminus \{x\}; \mathbb{Q}) \cong H^*(\mathbb{R}^n/G, \mathbb{R}^n/G \setminus \{0\}; \mathbb{Q}) \cong H^*_G(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; \mathbb{Q}).$$

Now

$$H^*(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; \mathbb{Q}) \cong \begin{cases} \mathbb{Q} & \text{if } * = n \\ 0 & \text{otherwise.} \end{cases}$$

and since $\mathcal{G}$ is oriented the action of G on $H^n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; \mathbb{Q})$ is trivial, so the result follows. □

2.3.1 Forms and de Rham cohomology

Cohomology with real coefficients can also be obtained as a de Rham cohomology. We take this as an excuse to introduce forms on orbifolds.

Definition 2.3.5. *Given an orbifold groupoid $\mathcal{G}$ let*

$$\Omega^p(\mathcal{G}) = \{\omega \in \Omega^p(G_0) : s^*\omega = t^*\omega\}$$

denote the set of p-forms of $\mathcal{G}$. The usual exterior derivative on G_0 defines a differential $d : \Omega^p(\mathcal{G}) \to \Omega^{p+1}(\mathcal{G})$. We call $(\Omega^*(\mathcal{G}), d)$ the de Rham complex of $\mathcal{G}$ and we denote by $H^*_{dR}(\mathcal{G})$ its cohomology, called the de Rham cohomology of $\mathcal{G}$.

It can be shown that the set of forms $\Omega^p(\mathcal{G})$ does not change with Morita equivalent groupoids, so this gives a good definition of forms on an orbifold. When our orbifold is a global quotient, $\mathcal{G} = G \ltimes X$, forms on $\mathcal{G}$ are the same as G-invariant forms on X. Indeed if $\omega \in \Omega^p(G_0) = \Omega^p(X)$ then (after identifying $T_{(g,x)}G \times X \cong T_xX$)

$$(s^*\omega)_{(g,x)} = \omega_x \text{ and } (t^*\omega)_{(g,x)} = (g^*\omega)_x.$$

Theorem 2.3.6. *Let $\mathcal{G}$ be an orbifold groupoid. Then*

$$H^*_{dR}(\mathcal{G}) \cong H^*(|\mathcal{G}|; \mathbb{R}) \cong H^*(\mathcal{G}; \mathbb{R}).$$

Proof. The first isomorphism is [Sat56, Theorem 1] and the second is proposition 2.3.3. □

With a notion of forms, we can finally define a symplectic obifold.

Definition 2.3.7. *A symplectic orbifold is an orbifold, presented by some groupoid $\mathcal{G}$, and a 2-form $\omega \in \Omega^2(\mathcal{G})$ such that $\omega_x \in \Lambda^2 T^*_x G_0$ is non-degenerate for every $x \in G_0$ and $d\omega = 0$.*

2.4 Morse homology of orbifolds

Recently, in [CH14], an orbifold Morse complex was introduced; earlier works in that direction were [LT97] and [Hep09]. This will be very important for us for two reasons. First, because some of the ideas can be adapted to construct an orbifold Floer complex, which we'll do in detail in section 4.4. And second because we'll use the isomorphism between the homology of this complex and the singular homology of the orbifold in the proof of 6.0.1. We will be quite brief in this exposition and refer the reader to [CH14]. We will also only consider the case of global quotients, as this is the case that will be important for us.

Let $\mathcal{X} = [X/G]$ be a global quotient orbifold where X is a compact oriented manifold and G is a finite group acting on X preserving its orientation. Consider a fixed G-invariant Riemannian metric $\mathfrak{g}$. Let $H : X \to \mathbb{R}$ be a G-invariant Morse function on X. This condition is generic amongst G-invariant functions (see [Was69, Lemma 4.8]). We denote by $\mathrm{Crit}_k(H)$ the set of critical points of H of index k and $\mathrm{Crit}(H) = \bigcup_{k \geq 0} \mathrm{Crit}_k(H)$. Recall that for each critical point $x \in \mathrm{Crit}(H)$ we have a stable manifold

$$W^s_H(x) = \{y \in X : \lim_{t \to +\infty} \varphi_t(y) = x\}$$

and an unstable manifold

$$W^u_H(x) = \{y \in X : \lim_{t \to -\infty} \varphi_t(y) = x\}$$

where φ_t denotes the flow at time t of ∇H (note that the gradient is computed from the Riemannian metric $\mathfrak{g}$). The index of x is the dimension of the stable manifold. A striking fact is that the correct Morse complex is not generated by all the critical points, but only by the so called orientable critical points.

Definition 2.4.1. *Let $x \in \mathrm{Crit}(H)$. We say that x is an orientable critical point if the G_x action on the unstable manifold $W^u_H(x)$ is orientation preserving. We denote by $\mathrm{Crit}(H)^+$ the set of orientable critical points and by $\mathrm{Crit}(H)^-$ the set of non-orientable critical points.*

Let now R be a ring. The usual Morse complex of H (computing the homology of X, not $\mathcal{X}$) is the R-module generated by the critical points graded by their index

$$CM_k(X, H; R) = \bigoplus_{x \in \mathrm{Crit}_k(H)} R \cdot x.$$

For coherence with notation we'll use later we write $\widetilde{CM}_k(\mathcal{X}, H; R)$ instead of $CM_k(X, H; R)$. This complex decomposes in a positive and negative part, that is,

$$\widetilde{CM}_k(\mathcal{X}, H; R) \cong \widetilde{CM}_k(\mathcal{X}, H; R)^+ \oplus \widetilde{CM}_k(\mathcal{X}, H; R)^-$$

where

$$\widetilde{CM}_k(\mathcal{X}, H; R)^\pm = \bigoplus_{x \in \mathrm{Crit}_k(H)^\pm} R \cdot x.$$

Since H is G-invariant, if x is a critical point then so is gx for any $g \in G$; hence there is an action of G on $\mathrm{Crit}(H)$ and this action extends R-linearly to $\widetilde{CM}_k(\mathcal{X}, H; R)$. It's easy to see that this action sends (non-)orientable critical points to (non-)orientable critical points. Finally the Morse complex of $\mathcal{X}$ is defined to be the G-invariant part

$$CM_k(\mathcal{X}, H; R) = \left(\widetilde{CM}_k(\mathcal{X}, H; R)^+\right)^G. \tag{2.1}$$

The differential on $\widetilde{CM}_*(\mathcal{X}, H; R)$ is defined by counting gradient flow trajectories between critical points. To do so we define the moduli space

$$\widehat{\mathcal{M}}\left(x^-, x^+; H, \mathfrak{g}\right) = \left\{u \in C^\infty(\mathbb{R}, X) \colon \dot{u}(s) = \nabla H(u(s)) \text{ and } \lim_{s \to \pm\infty} u(s) = x^\pm\right\}. \tag{2.2}$$

This moduli space admits an $\mathbb{R}$-action by translation, that is, $(\sigma \cdot u)(s) = u(s + \sigma)$ for $\sigma \in \mathbb{R}$; we denote by $\mathcal{M}\left(x^-, x^+; H, \mathfrak{g}\right)$ the quotient $\widehat{\mathcal{M}}\left(x^-, x^+; H, \mathfrak{g}\right)/\mathbb{R}$. Note that the map $u \mapsto u(0) \in X$ identifies the moduli space $\widehat{\mathcal{M}}(x^-, x^+)$ with

$$W^u_H(x^-) \cap W^s_H(x^+).$$

We assume that the pair $(H, \mathfrak{g})$ is Morse-Smale, meaning that all the intersections $W^u_H(x^-) \cap W^s_H(x^+)$ for $x^-, x^+ \in \mathrm{Crit}(H)$ are transverse. Then $\widehat{\mathcal{M}}(x^-, x^+)$ is a manifold of dimension $|x^+| - |x^-|$ (where $|x|$ denotes the index of a critical point x). If $|x^+| - |x^-| = 1$ then $\mathcal{M}(x^-, x^+)$ is a finite set. Finally, the differential $\partial = \partial_{H,\mathfrak{g}} : \widetilde{CM}_k(\mathcal{X}, H; R) \to \widetilde{CM}_{k-1}(\mathcal{X}, H; R)$ is defined on generators $x^+ \in \mathrm{Crit}_k(H)$ by

$$\partial x^+ = \sum_{x^- \in \mathrm{Crit}_{k-1}(H)} \left(\sum_{u \in \mathcal{M}(x^-, x^+)} \nu(u)\right) x^- \tag{2.3}$$

where $\nu(u) \in \{+1, -1\}$ are determined as we will now explain: for each $x \in \mathrm{Crit}(H)$ we fix an orientation of $W^u_f(x)$. Since X is oriented this also determines an orientation of $W^s_f(x)$, and this induces an orientation on $\widehat{\mathcal{M}}(x^-, x^+) = W^u_f(x^-) \cap W^s_f(x^+)$. Each $u \in \mathcal{M}(x^-, x^+)$ corresponds to a connected component $u(\mathbb{R}) \subseteq \widehat{\mathcal{M}}(x^-, x^+)$. If the orientation given in this way is the one induced by the standard orientation of $\mathbb{R}$ via u we set $\nu(u) = 1$; otherwise $\nu(u) = -1$.

It's a fact that the differential preserves $CM(\mathcal{X},H;R)\subseteq \widetilde{CM}(\mathcal{X},H;R)$. This is proved in [CH14, Lemmas 2.6 and 2.7]. Thus $(CM_*(\mathcal{X},H;J),\partial)$ is a complex and we can define its homology

$$HM_k(\mathcal{X},H,\mathfrak{g};R)=\frac{\ker(\partial: CM_k\to CM_{k-1})}{\operatorname{im}(\partial: CM_{k+1}\to CM_k)}.$$

Remark 2.4.2. Our conventions differ from the ones in [CH14] in two aspects: our moduli space consists of gradient flow trajectories instead of negative gradient flow trajectories, and our differential is from x^+ to x^-. These differences cancel out.

Remark 2.4.3. There are at least two good reasons for why we should only consider the complex generated by orientable orbits. First, because otherwise the differential wouldn't preserve $\widetilde{CM}(\mathcal{X})^G$; we will explain this better in section 4.1.1 in the context of Floer homology. Second, the topology of the sub-level sets doesn't change when we cross a non-orientable critical point; this was observed in [LT97].

The Morse homology of an orbifold is isomorphic to its singular homology as long as we use coefficients in $\mathbb{Q}$ (or other characteristic 0 field):

Theorem 2.4.4. *Suppose that R is a field of characteristic 0. Then the homology of the Morse complex of $\mathcal{X}$ is isomorphic to the singular homology of $\mathcal{X}$ with coefficients in R*

$$HM_k(\mathcal{X},H,\mathfrak{g};R)\cong H_k(\mathcal{X};R).$$

Proof. See [CH14, Theorem 2.9]. □

Finally, we would like to note that this construction is naturally homological. Not only because our the differential we defined has degree -1, but also because in theorem 2.4.4 the isomorphism is with singular homology; of course with coefficients in a field of characteristic 0 $H_k(\mathcal{X};R)\cong H^k(\mathcal{X};R)$, but if $\mathcal{X}$ were smooth the isomorphism with singular homology would still hold with coefficients in $R=\mathbb{Z}$. We could also define Morse cohomology by defining the differential using flow trajectories of $-\nabla H$ instead of ∇H. If $x\in \operatorname{Crit}_k(H)$ then $x\in\operatorname{Crit}_{n-k}(-H)$, so we have a natural Poincaré duality in Morse homology:

$$HM_k(\mathcal{X},H,\mathfrak{g};R)\cong HM^{n-k}(\mathcal{X},-H,\mathfrak{g};R).$$

Although with coefficients in a field of characteristic 0 homology and cohomology are the same, we will try to always use the natural version; for instance the statement of theorem 6.0.1 follows this philosophy.

2.5 Chen-Ruan cohomology

Chen and Ruan introduced in [CR04] a new cohomology theory for orbifolds, now called Chen-Ruan cohomology. This is fundamentally different from the topological cohomology we described earlier: it encodes a lot of information about the singularities of the orbifold. Chen and Ruan were inspired by physics and by models for string theory which were being constructed over orbifolds, namely in [DHVW85]. Their cohomology theory was seen as the classical part of a quantum cohomology for orbifolds, constructed using the space of morphisms from (orbifold) Riemann surfaces to our orbifold. Chen-Ruan cohomology has been playing a very important role in the development of mathematics (and physics) in the last 20 years; for instance, open-closed mirror symmetry predicts that the quantum cohomology of an orbifold can be "seen" from its mirror.

Although the product in Chen-Ruan cohomology is reasonably complicated, involving a Gromov-Witten theory for orbifolds, the definition of Chen-Ruan cohomology as a graded vector space is actually quite elementary, and this is only what we'll need. Indeed, if we ignore the grading, Chen-Ruan cohomology of an orbifold is simply the singular cohomology of its inertia orbifold – the space of topologically constant morphisms $S^1 \to \mathcal{X}$, see 2.2.6. In a philosophical way, we're replacing the cohomology of the space of points (particles) by the cohomology of the space of constant loops (strings).

What happens to the grading is more interesting. We need to shift the usual cohomology grading in each of the connected components of $\Lambda\mathcal{X}$, called the twisted sectors of $\mathcal{X}$.

2.5.1 Twisted sectors

Let us parametrize the twisted sectors, that is, the connected components of $\Lambda\mathcal{G}$. Recall that

$$|\Lambda\mathcal{G}| = \{(x, (g)_{G_x}) : x \in |\mathcal{G}| \text{ and } g \in G_x\}$$

where $(g)_{G_x}$ is the conjugacy class of g in G_x. We define a relation $\approx$ in $\Lambda\mathcal{G}$ as follows: given an orbifold chart $(\widetilde{U}, G, \phi)$ and $x, y \in \phi(\widetilde{U}) = U$ consider lifts of x, y to $\widetilde{x}, \widetilde{y} \in U$. Then G_x and G_y are naturally identified with the isotropy subgroups $G_{\widetilde{x}}, G_{\widetilde{y}} \subseteq G$. Then $\approx$ is the equivalence relation generated by

$$(x, (g)_{G_x}) \approx \left(y, (g')_{G_y}\right) \text{ if } g \text{ and } g' \text{ are conjugate in } G.$$

Given $(x, (g)_{G_x}) \in |\Lambda\mathcal{G}|$ we denote by (g) its equivalence class with respect to $\approx$ and we write T for the set of equivalence classes.

Definition 2.5.1. *Given $(g) \in T$ we let*

$$\mathcal{G}_{(g)} = \mathcal{G} \ltimes \{g' \in S_{\mathcal{G}} : (x, (g')_{G_x}) \in (g)\}.$$

The orbifolds $\mathcal{G}_{(g)}$, for each $(g) \in T$, $g \neq 1$, are called the twisted sectors of $\mathcal{G}$ and $\mathcal{G}_{(1)}$ is called the untwisted sector.

It can easily be seen that the underlying space $|\mathcal{G}_{(g)}|$ of the twisted sectors, together with the untwisted sector, form the connected components of $|\Lambda\mathcal{G}|$. Moreover it's also clear that the untwisted sector $\mathcal{G}_{(1)}$ is naturally identified with the original groupoid $\mathcal{G}$. The twisted sectors appear from isotropy which occurs in the presence of singularities, so the twisted sectors arise from the singular set.

An enlightening example is that of global quotients or more generally of quotients by Lie groups, that is, $\mathcal{G} = G \ltimes X$ where G is a finite (or compact Lie) group acting (almost freely) on X. In this case we saw in proposition 2.2.7 that the inertia orbifold is presented by the groupoid

$$\bigsqcup_{(g)} C(g) \ltimes X^g.$$

If each fixed point set X^g is connected, then the twisted sectors are precisely $\mathcal{G}_{(g)} = C(g) \ltimes X^g$ and T is the set of conjugacy classes (g) of G for which $X^g \neq \emptyset$. If the fixed point sets aren't connected there is a twisted sector for each connected component.

2.5.2 Degree shifting numbers

We now define the index shifting numbers of Chen-Ruan cohomology. These are also known by the names of age-grading and fermionic degree shifting numbers. From now on assume that $\mathcal{G}$ admits an almost complex structure; similarly to how we defined forms on $\mathcal{G}$, an almost complex structure on $\mathcal{G}$ is an almost complex structure J on G_0 such that $s^*J = t^*J$.

Let $g \in S_{\mathcal{G}}$ be an arrow with $s(g) = t(g) = x$. Choose an orbifold chart $(\widetilde{U}, G_x, \phi)$ with $\widetilde{U}$ embedded in G_0. Then $g \in G_x$ is a map $g : \widetilde{U} \to \widetilde{U}$. Its differential at x is a map $(dg)_x : T_xG_0 \to T_xG_0$. The almost complex structure J_x endows T_xG_0 with the structure of a complex vector space, hence identifying T_xG_0 with $\mathbb{C}^n$ where $2n = \dim \mathcal{G}$; the condition that $s^*J = t^*J$ implies that $(dg)_x$ preserves the almost complex structure J_x on T_xG_0, so it can be regarded as a linear transformation of complex vector spaces or, equivalently, a matrix in $GL(n, \mathbb{C})$.

Since g has finite order, say $m \in \mathbb{Z}_{\geq 1}$, the eigenvalues of $(dg)_x$ (always regarded as a complex linear transformation) are of the form $e^{2\pi i \lambda_1}, \dots e^{2\pi i \lambda_n}$ where $\lambda_j \in \mathbb{Q}$ are such that $m\lambda_j \in \mathbb{Z}$. We then define a map $\iota : S_{\mathcal{G}} \to \mathbb{Q}$ by mapping $g \in S_{\mathcal{G}}$ to

$$\iota(g) = \sum_{j=1}^{n} \{\lambda_j\} \in \mathbb{Q} \tag{2.4}$$

where $\{\lambda\}$ denotes fractional part of λ. Since eigenvalues don't change under conjugation, ι is invariant with respect to the $\mathcal{G}$-action on $S_{\mathcal{G}}$ and thus induces a map (still called ι) $\iota : |\Lambda\mathcal{G}| \to \mathbb{Q}$. By continuity of eigenvalues, the map ι is continuous and since it takes values in $\mathbb{Q}$ is must be locally constant. Hence, ι is constant in each of the (un)twisted sectors, so we denote by $\iota_{(g)}$ the value that ι takes in the (un)twisted sector $|\mathcal{G}_{(g)}|$. In particular it's clear that $\iota_{(1)} = 0$. We can define Chen-Ruan cohomology.

Definition 2.5.2. *Let R be a field of characteristic 0. We define the Chen-Ruan cohomology as the $\mathbb{Q}$-graded vector space*

$$H^*_{CR}(\mathcal{G}; R) = \bigoplus_{(g) \in T} H^{*-2\iota_{(g)}}\left(\mathcal{G}_{(g)}; R\right).$$

A few notes about the definition. First, there is no reason at this point to take coefficients in a field of characteristic 0 and not in a general ring, but that is necessary if we want to define a product structure. Also, by taking coefficients in a field of characteristic 0 we can use propositions 2.3.6 and 2.4.4 to interpret the cohomology $H^*(\mathcal{G}_{(g)}; R)$ either as cohomology of $B\mathcal{G}_{(g)}$, cohomology of $|\mathcal{G}_{(g)}|$, de Rham cohomology or Morse (co)homology.

Note also that the summand corresponding to $(1) \in T$ is just the usual cohomology of the untwisted sector $\mathcal{G}_{(1)} = \mathcal{G}$. So in a way Chen-Ruan cohomology adds to the usual cohomology contributions from the singularities. Of course when the orbifold is smooth Chen-Ruan cohomology reduces to singular cohomology.

The grading for sure looks mysterious, especially the fact that it can be rational. The case in which this degree shifting numbers are integers forms an important class of orbifolds.

Definition 2.5.3. *An almost complex orbifold of dimension $2n$ presented by $\mathcal{G}$ is Gorenstein if one of the following 3 equivalent conditions holds:*

1. *For every $(g) \in T$ the degree shifting number $\iota_{(g)}$ is an integer.*
2. *For each $x \in G_0$ and $g \in G_x$ the complex linear transformation $(dg)_x$ has determinant 1.*
3. *The determinant bundle $\Lambda^n_{\mathbb{C}} T\mathcal{G}$ is a honest bundle.*

But even in this case the meaning of the degree shifting numbers is not clear at all. We can give two reasons that justify their presence. The first is that the product structure that can be defined on H^*_{CR}

respects the grading constructed this way. It seems that this was the original motivation to define the grading in this way.

Theorem 2.5.4. *Let $\mathcal{G}$ a groupoid presenting an almost complex orbifold. Then the $\mathbb{Q}$-graded vector space $H^*_{CR}(\mathcal{G})$ can be given an associative and unital algebra structure, with a product that respects the grading. Moreover, the restriction of this product to the summand $H^*(\mathcal{G})$ is the usual cup product.*

This algebra structure was constructed by Chen and Ruan in [CR04]. It is constructed as the classical limit of a "natural" generalization of the quantum cup product to orbifolds. As in quantum cohomology, this product is defined using a 3-point function which is essentially an orbifold Gromov-Witten invariant – the theory of these was developed in [CR02]. Formally, this 3-point function is defined by integrating over a virtual fundamental class of the moduli space $\overline{\mathcal{M}}_3(\mathcal{G})$ of topologically constant maps $S^2 \to \mathcal{G}$ with 3 possibly singular marked points on S^2. The numbers $\iota_{(g)}$ appear in a formula for the virtual dimension of the connected components of the moduli space $\overline{\mathcal{M}}_3(\mathcal{G})$. Details for this very interesting topic can be found in [CR04, CR02, ALR07].

Another very good reason for this grading is that it makes the crepant resolution conjecture true (at the level of graded vector spaces, at least). We will discuss this briefly later, see theorem 2.5.7. Finally, another nice thing (and more down to earth) about this grading is that it gives a Poincaré duality for Chen-Ruan cohomology.

Proposition 2.5.5. *Let $\mathcal{G}$ present a compact almost complex orbifold of dimension $2n$. Then we have*

$$\dim H^*_{CR}(\mathcal{G}) = \dim H^{2n-*}_{CR}(\mathcal{G}).$$

Proof. Let $I : |\Lambda\mathcal{G}| \to |\Lambda\mathcal{G}|$ be the involution sending

$$(x, (g)_{G_x}) \mapsto \left(x, (g^{-1})_{G_x}\right).$$

Restricted to $\mathcal{G}_{(g)}$ it induces an isomorphism $|\mathcal{G}_{(g)}| \overset{\cong}{\to} |\mathcal{G}_{(g^{-1})}|$. Since

$$\{\lambda\} + \{-\lambda\} = \begin{cases} 0 & \text{if } \lambda \in \mathbb{Z} \\ 1 & \text{otherwise} \end{cases}$$

we see that

$$\iota_{(g)} + \iota_{(g^{-1})} = \#\{\text{eigenvalues of } (dg)_x \text{ different from } 1\} = n - \frac{1}{2}\dim \mathcal{G}_{(g)}.$$

By Poincaré duality for (singular) cohomology of orbifolds (see 2.3.4) we have

$$\dim H^{*-2\iota_{(g)}}(\mathcal{G}_{(g)}) = \dim H^{*-2n+\dim \mathcal{G}_{(g)}+2\iota_{(g^{-1})}}(\mathcal{G}_{(g)}) = \dim H^{2n-*-2\iota_{(g^{-1})}}(\mathcal{G}_{(g^{-1})})$$

and summing over $(g) \in T$ finishes the proof. □

2.5.3 Crepant resolution conjecture

The crepant resolution conjecture is certainly one of the big attractions of Chen-Ruan cohomology.

Definition 2.5.6. *Let $\mathcal{X}$ be a complex orbifold of dimension $2n$. A resolution of $\mathcal{X}$ is a holomorphic map $f : Y \to \mathcal{X}$ from a smooth complex manifold Y to $\mathcal{X}$ such that the restriction $f : f^{-1}(\mathcal{X}_{\text{reg}}) \to \mathcal{X}_{\text{reg}}$*

is an isomorphism and $f^{-1}(\mathcal{X}_{\text{sing}})$ is a (necessarily codimension 1) complex sub-manifold of Y, called the exceptional divisor. If we allow Y to also be an orbifold we say that we have a partial resolution.

A resolution is called crepant if it respects the canonical classes, that is

$$f^*(\Lambda^n_{\mathbb{C}} T\mathcal{X}) \cong \Lambda^n_{\mathbb{C}} TY.$$

We'll give some examples later. For a crepant resolution of $\mathcal{X}$ to exist, the bundle $\Lambda^n T\mathcal{X}$ must be a honest bundle (as pullbacks induce isomorphisms on the fibers), that is, $\mathcal{X}$ needs to be Gorenstein. When $\mathcal{X}$ is Gorenstein and has dimension at most 3, $\mathcal{X}$ always admits a crepant resolution, and if the dimension is at most 2 then this resolution is unique. However, in higher dimension the situation is much more complicated: for instance an orbifold as simple as $[\mathbb{C}^4/\{\mathrm{Id}, -\mathrm{Id}\}]$ doesn't admit a crepant resolution.

It was expected by string theorists that the string theory of an orbifold and the string theory of a crepant resolution should be somehow equivalent. This led to the prediction that the Chen-Ruan cohomology of an orbifold should be isomorphic to the usual cohomology of its resolution. As graded vector spaces, this was shown to be true by Yasuda in [Yas04] using the machinery of motivic integration due to Kontsevich and developed by Batyrev and others.

Theorem 2.5.7. *Let $Y \to \mathcal{X}$ be a crepant resolution of a complex Gorenstein orbifold $\mathcal{X}$. Then*

$$\dim H^*(Y) = \dim H^*_{CR}(\mathcal{X}).$$

Proof. See [Yas04, Theorem 1.5]. □

The equivalence of the product structures is a much more subtle question, still unanswered. In general, it's not true that $H^*(\mathcal{X})$ is isomorphic to $H^*(Y)$ as graded algebras. However, in [Rua06] Ruan formulated a precise way to add a quantum correction to the cup product on $H^*(Y)$; the new graded algebra is denoted by $H^*_\pi(Y)$.

Conjecture 2.5.8 (Crepant resolution conjecture)**.** *Let $Y \to \mathcal{X}$ be a crepant resolution of a complex Gorenstein orbifold $\mathcal{X}$. Then we have a graded algebra isomorphism*

$$H^*_\pi(Y) \cong H^*_{CR}(\mathcal{X}).$$

When Y is a hyperkähler the quantum correction vanishes and $H^*_\pi(Y) = H^*(Y)$. Even in this case the crepant resolution conjecture isn't known. Let's finish this chapter with a few examples of interesting crepant resolutions.

Example 2.5.9. The orbifold $[\mathbb{C}^2/(\mathbb{Z}/2)]$ admits a crepant resolution by the total space $K\mathbb{P}^1$ of the canonical bundle $\mathcal{O}_{\mathbb{P}^1}(2) \to \mathbb{P}^1$.

Example 2.5.10. When $G \subseteq SL(n, \mathbb{C})$ is a finite subgroup the orbifold $\mathbb{C}^n/G$ is Gorenstein. In this case the crepant resolution conjecture (or even theorem 2.5.7) is a form of the generalized McKay correspondence.

Example 2.5.11. The Kummer surface (see example 2.1.5) admits a crepant resolution Y, known as the $K3$ surface. The $K3$ surface and the torus $\mathbb{T}^4$ are the two only compact Calabi-Yau smooth surfaces. We can use theorem 2.5.7 to compute the Betti numbers of the $K3$ surface to be

$$\dim H^*(Y) = \begin{cases} 1 & \text{if } * = 0, 4 \\ 22 & \text{if } * = 2 \\ 0 & \text{otherwise} \end{cases}.$$

Example 2.5.12. If X is a complex manifold of dimension at most 2 then the symmetric product X^n/S_n (see example 2.1.6) admits a famous crepant resolution $X^{[n]}$ called the Hilbert scheme of points. For example if X is $\mathbb{C}^2$, $\mathbb{T}^4$ or a $K3$ surface then $X^{[n]}$ is hyperkähler and in all these cases the crepant resolution conjecture has been verified.

Chapter 3

Floer homology with g-periodic boundary conditions

When we try to define Floer homology for a global quotient $[X/G]$ we will consider Hamiltonian 1-periodic orbits in $[X/G]$. Such Hamiltonian loops will lift to Hamiltonian orbits $\gamma : [0,1] \to X$ that "close" in the quotient, that is, $\gamma(1) = g\gamma(0)$ for some $g \in G$ (see 2.2.5). Moreover the Floer cilinders will also lift to some maps $u : [0,1] \times \mathbb{R} \to X$ with a boundary condition $u(1,s) = gu(0,s)$ for $s \in \mathbb{R}$. In this section we consider a fixed symplectomorphism g and develop Floer homology based on Hamiltonian orbits and Floer trajectories with such boundary conditions, which we'll generally call g-periodic boundary conditions. Note that when $g = \mathrm{id}_X$ these conditions reduce to the usual 1-periodicity. This is actually a common extension of Floer homology (see for instance [DS94] or [FHS95]) since we can always reduce the usual Floer homology to Floer homology with g-periodic boundary conditions but $H = 0$ (see proposition 3.4.1).

3.1 Setup

Let (X, ω) be a compact symplectic manifold and $g : X \to X$ a symplectomorphism, that is, a diffeomorphism such that $g^*\omega = \omega$.

Recall that an almost complex structure J on X is a section of the endomorphism bundle $\mathrm{End}(TX) \to X$ such that for each $x \in X$ we have $J(x)^2 = -\mathrm{id}_{T_xX}$. An almost complex structure is said to be compatible with ω if

$$\langle v, w \rangle = \omega(v, Jw)$$

defines a Riemannian metric on X.

Denote by $\mathcal{J}(X, \omega)$ the set of time dependent almost complex structures $J = (J_t)_{t\in\mathbb{R}}$ such that each J_t is compatible with ω. Note that a time dependent almost complex structure gives a time dependent Riemannian metric $\langle v, w \rangle_t = \omega(v, J_t w)$. We denote

$$\mathcal{J}_g(X, \omega) = \{J = (J_t)_{t\in\mathbb{R}} \in \mathcal{J}(X, \omega) : J_t = g^* J_{t+1} \text{ for } t \in \mathbb{R}\}.$$

Here the pullback of an almost complex form J by g means the usual pullback of tensors when we regard J as a $(1,1)$ tensor (recall that $\mathrm{End}(TX) \cong TX \otimes T^*X$ as bundles over X). This means that $g^*J = g_*^{-1} J g_*$

or, more explicitly with base points,

$$(g^*J)(x) = (dg)^{-1}_{g(x)} J(g(x))(dg)_x.$$

Moreover we'll consider (time dependent) Hamiltonians $H : \mathbb{R} \times X \to \mathbb{R}$ also with a certain periodicity condition. More precisely, we define

$$C^\infty_g(\mathbb{R} \times X) = \{H \in C^\infty(\mathbb{R} \times X) : H_t = H_{t+1} \circ g \text{ for } t \in \mathbb{R}\}.$$

We write $H_t(x)$ for $H(t,x)$. A Hamiltonian H induces a Hamiltonian vector field X^H_t defined by the equation

$$\iota(X^H_t)\omega = dH_t.$$

We sometimes omit H and write simply X_t, if H is clear from the context. Note that

$$\langle \nabla_t H_t, Y \rangle_t = (dH_t)Y = \omega(X_t, Y) = \langle J_t X_t, Y \rangle_t$$

where the gradient ∇_t is calculated with respect to $\langle\,,\,\rangle_t$, thus

$$\nabla_t H_t = J_t X_t \text{ or, equivalently, } X_t = -J_t \nabla_t H_t \tag{3.1}$$

We let φ^H_t (or simply φ_t) be the flow of X^H_t, satisfying:

$$\varphi^H_0(x) = x \quad \text{and} \quad \frac{d\varphi^H_t(x)}{dt} = X^H_t(\varphi^H_t(x)).$$

Note that the condition $H_t = H_{t+1} \circ g$ implies that $X^H_t = g_* X^H_{t+1}$. We say that $\gamma : \mathbb{R} \to X$ is a Hamiltonian trajectory if $\gamma(t) = \varphi_t(\gamma(0))$ or, equivalently, if γ satisfies the ordinary differential equation $\dot\gamma(t) = X_t(\gamma(t))$. In this version of Floer homology with g-periodic boundary conditions we consider Hamiltonian orbits in

$$\mathcal{P}_g(H) = \{\gamma \in C^\infty([0,1], X) : \dot\gamma(t) = X^H_t(\gamma(t)) \text{ and } \gamma(1) = g(\gamma(0))\}.$$

Note that if $g = \mathrm{id}_X$ then these are the Hamiltonian 1-periodic orbits.

Remark 3.1.1. If $\gamma \in \mathcal{P}_g(H)$ then we can extend γ to $\mathbb{R}$ by continuity by asking that $\gamma(t+1) = g(\gamma(t))$. The condition that $\gamma(1) = g(\gamma(0))$ ensures that γ is continuous. Moreover this extension of γ still satisfies Hamilton's equation for every $t \in \mathbb{R}$ since $H \in C^\infty_g(\mathbb{R} \times X)$, so the extension of γ is actually smooth in $\mathbb{R}$. Something similar will also happen with solutions of Floer equation; in that case we can appeal to elliptic regularity to show that the extension is smooth.

Since Hamiltonian orbits are given by $\gamma(t) = \varphi_t(x_0)$ for some $x_0 \in X$ the set $\mathcal{P}_g(H)$ is in bijection with points $x_0 \in X$ such that $\varphi_1(x_0) = g(x_0)$, that is, fixed points of $\varphi_1^{-1} \circ g$; we observe that $\varphi_1^{-1} \circ g$ is a symplectomorphism, but might not be Hamiltonian. A condition that will be crucial in defining Floer homology is that of non-degeneracy:

Definition 3.1.2. *Given a Hamiltonian $H \in C^\infty_g(\mathbb{R} \times X)$ and $\gamma \in \mathcal{P}_g(H)$ we say that γ is non-degenerate if the linearized return map*

$$d(\varphi_1^{-1} \circ g)_{x_0} : T_{x_0}X \to T_{x_0}X$$

does not admit 1 *as an eigenvalue, where $x_0 = \gamma(0)$.*

We say that H satisfies the non-degeneracy condition if every $\gamma \in \mathcal{P}_g(H)$ is non-degenerate.

If H satisfies the non-degeneracy condition then the fixed points of $\varphi_1^{-1} \circ g$ form a discrete set (see [AD14, Lemma 6.5.11]), so $\mathcal{P}_g(X)$ is finite (because X is compact). We also note that the non-degeneracy condition is generic (see [HS95, Theorem 3.1] for a proof when $g = \mathrm{id}_X$).

We now consider Floer trajectories between these Hamiltonian orbits. These are maps $u : [0,1] \times \mathbb{R} \to X$ that

1. Satisfy Floer equation
$$\partial_s u + J_t(u)\left(\partial_t u - X_t^H(u)\right) = 0. \tag{3.2}$$
where t, s are the $[0,1]$ and $\mathbb{R}$ variables, respectively.

2. Tend to Hamiltonian orbits, that is, the limits
$$\gamma^{\pm}(t) = \lim_{s \to \pm\infty} u(t,s)$$
exist and define Hamiltonian orbits $\gamma^{\pm} \in \mathcal{P}_g(H)$. We usually write for brevity $\gamma^{\pm}(t) = u(t, \pm\infty)$.

3. For every $s \in \mathbb{R}$ we have $u(1,s) = g(u(0,s))$.

Remark 3.1.3. If we assume that H satisfies the non-degeneracy condition then the solutions of Floer equation (3.2) that tend to Hamiltonian orbits (condition (2) above) are precisely the ones that have finite energy
$$E(u) = \frac{1}{2}\int_{[0,1]\times\mathbb{R}} \left(|\partial_s u|^2 + |\partial_t u - X_t^H(u)|^2\right) dt\, ds < +\infty.$$
In that case the energy is given by
$$\int_{[0,1]\times\mathbb{R}} u^*\omega + \int_{[0,1]} H_t(\gamma^+(t))\, dt - \int_{[0,1]} H_t(\gamma^-(t))\, dt.$$
Moreover, if this happens the convergence $u(t,s) \to \gamma^{\pm}(t)$ when $s \to \pm\infty$ is uniform in t and $\partial_s u$ decays exponentially when $s \to \pm\infty$.

These are all very standard facts when $g = \mathrm{id}_X$ and they certainly follow from an easy adaptation of the proof in that case, although we don't know of an explicitly written proof in the literature. Alternatively we can use the proof of proposition 3.4.2 and appeal to known results in Lagrangian Floer homology, for instance [RS01, Theorem A].

Definition 3.1.4. *We define the moduli spaces*
$$\begin{aligned}\widehat{\mathcal{M}}_g(\gamma^-, \gamma^+; H, J) = \{u \in & C^\infty([0,1]\times\mathbb{R}, X) \,|\, u \text{ is a solution of (3.2)}, \\ & u(t, \pm\infty) = \gamma^{\pm}(t),\, u(1,s) = g(u(0,s))\}.\end{aligned} \tag{3.3}$$
If $\gamma^- \neq \gamma^+$ then $\widehat{\mathcal{M}}_g(\gamma^-, \gamma^+; H, J)$ admits an $\mathbb{R}$-action by translation in the s variable and we define
$$\mathcal{M}_g(\gamma^-, \gamma^+; H, J) = \widehat{\mathcal{M}}_g(\gamma^-, \gamma^+; H, J)/\mathbb{R}. \tag{3.4}$$

Sometimes we will omit the information about H, J and write only $\widehat{\mathcal{M}}_g(\gamma^-, \gamma^+)$, $\mathcal{M}_g(\gamma^-, \gamma^+)$. When we write $\widehat{\mathcal{M}}_g = \widehat{\mathcal{M}}_g(H,J)$, $\mathcal{M}_g = \mathcal{M}_g(H,J)$ we mean the unions of $\widehat{\mathcal{M}}_g(\gamma^-, \gamma^+; H, J)$ and $\mathcal{M}_g(\gamma^-, \gamma^+; H, J)$ over every pair of orbits $\gamma^-, \gamma^+ \in \mathcal{P}_g(H)$ (corresponding to all g-periodic solutions of Floer equation with finite energy).

3.2 Fredholm property and the relative index

In suitable transversality conditions the moduli spaces $\mathcal{M}_g(\gamma^-,\gamma^+)$ should be finite dimensional manifolds. This is controlled by the differential of the Floer operator.

The Floer operator $\mathcal{F}$ is defined as an operator

$$\mathcal{F} : C^\infty_g([0,1]\times\mathbb{R}, X) \to \bigcup_u C^\infty_g(u^*TX)$$

given by

$$\mathcal{F}(u) = \partial_s u + J_t(u)(\partial_t u - X_t(u)) \in C^\infty_g(u^*TX).$$

Here $C^\infty_g([0,1]\times\mathbb{R}, X)$ denotes the space of smooth maps $[0,1]\times\mathbb{R}\to X$ such that

$$u(1,s) = g(u(0,s)).$$

The space $C^\infty_g(u^*TX)$ is the space of sections of u^*TX with a g-boundary condition, that is, maps $\xi : [0,1]\times\mathbb{R}\to TX$ such that

$$\xi(t,s) \in T_{u(t,s)}X \text{ and } \xi(1,s) = (dg)_{u(0,s)}\xi(0,s).$$

We are interested in the linearization of $\mathcal{F}$ at some point $u \in C^\infty_g([0,1]\times\mathbb{R}, X)$ (and in particular at $u \in \widehat{\mathcal{M}}_g$). The tangent space of $C^\infty_g([0,1]\times\mathbb{R}, X)$ at u can be identified with $C^\infty_g(u^*TX)$; indeed given a curve

$$]-\varepsilon,\varepsilon[\ni\tau\mapsto u_\tau \in C^\infty_g([0,1]\times\mathbb{R}, X)$$

the vector tangent to it at $\tau = 0$ is defined by

$$\xi(t,s) = \frac{d}{d\tau}u_\tau(t,s)_{|\tau=0} \in T_{u(t,s)}X$$

and clearly $\xi \in C^\infty_g(u^*TX)$. So $(d\mathcal{F})_u$ is an operator defined on $C^\infty_g(u^*TX)$. Let $W^{k,p}_g(u^*TX)$ be the completion of the compactly supported sections in $C^\infty_g(u^*TX)$ with respect to the Sobolev $W^{k,p}$ norm, and let $L^p_g(u^*TX) = W^{0,p}_g(u^*TX)$. Then we can extend by continuity $(d\mathcal{F})_u$ to an operator

$$D_u : W^{1,p}_g(u^*TX) \to L^p_g(u^*TX).$$

To study this operator we would like to write it in local coordinates, and we can do this by trivializing the bundle u^*TX in an appropriate way.

Proposition 3.2.1. *Given a symplectomorphism g of (X,ω), $J \in \mathcal{J}_g(X,\omega)$ and $u \in C^\infty_g([0,1]\times\mathbb{R}, X)$ there is a trizialization*

$$\Psi(t,s) : \mathbb{R}^{2n} \to T_{u(t,s)}X \text{ for each } (t,s) \in [0,1]\times\mathbb{R}$$

preserving the symplectic form and almost complex structure (where on $\mathbb{R}^{2n}$ these are the standard ones) such that

$$\Psi(1,s) = (dg)_{u(0,s)}\Psi(0,s).$$

Moreover, if $\partial_s u$ satisfies an exponential decay condition (which is the case if u is a finite energy solution of Floer equation) then

$$\|\Psi\|_{1,\infty}, \|\Psi^{-1}\|_{1,\infty} < \infty.$$

In the proposition $\|\Psi\|_{1,\infty}$ denotes

$$\|\Psi\|_{1,\infty} = \sup_{(t,s)} \left(\|\Psi(t,s)\| + \|D\Psi(t,s)\|\right).$$

Proof. We consider the total space $u^*TX/\sim$ where we identify $T_{u(0,s)}X$ and $T_{u(1,s)}X$ via

$$dg : T_{u(0,s)}X \to T_{u(1,s)}X.$$

The the projection map $u^*TX \to [0,1] \times \mathbb{R}$ induces a projection $u^*TX/\sim \to S^1 \times \mathbb{R}$. This is a $U(n)$ bundle over $S^1 \times \mathbb{R}$ since it inherits the symplectic form and almost complex structures of u^*TM, as g is a symplectomorphism and $J \in \mathcal{J}_g(X, \omega)$, respectively; note that the almost complex structure in u^*TX at $(t,s) \in [0,1] \times \mathbb{R}$ is given by $J_t(u(t,s))$. It's well known, and we'll see in 4.2.4, that any $U(n)$-bundle over $S^1 \times \mathbb{R}$ is trivial. Thus there is a trivialization of $U(n)$-bundles

$$\widetilde{\Psi} : S^1 \times \mathbb{R} \times \mathbb{R}^{2n} \to u^*TX/\sim.$$

This lifts to a trivialization

$$\Psi : [0,1] \times \mathbb{R} \times \mathbb{R}^{2n} \to u^*TX$$

such that $\Psi(1,s,v) \sim \Psi(0,s,v)$, that is, $\Psi(1,s,v) = (dg)_{u(0,s)}\Psi(0,s,v)$. Hence

$$\Psi(t,s) = \Psi(t,s,\cdot) : \mathbb{R}^{2n} \to T_{u(t,s)}X$$

satisfies the condition required. It preserves the symplectic form and almost complex structure since $\widetilde{\Psi}$ is a $U(n)$-bundle isomorphism.

For the last assertion we refer the reader interested in the details to [dS18, Section 8.1] (where this is done only in the case $g = \mathrm{id}$). But the key point is that the exponential decay condition on $\partial_s u$ ensures that u can be extended to a map $\overline{u} : [0,1] \times \overline{\mathbb{R}} \to X$ where $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ is the two-point compactification of X with a differential structure pulled-back from $[-1,1]$ via an appropriate homeomorphism $\overline{\mathbb{R}} \to [-1,1]$. Then we pick a trivialization

$$S^1 \times \overline{\mathbb{R}} \times \mathbb{R}^{2n} \to \overline{u}^*TX/\sim$$

and restrict it to $S^1 \times \mathbb{R}$. And now compactness of $S^1 \times \overline{\mathbb{R}}$ gives the bound. □

We fix now a trivialization Ψ as in proposition 3.2.1; this induces a correspondence between sections of u^*TX and sections of the trivial bundle $[0,1] \times \mathbb{R} \times \mathbb{R}^{2n} \to [0,1] \times \mathbb{R}$, that is, maps $[0,1] \times \mathbb{R} \to \mathbb{R}^{2n}$; that is, we associate to $\xi \in C^\infty(u^*TX)$ the map

$$[0,1] \times \mathbb{R} \ni (t,s) \mapsto \Psi(t,s)^{-1}\xi(t,s) \in \mathbb{R}^{2n}.$$

We note that if $\xi \in C^\infty_g(u^*TX)$ then

$$\Psi(1,s)^{-1}\xi(1,s) = \Psi(1,s)^{-1}(dg)_{u(0,s)}\xi(0,s) = \Psi(0,s)^{-1}\xi(0,s).$$

Hence Ψ induces a correspondence between $C^\infty_g(u^*TX)$ and $C^\infty(S^1 \times \mathbb{R}, \mathbb{R}^{2n})$. If $\partial_s u$ decays exponentially then the last part of proposition 3.2.1 implies that Ψ also induces isomorphisms (of Banach spaces)

$$W^{1,p}(u^*TX) \cong W^{1,p}(S^1 \times \mathbb{R}, \mathbb{R}^{2n}) \text{ and } L^p(u^*TX) \cong L^p(S^1 \times \mathbb{R}, \mathbb{R}^{2n}).$$

Under this isomorphisms, D_u takes the form of a perturbed Cauchy-Riemann operator. These are very well understood: under a non-degeneracy assumption they are Fredholm and we know how to compute their Fredholm indices – see [dS18, Section 7], [AD14, Sections 8.7-8.9].

Definition 3.2.2. *Let $S : S^1 \times \mathbb{R} \to M_{2n\times 2n}(\mathbb{R})$ be continuous. We denote by $L_S : W^{1,p}(S^1 \times \mathbb{R}, \mathbb{R}^{2n}) \to L^p(S^1 \times \mathbb{R}, \mathbb{R}^{2n})$ the perturbed Cauchy-Riemann operator given by*

$$L_S = \partial_s + J_0 \partial_t + S$$

where J_0 is the standard complex structure on $\mathbb{C}^n \times \mathbb{R}^{2n}$.

We say that such S is admissible if

1. *S can be extended by continuity to $S^1 \times \overline{\mathbb{R}}$;*
2. *The limits $S^{\pm}(t) = S(t, \pm\infty)$ are paths of symmetric matrices;*
3. *If we let $R^{\pm} : [0, 1] \to M_{2n\times 2n}(\mathbb{R})$ be the solutions of the ordinary differential equation*

$$R^{\pm}(0) = I \text{ and } \frac{d}{dt}R^{\pm}(t) = J_0 S^{\pm}(t)R^{\pm}(t)$$

then the matrices $R^{\pm}(1)$ don't have 1 as an eigenvalue.

Proposition 3.2.3. *Let Ψ be a trivialization as in proposition 3.2.1. Then through this trivialization D_u is a perturbed Cauchy-Riemann operator, that is,*

$$D_u \circ \Psi = \Psi \circ L_S \tag{3.5}$$

where S is defined by

$$\Psi S = \nabla_s \Psi + J(\nabla_t \Psi - \nabla_\Psi X_t^H) + (\nabla_\Psi J)(\partial_t u - X_t^H). \tag{3.6}$$

Moreover, if we denote $\Psi^{\pm}(t) = \lim_{s\to\pm\infty} \Psi(t, s)$ and

$$\Phi^{\pm}(t) = \Psi^{\pm}(t)^{-1}(d\varphi_t^H)_{\gamma^{\pm}(0)}\Psi^{\pm}(0) \in Sp(\mathbb{R}^{2n}) \tag{3.7}$$

then

$$\dot{\Phi}^{\pm} = J_0 S^{\pm} \Phi^{\pm}. \tag{3.8}$$

Proof. This is a straightforward computation, see for instance [DS94, Theorem 2.2]. ☐

The known theory of such operators, together with proposition 3.2.3, gives a Fredholm property for the operator D_u when we have a non-degeneracy assumption.

Theorem 3.2.4 (Fredholm property)**.** *Suppose that $\gamma^-, \gamma^+ \in \mathcal{P}_g(H)$ are non-degenerate (see definition 3.1.2) and $u \in \widehat{\mathcal{M}}_g(\gamma^-, \gamma^+)$. Then the operator D_u is Fredholm and has index*

$$\operatorname{ind} D_u = \mu_{CZ}(\Phi^+) - \mu_{CZ}(\Phi^-) \equiv \mu(u)$$

where $\Phi^{\pm}$ are defined by (3.7).

Proof. It's well known that when S is admissible (see definition 3.2.2) then L_S is a Fredholm operator with index $\operatorname{ind} L_S = \mu_{CZ}(R^+) - \mu_{CZ}(R^-)$ ([dS18, Theorem 7.9.2] or [AD14, Proposition 8.7.1, Theorem 8.8.1]).

By equation (3.8) we have that $R^{\pm} = \Phi^{\pm}$, so the non-degeneracy condition of S is that the following matrices don't have 1 as eigenvalue:

$$\Phi^{\pm}(1) = \Psi^{\pm}(1)^{-1}(d\varphi_1)_{x_0^{\pm}}\Psi^{\pm}(0) = \Psi^{\pm}(0)^{-1}(dg)^{-1}_{x_0^{\pm}}(d\varphi_1)_{x_0^{\pm}}\Psi^{\pm}(0)$$

Here $x_0^\pm = \gamma^\pm(0)$. But this happens if and only if $\gamma^\pm$ are non-degenerate.

Since φ_t are symplectomorphisms it's clear that $\Phi^\pm(t)$ is a path of matrices in $Sp(n)$, hence by (3.8) $J_0 S^\pm(t)$ is in the Lie algebra $\mathfrak{sp}(n) = \{T \in M_{2n\times 2n}(\mathbb{R}) : (J_0T)^t = J_0T\}$ and thus $S^\pm(t)$ is symmetric. We conclude that S is admissible and the cited theory on the perturbed Cauchy-Riemann operators shows the result. ☐

A key point in the definition of Floer homology is that we need the moduli spaces above defined to be finite dimensional manifolds. This happens when the differential D_u is surjective. The proof of this with our g-periodicity condition is a straightforward adaptation of the proof in the classical 1-periodic situation.

Theorem 3.2.5. *Suppose $\gamma^-, \gamma^+ \in \mathcal{P}_g(H)$ are non-degenerate. If D_u is a surjective operator for every $u \in \widehat{\mathcal{M}}_g(\gamma^-, \gamma^+)$ then $\widehat{\mathcal{M}}(\gamma^-, \gamma^+)$, $\mathcal{M}(\gamma^-, \gamma^+)$ are smooth manifolds; moreover, their local dimensions at $u \in \widehat{\mathcal{M}}_g(\gamma^-, \gamma^+)$ are*

$$\dim_u \widehat{\mathcal{M}}(\gamma^-, \gamma^+) = \mu_{CZ}(\Phi^+) - \mu_{CZ}(\Phi^-)$$

and

$$\dim_u \mathcal{M}(\gamma^-, \gamma^+) = \mu_{CZ}(\Phi^+) - \mu_{CZ}(\Phi^-) - 1$$

where $\Phi^\pm$ are defined by 3.7.

Sketh. Essentially this result is an application of transversality in Banach manifolds – see [Lan99, Section II.§2]. However, we have to choose the said Banach manifold in an appropriate manner, as done in [dS18, Notation 8.2.1] or [AD14, Definition 8.2.2]) for $g = \mathrm{id}_X$. The correct space of cylinders to consider is $\mathcal{B}_g^{1,p}(\gamma^-, \gamma^+)$ consisting of "$W^{1,p}$-maps" (with $p > 2$) $u : [0,1] \times \mathbb{R} \to X$ such that

$$u(1, s) = g(u(0, s))$$

and

$$u(t, s) \longrightarrow \gamma^\pm(t) \text{ when } s \to \pm\infty.$$

To be precise,

$$\begin{aligned} \mathcal{B}_g^{1,p}(\gamma^-, \gamma^+) = \{\exp_u(Y)\colon & u \in C_g^\infty([0,1]\times\mathbb{R}, X),\ u(t, \pm\infty) = \gamma^\pm(t), \\ & \partial_s u, \partial_t u - X_t(u) \text{ satisfy exponential decay}, \\ & Y \in W_g^{1,p}(u^*TX), \|Y\| < r_u\} \end{aligned} \tag{3.9}$$

where exp means the geodesic exponential and for each u in the conditions stated $r_u > 0$ is the injectivity radius of $\exp_u$. Note that by requiring $p > 2$ we have a Sobolev embedding $W_g^{1,p}(u^*TX) \hookrightarrow C_g^0(u^*TX)$, so the expression $\exp_u(Y)$ makes sense as a globally defined (continuous) function.

This is a Banach manifold locally modelled by $W_g^{1,p}(u^*TX)$. The Floer differential extends by continuity to an operator

$$\mathcal{F} : \mathcal{B}_g^{1,p}(\gamma^-, \gamma^+) \to \mathcal{E}$$

where $\mathcal{E}$ is a bundle over $\mathcal{B}_g^{1,p}(\gamma^-, \gamma^+)$ where the fiber over $u \in \mathcal{B}_g^{1,p}(\gamma^-, \gamma^+)$ is $L_g^2(u^*TX)$; this is also a Banach manifold and $\mathcal{F}$ is a smooth map between the said Banach manifolds.

We need to show that $\mathcal{F}$ is transverse to the zero section of $\mathcal{E}$. Suppose that $\mathcal{F}(u) = 0$. The tangent space of $\mathcal{E}$ at $(u, 0)$ splits as

$$T_{(u,0)}\mathcal{E} = T_u\mathcal{B}_g^{1,p}(\gamma^-, \gamma^+) \oplus L_g^2(u^*TX)$$

and the differential $(d\mathcal{F})_u$ is

$$(\mathrm{id}, D_u) : T_u\mathcal{B}_g^{1,p}(\gamma^-, \gamma^+) \to T_{(u,0)}\mathcal{E} = T_u\mathcal{B}_g^{1,p}(\gamma^-, \gamma^+) \oplus L_g^2(u^*TX).$$

Note that $T_u\mathcal{B}_g^{1,p}(\gamma^-, \gamma^+)$ is the tangent space to the zero section and that $\pi_2 \circ (d\mathcal{F})_u = D_u$ where

$$\pi_2 : T_{(u,0)}\mathcal{E} \to T_{(u,0)}\mathcal{E}/T_u\mathcal{B}_g^{1,p}(\gamma^-, \gamma^+) = L_g^2(u^*TX)$$

is the projection map. Since $D_u = \pi_2 \circ (d\mathcal{F})_u$ is a surjective Fredholm operator it follows that the space of solutions of Floer equation

$$\{u \in \mathcal{B}_g^{1,p}(\gamma^-, \gamma^+) : \mathcal{F}(u) = 0\} = \mathcal{F}^{-1}(\text{zero section of } \mathcal{E})$$

is a smooth manifold of dimension $\mu(u) = \operatorname{ind} D_u$. By elliptic regularity any such solution is smooth, so is in $\widehat{\mathcal{M}}_g(\gamma^-, \gamma^+)$. On the other hand, $\widehat{\mathcal{M}}_g(\gamma^-, \gamma^+) \subseteq \mathcal{B}^{1,p}(\gamma^-, \gamma^+)$ because finite energy solutions of Floer equation are such that $\partial_s u$, $X_t(u) - \partial_t u$ have an exponential decay. □

Remark 3.2.6. By local dimension at u we mean the following: the space $\widehat{\mathcal{M}}_g(\gamma^-, \gamma^+)$ is a disjoint union of connected smooth manifolds of possibly different dimensions each, and the component where u lies has the said dimension.

Remark 3.2.7. The difference $\mu(u) = \mu_{CZ}(\Phi^+) - \mu_{CZ}(\Phi^-)$ does not depend on the choice of trivialization Ψ, since it's the Fredholm index of D_u. However, in general the indices $\mu_{CZ}(\Phi^\pm)$ don't depend only on the orbit $\gamma^\pm$, but they do depend in an important way on the trivialization Ψ of u^*TX we chose. This makes unclear how to assign an absolute grading to orbits in $\widehat{\mathcal{M}}_g$, and without further assumptions this is not possible. For instance if $c_1(X) \neq 0$ it's possible that we have $u, v \in \widehat{\mathcal{M}}_g(\gamma^-, \gamma^+)$ with $\mu(u) \neq \mu(v)$. We will discuss this issue in greater detail in section 4.2.

Theorem 3.2.5 motivates the definition of a regular pair.

Definition 3.2.8. *We say that $(H, J) \in C_g(\mathbb{R} \times X) \times \mathcal{J}_g(X, \omega)$ is a regular pair if H satisfies the non-degeneracy condition 3.1.2 and D_u is surjective for every $u \in \widehat{\mathcal{M}}_g(H, J)$.*

3.3 Floer complex

We would now like to define the Floer complex and its differential. For this we fix a (possibly Novikov) ring R and we let the Floer complex CF be the R-module generated by $\mathcal{P}_g(H)$, that is,

$$CF(X, g, H; R) = \bigoplus_{\gamma \in \mathcal{P}_g(H)} R \cdot \gamma.$$

Defining the differential is the hard part. We would like to do this by counting trajectories $u \in \mathcal{M}_g(\gamma^-, \gamma^+; H, J)$ with $\mu(u) = 1$, but to do so we need that this is a finite set. In general this is not guaranteed just by asking that (H, J) is a regular pair, but one needs some compactness results which don't hold for general symplectic manifolds. The general tool to prove such results is Gromov compactness theorem, which we now describe before discussing more specifically the problem of constructing the differential. The original Gromov compactness for (pseudo-)holomorphic was proved by Gromov in the seminal paper [Gro85]. Here we describe a statement as used in [HS95]. For a more complete statement (and proof) we refer to [Zil02].

Definition 3.3.1 (Convergence modulo bubbling). *Let $\gamma^-, \gamma^+ \in \mathcal{P}_g(H)$ and let $u_\nu \in \widehat{\mathcal{M}}_g(\gamma^-, \gamma^+)$ be a sequence of solutions of Floer equation. We say that u_ν converges directly modulo bubbling to $u \in \widehat{\mathcal{M}}_g(\gamma^-, \gamma^+)$ if there is finite set $Z \subseteq [0,1] \times \mathbb{R}$ such that, in every compact set contained in $[0,1] \times \mathbb{R} \backslash Z$ the sequence u_ν (and its derivatives) converges uniformly to u (and its derivatives). We call Z the set of singularities.*

Given Hamiltonian orbits $\gamma^- = \gamma^0, \gamma^1, \dots, \gamma^m = \gamma^+$ and Floer trajectories $u^j \in \widehat{\mathcal{M}}_g(\gamma^{j-1}, \gamma^j)$, $j = 1, \dots, m$, we say that u_ν converges to $(u^1, \dots, u^m)$ modulo bubbling if the following happens: for every sequence $s_\nu \in \mathbb{R}$ the sequence $v_\nu(t,s) = u_\nu(t, s + s_\nu)$ converges directly modulo bubbling to $u^j(t, s + s^j)$ for some $j = 1, \dots, m$ and $s_j \in \mathbb{R}$ or converges modulo bubbling to γ^j for some $j = 0, \dots, m$. Moreover every u^j can be approximated by u_ν in this way.

Note that there is a slight subtlety in this definition. Convergence modulo bubbling of u_ν to a single $u = u^1$, defined in the second paragraph, is not the same as direct convergence, defined in the first paragraph: it is direct convergence of u_ν to some translation of u in the s variable. Indeed u_ν converges to u if the classes of u_ν in $\mathcal{M}_g(\gamma^-, \gamma^+) = \widehat{\mathcal{M}}_g(\gamma^-, \gamma^+)/\mathbb{R}$ converge to the class of u in the topology of $\mathcal{M}_g(\gamma^-, \gamma^+)$, and u_ν converges directly to u if it converges in $\widehat{\mathcal{M}}_g(\gamma^-, \gamma^+)$.

We now state a version of Gromov-Floer compactness that's enough for our purposes. Recall that a J-holomorphic sphere is a map $v : (S^2, j_0) \to (X, J)$ such that $dv \circ j_0 = J \circ dv$ (here j_0 denotes the standard complex structure on $S^2 \cong \mathbb{C}P^1$). The energy of a holomorphic sphere is

$$E(v) = \int_{S^2} v^*\omega = \frac{1}{2} \int_{S^2} |dv|^2 \geq 0.$$

Theorem 3.3.2 (Gromov-Floer compactness). *Let $\gamma^-, \gamma^+ \in \mathcal{P}_g(H)$ and let $u_\nu \in \widehat{\mathcal{M}}_g(\gamma^-, \gamma^+)$ be a sequence of solutions of Floer equation with bounded energy $E(u_\nu)$ (see remark 3.1.3) and constant index $\mu(u_\nu) = \mu$.*

Then there is a subsequence of u_ν that converges modulo bubbling to $(u^1, \dots, u^m)$. Moreover if $Z = \{z_1, \dots, z_\ell\}$ is the set of singularities of the subsequence, then there are J-holomorphic spheres $v_1, \dots, v_\ell : S^2 \to X$ such

$$\lim_{\nu \to \infty} E(u_\nu) = \sum_{j=1}^{m} E(u^j) + \sum_{k=1}^{\ell} E(v^k) \qquad \mu = \sum_{j=1}^{m} \mu(u^j) + 2 \sum_{k=1}^{\ell} \langle c_1(TX), v^k \rangle \tag{3.10}$$

and $u^1, \dots, u^m, v^1, \dots, v^\ell$ is a connected family, that is,

$$\bigcup_{j=1}^{m} u^j([0,1] \times \overline{\mathbb{R}}) \cup \bigcup_{k=1}^{\ell} v^j(S^2)$$

is connected.

We see from Gromov-Floer compactness that there are essentially two obstructions to the 0-dimensional manifold $\{u \in \mathcal{M}_g(\gamma^-, \gamma^+); H, J) : \mu(u) = 1\}$ being compact/finite. The first is that the energy of such solutions may be unbounded, and so we cannot apply the theorem to get convergent subsequences. The second is the existence of bubbles. We begin by stating precisely what we mean by the non-existence of bubbles for our purposes:

Definition 3.3.3. *Let (H, J) be a regular pair. We say that (H, J) has no-bubbling if for every sequence $u_\nu \in \widehat{\mathcal{M}}_g(\gamma^-, \gamma^+)$ with bounded energy and $\mu(u_\nu) = \mu$ for $\mu = 1$ or $\mu = 2$ there is a subsequence converging without bubbling. More precisely:*

1. If $\mu = 1$ then a subsequence of u_ν converges to $u \in \widehat{\mathcal{M}}_g(\gamma^-, \gamma^+)$ with $\mu(u) = 1$.

2. If $\mu = 2$ then a subsequence of u_ν converges either to $u \in \widehat{\mathcal{M}}_g(\gamma^-, \gamma^+)$ with $\mu(u) = 2$ or to (u^1, u^2) with $u^1 \in \widehat{\mathcal{M}}_g(\gamma^-, \gamma^1)$, $u^2 \in \widehat{\mathcal{M}}_g(\gamma^1, \gamma^+)$ for some $\gamma^1 \in \mathcal{P}_g(H)$ and $\mu(u^1) = \mu(u^2) = 1$.

Part 1. of the no-bubbling assumption says that

$$\{u \in \mathcal{M}_g(\gamma^-, \gamma^+) : \mu(u) = 1, E(u) \leq C\} \subseteq \mathcal{M}_g(\gamma^-, \gamma^+)$$

is compact, and hence finite since it's a 0-dimensional manifold. Part 2. says that we can compactify the 1-dimensional manifold

$$\{u \in \mathcal{M}_g(\gamma^-, \gamma^+) : \mu(u) = 2, E(u) \leq C\}$$

by adding a boundary formed by pairs $(u^1, u^2) \in \mathcal{M}_g(\gamma^-, \gamma^1) \times \mathcal{M}_g(\gamma^1, \gamma^+)$ for some γ^1 such that $\mu(u^1) = \mu(u^2) = 1$ and $E(u^1) + E(u^2) \leq C$. These are the conditions we should need to define a differential ∂ (with Novikov coefficients) and to prove that $\partial^2 = 0$, respectively.

Even if we assume that there are no bubbles, the moduli space $\{u \in \mathcal{M}_g(\gamma^-, \gamma^+) : \mu(u) = 1\}$ can still be infinite if we don't bound the energy of its elements. Further impositions can be made on (X, ω) so that we can bound the energy (the simplest of which is ω being atoroidal) of such trajectories, but we can also use the algebraic formalism of Novikov rings to contour this problem and define a differential even without energy bounds.

Definition 3.3.4 (Novikov ring)**.** *Let R be a ring. We define the universal Novikov ring $\Lambda^{\mathrm{univ}}(R)$ with coefficients in R to be the ring of formal series*

$$\sum_{j=1}^{\infty} c_j T^{\lambda_j}$$

where $c_j \in R$ and $\lambda_j \in \mathbb{R}$ are such that $\lambda_j \to +\infty$.

If (X, ω) is a symplectic manifold we define its Novikov ring $\Lambda_\omega(X; R)$ to be the ring of formal series

$$\sum_{A \in H_2(X;\mathbb{Z})} c_A e^A$$

with $c_A \in \mathbb{R}$ such that for every $c \in \mathbb{R}$ we have

$$\#\{A \in H_2(X;\mathbb{Z}) : c_A \neq 0 \text{ and } \omega(A) < c\} < \infty.$$

Note that we have a canonical homomorphism from the Novikov ring of a symplectic manifold to the universal Novikov ring given by

$$e^A \mapsto T^{\omega(A)}.$$

In our definitions we'll use the universal Novikov ring, but we could also use the one corresponding to our symplectic manifold. The latter has the advantage that it can be used to correct grading problems when $c_1(X) \neq 0$ and we need to consider the homotopy class relative to the boundary of solutions of Floer equation; on the other hand, it would require us to pick a fixed non-canonical homotopy between each pair of Hamiltonian orbits γ^-, γ^+ as done in [BO09]. To keep the exposition slightly cleaner we stick to the universal Novikov ring, since when we consider gradings we'll assume that the first Chern class vanishes. When R is a field (we'll use $R = \mathbb{Q}$) the Novikov rings with coefficients in R are also fields (see [HS95, Theorem 4.1]).

The last ingredient needed to define the Floer complex in characteristic different from 2 are coherent orientations. These were introduced in the first place in [FH93], only in the case $g = 1$, but a generalization to our setup is straightforward. Coherent orientations essentially consist in orienting all the moduli spaces $\mathcal{M}_g(\gamma^-, \gamma^+)$ in a way that's coherent with the gluing construction and thus allows us to define an orientation in the compactified moduli space. For now all the reader has to retain is that when $u \in \widehat{\mathcal{M}}_g(\gamma^-, \gamma^+)$ has relative index $\mu(u) = 1$ the moduli space $\mathcal{M}_g(\gamma^-, \gamma^+)$ has local dimension 0 at u, so an orientation is simply an assignment of a sign $\nu(u) \in \{+1, -1\}$. Later in section 4.2 we'll give a more detailed explanation of coherent orientations in a more specific setting.

Definition 3.3.5. *Let (X, ω) be a symplectic manifold, $g : X \to X$ a symplectomorphism, $H \in C_g(\mathbb{R} \times X)$ a Hamiltonian and $J \in \mathcal{J}_g(X, \omega)$ an almost complex structure.*

Assume that (H, J) is a regular pair 3.2.8 (in particular H satisfies the non-degeneracy condition) and has no-bubbling 3.3.3. Then we define the Floer complex $CF(X, g, H; \Lambda)$ with coefficients in the Novikov ring $\Lambda = \Lambda^{\text{univ}}(R)$ to be the Λ-module generated by $\mathcal{P}_g(H)$:

$$CF(X, g, H; \Lambda) = \bigoplus_{\gamma \in \mathcal{P}_g(H)} \Lambda \cdot \gamma.$$

We define a differential $\partial = \partial_{H,J} : CF(X, g, H; \Lambda) \to CF(X, g, H; \Lambda)$ by counting (with signs) isolated Floer trajectories between Hamiltonian orbits corresponding to generators. More precisely let ∂ be the Λ-linear map defined on generators $\gamma^+ \in \mathcal{P}_g(H)$ by

$$\partial \gamma^+ = \sum_{\gamma^- \in \mathcal{P}_g(H)} \left(\sum_{\substack{u \in \mathcal{M}_g(\gamma^-, \gamma^+; H, J) \\ \mu(u) = 1}} \nu(u) T^{\omega(u)} \right) \gamma^-. \tag{3.11}$$

Note that, by the no-bubbling assumption, for fixed γ^-, γ^+, there is only a finite number of Floer trajectories $u \in \mathcal{M}_g(\gamma^-, \gamma^+; H, J)$ with $\mu(u) = 1$ and $\omega(u) \leq C$, so

$$\sum_{\substack{u \in \mathcal{M}_g(\gamma^-, \gamma^+; H, J) \\ \mu(u) = 1}} \nu(u) T^{\omega(u)}$$

is really an element of Λ.

We expect that when no-bubbling occurs the definition of the Floer complex indeed gives a complex, that is, $\partial^2 = 0$, and here we sketch a (incomplete) proof of this fact. First, proving this is equivalent to proving that for any fixed $\gamma^-, \gamma^+ \in \mathcal{P}_g(H)$ we have

$$\sum_{(u_1, u_2)} \nu(u_1) \nu(u_2) T^{\omega(u_1) + \omega(u_2)} = 0 \tag{3.12}$$

where (u_1, u_2) runs through the broken trajectories between γ^- and γ^+. To be precise, a broken trajectory is a pair $(u^1, u^2) \in \mathcal{M}_g(\gamma^-, \gamma^1) \times \mathcal{M}_g(\gamma^1, \gamma^+)$ for some $\gamma^1 \in \mathcal{P}_g(H)$ such that $\mu(u^1) = \mu(u^2) = 1$. Now we expect that a gluing property (as in [dS18, Chapter 10] or [AD14, Chapter 9]) can be used to show that by adding the broken trajectories to

$$\{u \in \mathcal{M}_g(\gamma^-, \gamma^+) : \mu(u) = 2\}$$

we get a 1-dimensional manifold $\overline{\mathcal{M}}$ with boundary, and the boundary is precisely the space of broken trajectories. Moreover, this manifold should be oriented and the orientation induced on the boundary should coincide with the orientations given by the assignment of the numbers $\nu(u^j)$; this is ensured by a choice of coherent orientations of the moduli spaces (see [FH93] for this discussion in the the case $g = 1$).

Also, the no-bubbling hypothesis 3.3.3 shows that if we consider only trajectories $u \in \overline{\mathcal{M}}$ with bounded energy (that is, we impose that $E(u) \leq C$ for some fixed $C \in \mathbb{R}$) we get a compact subset of $\overline{\mathcal{M}}$. Now each connected component of $\overline{\mathcal{M}}$ has constant energy; indeed by remark 3.1.3 the energy only depends on $\int_{[0,1]\times\mathbb{R}} u^*\omega$, and this quantity only depends on the homotopy class of u relative to the boundary (because ω is closed), and in each connected component of $\overline{\mathcal{M}}$ this homotopy class relative to the boundary is the same. So it follows that $\overline{\mathcal{M}}$ is a countable union of copies of S^1 and $[0, 1]$. Thus each broken trajectory (u^1, u^2) corresponds to a point in the border of one of the segments $[0, 1]$, so we can pair the broken trajectories. If (u^1, u^2) and (v^1, v^2) are the two boundary points of some segment, then by the coherence of the orientations we have

$$\nu(u^1)\nu(u^2) = -\nu(v^1)\nu(v^2).$$

Moreover, since $u^1\#u^2$ and $v^1\#v^2$ are homotopic relative to the boundary (by the argument above) then $\omega(u^1) + \omega(u^2) = \omega(v^1) + \omega(v^2)$. Hence each of these pairs of broken trajectories cancel out in the sum of equation (3.12) and the sum is indeed 0.

There are details in this proof to fill out. Namely the gluing construction should be adapted to the g-periodic case and we have to deal with coherent orientation. We won't take care of these details here, but we expect that they are a straightforward adaptation of the existing literature. So from now one we will assume that this holds.

Assumption 3.3.6. *We assume that whenever we are in the conditions of definition 3.3.5, including the no-bubbling condition, we have $\partial^2 = 0$.*

Once again, we stress that this is stated as an assumption only because we did not fill the details, but we really expect it to be true and proven by filling the details of the argument we just sketched. For instance in [DS94] this is also implicitly assumed to be true (at least for monotone symplectic manifolds, where no-bubbling is assured, see 3.3.9) but without a proof. So in ideal circumstances we can define Floer homology.

Definition 3.3.7. *Suppose we are in the conditions of definition 3.3.5 and assume 3.3.6. Then we define the Floer homology $HF(X, g, H, J; \Lambda)$ with coefficients in $\Lambda = \Lambda^{\mathrm{univ}}(R)$ to be the homology of the Floer complex $(CF(X, g, H; \Lambda), \partial_{H,J})$, that is,*

$$HF(X, g, H, J; \Lambda) = \frac{\ker \partial}{\operatorname{im} \partial}.$$

3.3.1 Monotone manifolds

A situation in which we're guaranteed to have no-bubbling is that of monotone symplectic manifolds.

Definition 3.3.8. *We say that a symplectic manifold (X, ω) is monotone if*

$$\langle [\omega], \pi_2(X)\rangle = \lambda\langle c_1(TX), \pi_2(X)\rangle$$

for some $\lambda \in \mathbb{R}_{\geq 0}$, where $\langle\, ,\rangle$ denotes the pairing between cohomology and $\pi_2(X)$ given by integrating over spheres.

We'll show that for monotone manifolds the no-bubbling is automatic.

Proposition 3.3.9. *Let (X,ω) be a monotone manifold. Then any regular pair (H,J) has no-bubbling.*

Proof. By Gromov-Floer compactness there is a subsequence converging modulo bubbling to $u^1,\dots,u^m$ with bubbles $v^1,\dots,v^\ell$. Moreover we have the relations in (3.10). Since (X,ω) is monotone

$$\lambda\langle c_1(TX),v^j\rangle = \langle[\omega],v^j\rangle = E(v_j) > 0$$

where the equality $\langle[\omega],v^j\rangle = E(v^j)$ holds because v^j is J-holomorphic. Thus if there is bubbling (i.e. $\ell > 0$) we get

$$\mu = \sum_{j=1}^{m}\mu(u^j) + 2\sum_{k=1}^{\ell}\langle c_1(TX),v^k\rangle \geq \sum_{j=1}^{m}\mu(u^j)+2$$

so $\sum_{j=1}^m \mu(u^j)\leq 0$. But since (H,J) is regular we must have $\mu(u^j)\geq 1$, otherwise the local dimension of $\mathcal{M}_g(\gamma^-,\gamma^+)$ is $\mu(u^j)-1<0$ by 3.2.5, a contradiction. So there are no bubbles and $\mu(u^j)\geq 1$, and since $\mu\leq 2$ we either have $m=1$ and $\mu(u^1)=1,2$ or $m=2$ and $\mu(u^1)=\mu(u^2)=1$. □

So for monotone manifolds we can define the Floer complex. In some conditions we can even do better and bound the energy of solutions of Floer equation with fixed limits.

Proposition 3.3.10. *Suppose that (X,ω) is aspherical, that is, it's monotone with constant $\lambda=0$, and suppose that $\pi_1(X)=0$. Under these conditions, if $u,v\in\widehat{\mathcal{M}}_g(\gamma^-,\gamma^+)$ then we have $E(u)=E(v)$.*

Proof. Let $w=u\#(-v):[0,1]\times S^1\to X$ be the map obtained by gluing the maps $u,v:[0,1]\times\mathbb{R}\to X$ where S^1 is regarded as the union of two copies of $\overline{\mathbb{R}}$ with their extremes $\pm\infty$ identified. By remark 3.1.3

$$E(u)-E(v)=\int_{[0,1]\times S^1} w^*\omega.$$

Since $\pi_1(X)=0$ we have a capping disk $\sigma_0:D^2\to X$ with $(\sigma_0)_{|\partial D^2}=w_{|\{0\}\times S^1}$. Define a second capping disk $\sigma_1:D^2\to X$ by $\sigma_1=g\circ\sigma_0$; since $w(1,s)=gw(0,s)$ we have $(\sigma_1)_{|\partial D^2}=w_{|\{1\}\times S^1}$. By assembling all these together we get a map from the sphere $f=\sigma_0\#w\#\overline{\sigma}_1:S^2\to X$, where the sphere is regarded as the union along the boundaries of the cylinder (domain of w) with two copies of the disk (domains of σ_0 and σ_1). Since g preserves ω we have

$$\int_{D^2}\sigma_0^*\omega=\int_{D^2}\sigma_1^*\omega$$

and thus by asphericity

$$E(u)-E(v)=\int_{[0,1]\times S^1} w^*\omega=\int_{S^2} f^*\omega=0. \quad \square$$

When we can bound the energy we can define the complex with any coefficients R (not necessarily a Novikov ring). The chain complex is the R-module generated by $\mathcal{P}_g(H)$ and the differential is defined as

$$\partial\gamma^+=\sum_{\gamma^-\in\mathcal{P}_g(H)}\left(\sum_{\substack{u\in\mathcal{M}_g(\gamma^-,\gamma^+;H,J)\\ \mu(u)=1}}\nu(u)\right)\gamma^-.$$

The coefficient of γ^- is now given by a finite sum; this is because every $u\in\mathcal{M}_g(\gamma^-,\gamma^+;H,J)$ has the same energy and thus Gromov-Floer compactness (with the no-bubbling we proved) says that there is a finite number of such u with $\mu(u)=1$.

3.3.2 Manifolds with trivial first Chern class

In this section we suppose that X is Calabi-Yau, that is, the first Chern class of X vanishes: $c_1(TX) = 0$. In this case, the arguments we used in the monotone case fail. For instance it was fundamental in the proof of no-bubbling 3.3.9 that $\langle c_1(TX), v\rangle > 0$ for holomorphic spheres v. Indeed, in general it's not true that any regular pair has no-bubbling.

However, in [HS95] Hofer and Salamon proved that if X is Calabi-Yau then a generic pair (H, J) is regular and has no-bubbling, in the case $g = id_X$. The key point of the argument is a dimensional analysis that shows that, in generic conditions, J-holomorphic spheres don't intersect Floer trajectories with $\mu(u) \leq 2$. Let us give the high level idea behind Hofer-Salamon proof. Once again we won't give rigorous details, but we hope that this is enough to convince the reader that the no-bubbling assumption is reasonable, in the sense that it should include important examples.

We denote

$$M_0(J) = \{x \in X : x \in \text{im } v \text{ for some } J\text{-holomorphic sphere } v\}$$

and

$$M_2(H, J) = \{x \in X : x \in u([0, 1] \times \overline{\mathbb{R}}) \text{ for some } u \in \mathcal{M}_g(H, J) \text{ such that } \mu(u) \leq 2\}.$$

If $M_0(J) \cap M_2(H, J) = \emptyset$ then we have no-bubbling; indeed if there are bubbles in 3.3.2 we would get a contradiction with the fact that the bubbles and the Floer trajectories form a connected set. And indeed this happens in generic conditions, as $M_0(J)$ is expected to have codimension 4 and $M_2(H, J)$ to have dimension 3.

For a generic J, the space $\widehat{\mathcal{M}}(A, J)$ of J-holomorphic spheres in homology class $A \in H_2(X; \mathbb{Z})$ is a manifold of dimension

$$\dim \widehat{\mathcal{M}}(A, J) = 2n + c_1(A) = 2n$$

(see [MS12, Theorem 3.1.15]). Now $M_0(J)$ is the image of the evaluation map

$$\bigcup_{A \in H_2(X)} \widehat{\mathcal{M}}(A, J) \times_G S^2 \to X$$

where $G = PSL(2, \mathbb{C})$ is the group of conformal automorphisms of S^2. Since G has dimension 6 this image is a set of dimension $2n + 2 - 6 = 2n - 4$, in the sense that it's the image of a (second countable) manifold of dimension $2n - 4$ through a smooth map.

On the other hand if (H, J) is regular then $M_2(H, J)$ is the image of the evaluation map

$$\left\{u \in \widehat{\mathcal{M}}_g(H, J) : \mu(u) \leq 2\right\} \times_{\mathbb{R}} ([0, 1] \times \mathbb{R})$$

and the set $\{u \in \widehat{\mathcal{M}}_g(H, J) : \mu(u) \leq 2\}$ is the union of manifolds of dimension at most 2. Thus $M_2(H, J)$ is a set dimension $3 = 2 + 2 - 1$.

Since $(2n - 4) + 3 < 2n$ it's expected that for a generic Hamiltonian H not only the pair (H, J) is generic but we also have $M_0(J) \cap M_2(H, J) = \emptyset$. This is made precise, for $g = 1$, in [HS95, Theorem 3.2], and adapting the proof to a general g is straightforward.

3.4 Relation to Lagrangian Floer homology

We now discuss how we can interpret the construction of Floer homology with g-periodic boundary conditions via Lagrangian Floer homology. We won't really use the results of this section, but it should be noted that in principle the general theory of Lagrangian Floer homology developed in [FOOO09a, FOOO09b] can be used to deal with general symplectic manifolds where we can't avoid bubbling and have to work with virtual fundamental cycles and abstract perturbation theory.

We begin by showing that our construction of Floer homology with g-periodic boundary conditions can be reduced to the case $H = 0$ by changing g. In this case our Floer trajectories are holomorphic strips. This is actually a very good reason to consider Floer homology with g-periodic boundary conditions (and not only $g = id_X$), since it shows that if we perform this reduction starting with the usual Hamiltonian Floer homology (that is, with $g = id_X$) we naturally arrive at Floer homology of a Hamiltonian symplectomorphism g.

Proposition 3.4.1. *Suppose we are in the conditions of 3.3.5 to define the Floer chain complex. Let as usual φ_t be the flow of X_t. Then, if we choose compatible orientations on the relevant moduli spaces, there is a canonical isomorphism of chain complexes*

$$(CF(X, g, H; \Lambda), \partial_{H,J_t}) \cong (CF(X, \varphi_1^{-1} \circ g, H = 0; \Lambda), \partial_{H=0,\varphi_t^* J_t}).$$

Proof. First, there is a bijection between the sets of generators of the chain complexes $\mathcal{P}_g(H)$ and $\mathcal{P}_{\varphi_1^{-1} \circ g}(0)$; indeed the former is the set of orbits $\gamma(t) = \varphi_t(x_0)$ for some fixed point x_0 of $\varphi_1^{-1} \circ g$ and the latter is the set of constant orbits $c^{x_0}(t) = x_0$ for x_0 fixed point of $\varphi_1^{-1} \circ g$.

To prove the claimed isomorphism it's enough to show that we can identify canonically the moduli spaces

$$\widehat{\mathcal{M}}_g \left(\gamma^-, \gamma^+; H, J\right) \text{ and } \widehat{\mathcal{M}}_{\varphi_1^{-1} \circ g} \left(c^{x_0^-}, c^{x_0^-}; H = 0, \varphi_t^* J\right)$$

where $x_0^\pm = \gamma^\pm(0)$. Indeed given $u \in \widehat{\mathcal{M}}_g(\gamma^-, \gamma^+; H, J)$ we let $v(t, s) = \varphi_t^{-1}(u(t, s))$. It's straightforward to see that v satisfies the required boundary conditions, so we're left with showing that v obeys Floer equation (3.2) with Hamiltonian $H = 0$ and almost complex structure $\varphi_t^* J_t$. Indeed

$$\partial_s v(t, s) = (d\varphi_t)^{-1} \partial_s u(t, s)$$

and

$$\partial_t v(t, s) = (d\varphi_t)^{-1} \partial_t u(t, s) - (d\varphi_t)^{-1} X_t(u(t, s)).$$

Thus, since u obeys equation 3.2, then

$$\partial_s v + (\varphi_t^* J_t)(v) \partial_t v = (d\varphi_t)^{-1} \left(\partial_s u + J_t(u)(\partial_t u - X_t(u))\right) = 0$$

so $v \in \widehat{\mathcal{M}}_{\varphi_1^{-1} \circ g} \left(c^{x_0^-}, c^{x_0^-}; H = 0, \varphi_t^* J\right)$. Conversely if $v \in \widehat{\mathcal{M}}_{\varphi_1^{-1} \circ g} \left(c^{x_0^-}, c^{x_0^-}; H = 0, \varphi_t^* J\right)$ then letting $u(t, s) = \varphi_t(v(t, s))$ we have $u \in \widehat{\mathcal{M}}_g \left(\gamma^-, \gamma^+; H, J\right)$. □

We now explain the way to formulate Floer homology with g-periodic boundary conditions in terms of Lagrangian Floer homology. A quick and not so extensive reference for it is [Ped18], and a much more complete one is [FOOO09a, FOOO09b].

Proposition 3.4.2. *Suppose we are in the conditions of 3.3.5 to define the Floer chain complex and $H = 0$. Consider the symplectic manifold $(X \times X, (-\omega) \oplus \omega)$ and its Lagrangian submanifolds*

$$L_1 = \{(x, x) : x \in X\} \text{ and } L_g = \{(x, g(x)) : x \in X\}.$$

Moreover let $\widetilde{J}_t$ be the almost complex structure

$$\widetilde{J}_t = \left(-J_{\frac{1-t}{2}}\right) \oplus J_{\frac{1+t}{2}}$$

on $X \times X$, which is compatible with $(-\omega) \oplus \omega$. Then, after choosing compatible orientations on the relevant moduli spaces, there is a canonical isomorphism of chain complexes

$$(CF(X, g, H = 0; \Lambda), \partial_{H=0,J_t}) \cong (CLF(X \times X, L_1, L_g; \Lambda), \partial_{\widetilde{J}_t})$$

where the latter is the Lagrangian Floer complex.

Proof. Once again the generators of $CF(X, g, H = 0; \Lambda)$ are in bijection with fixed points x_0 of g. Similarly $CLF(X \times X, L_1, L_g; \Lambda)$ is generated by points in the intersection $L_1 \cap L_g$, which have the form $(x_0, x_0) = (x_0, g(x_0))$ for x_0 a fixed point of g. To prove the result it's enough to exhibit a correspondence between the moduli spaces

$$\widehat{\mathcal{M}}_g \left(c^{x_0^-}, c^{x_0^+}; H = 0, J\right) \text{ and } \widehat{\mathcal{M}}^{\text{lag}} \left((x_0^-, x_0^-), (x_0^+, x_0^+); L_1, L_g, \widetilde{J}\right).$$

Indeed suppose that $u \in \widehat{\mathcal{M}}_g \left(c_{x_0^-}, c_{x_0^+}; H = 0, J\right)$; this is a J-holomorphic strip $u : [0, 1] \times \mathbb{R}$ with boundary conditions $u(t, \pm\infty) = x_0^\pm$ and $u(1, s) = g(u(0, s))$. Now we define $v : [0, 1] \times \mathbb{R} \to X \times X$ by

$$v(t, s) = \left(u\left(\frac{1-t}{2}, \frac{s}{2}\right), u\left(\frac{1+t}{2}, \frac{s}{2}\right)\right).$$

It's straightforward to check that v is $\widetilde{J}$-holomorphic, that $v(0, s) \in L_1, v(1, s) \in L_g$ and $v(t, \pm\infty) = (x_0^\pm, x_0^\pm)$ so $v \in \widehat{\mathcal{M}}^{\text{lag}} \left((x_0^-, x_0^-), (x_0^+, x_0^+); L_1, L_g, \widetilde{J}\right)$. Conversely, given

$$v = (v_1, v_2) \in \widehat{\mathcal{M}}^{\text{lag}} \left((x_0^-, x_0^-), (x_0^+, x_0^+); L_1, L_g, \widetilde{J}\right)$$

we can recover u by defining

$$u(t, s) = \begin{cases} v_1(1 - 2t, 2s) & \text{if } t \in [0, 1/2] \\ v_2(-1 + 2t, 2s) & \text{if } t \in [1/2, 1]. \end{cases}$$

The boundary condition $v(0, s) \in L_1$ makes sure that this gluing is C^1, and hence smooth by elliptic regularity. It's straightforward to show that u is J-holomorphic and obeys the required boundary conditions, so $u \in \widehat{\mathcal{M}}_g \left(c^{x_0^-}, c^{x_0^+}; H = 0, J\right)$. $\square$

Remark 3.4.3. In Lagrangian Floer homology the possibility of orienting in a coherent way the relevant moduli spaces, and hence defining Lagrangian Floer homology in characteristic not 2, is non-trivial and discussed in [FOOO09b] (in particular see Theorem 8.1.14). The condition to be able to do so is that the Lagrangian submanifolds $L_1, L_2 \subseteq Y$ are relatively spin, that is, they are both orientable and there is a class $st \in H^2(Y; \mathbb{Z}/2)$ such that $st_{|L_j} = w_2(TL_j)$. In the case we're interested in, of proposition 3.4.2, this is automatic; L_1, L_g are both diffeomorphic to X, which is orientable since it's symplectic, and we can take $st = \pi_1^* w_2(TX)$ where $\pi_1 : X \times X \to X$ is the projection onto the first component.

Remark 3.4.4. The sub-manifold L_g is the fixed point set of the anti-symmetric involution

$$X \times X \ni (x, y) \mapsto (g^{-1}y, gx) \in X \times X.$$

By [Oh93, Example (i)] if X is monotone then L_g is a monotone Lagrangian in $X \times X$. If moreover we assume that $\pi_1(X)$ is torsion the results of [Oh93] apply and the Lagrangian Floer homology is well defined without Novikov rings (compare with 3.3.10).

Chapter 4

Floer homology of global quotient orbifolds

4.1 Introduction

In this section we will define, under suitable conditions, the Floer homology of global quotient orbifolds. To define the Floer complex of such an orbifold we will use our previous construction of Floer homology for g-periodic Hamiltonian orbits. We let (X, ω) be a compact symplectic manifold of dimension $2n$ and let G be a group acting on (X, ω) by symplectomorphisms. Then the quotient orbifold $\mathcal{X} = [X/G]$ is a symplectic orbifold $(\mathcal{X}, \omega)$.

We denote by $\mathcal{J}_G(X, \omega)$ the set of G-invariant time dependent almost complex structures on X; more precisely

$$\mathcal{J}_G(X, \omega) = \bigcap_{g \in G} \mathcal{J}_g(X, \omega)$$

where we defined $\mathcal{J}_g(X, \omega)$ in section 3. Alternatively, $(J_t)_{t\in\mathbb{R}} \in \mathcal{J}_G(X, \omega)$ if $J_{t+1} = J_t$ and $g^* J_t = J_t$ for every $g \in G$. Such a family of almost complex structure corresponds to a 1-periodic family of almost complex structures on $\mathcal{X}$. As in the non-equivariant case this set is always contractible and non-empty.

Proposition 4.1.1. *Let G be a finite group acting on (X, ω) by symplectomorphisms. Then the set $\mathcal{J}_G(X, \omega)$ is contractible and non-empty.*

Proof. Given a (time dependent, 1-periodic) almost complex structure $(J_t)_{t\in\mathbb{R}} \in \mathcal{J}(X, \omega)$ not necessarily G-invariant we can assign a (time dependent, 1-periodic) Riemannian metric $\langle\, ,\, \rangle_t$ determined by

$$\omega(u, v) = \langle J_t u, v \rangle_t$$

and hence we get a map

$$j : \mathcal{J}(X, \omega) \to \mathrm{Maps}(S^1, Riem(X)).$$

The map j has a homotopy inverse r (see [MS17, Proposition 2.50 ii)]). Since both j and r are G-equivariant it follows that

$$\mathcal{J}_G(X, \omega) \simeq \mathrm{Maps}_G(S^1, Riem(X))$$

where the right hand side are G-invariant (time dependent) Riemannian metrics. The space of such Riemannian metrics admits an obvious convex structure, and thus is contractible.

To prove existence just take a constant family $\langle\, ,\, \rangle_t$ of equivariant Riemannian metrics, which exists by a simple averaging argument, and apply r. $\square$

Similarly we denote by $C_G(\mathbb{R} \times X)$ the set of 1-periodic time dependent G-invariant Hamiltonians

$$C_G(\mathbb{R} \times X) = \bigcap_{g \in G} C_g(\mathbb{R} \times X).$$

Equivalently, a Hamiltonian $H \in C_G(\mathbb{R} \times X)$ is a smooth map $H : S^1 \times X \to \mathbb{R}$ such that $H_t(x) = H_t(g(x))$ where as usual we denote $H_t(x) = H(t, x)$ for $(t, x) \in S^1 \times X$. Here S^1 is $\mathbb{R}/\mathbb{Z}$ and we will consider t to be interchangeably in S^1 or $\mathbb{R}$. Such a Hamiltonian induces a 1-periodic Hamiltonian in the quotient $\mathcal{X} = [X/G]$.

We would now like to define a Floer homology $HF(\mathcal{X}, \omega, H, J; \Lambda)$ with coefficients in a (Novikov) ring Λ. First, we recall from proposition 2.2.5 that the loop space of $\mathcal{X}$ is given by

$$\{(\gamma, g) \in C^\infty([0, 1], X) \times G : \gamma(1) = g\gamma(0)\}/G$$

where G acts on a pair (γ, g) by $h \cdot (\gamma, g) = (h\gamma, hgh^{-1})$. We write $(\gamma, g) \sim (\gamma', g')$ if (γ, g) and (γ', g') are in the same orbit with respect to the action of G. Instead of (γ, g) we will write γ_g, or only γ if g is clear (or has been made clear); the class of γ_g is denoted by $[\gamma_g]$. Now a loop $[\gamma_g]$ in $\mathcal{X}$ is a Hamiltonian orbit if and only if its lift $\gamma : [0, 1] \to X$ is a Hamiltonian orbit of H. We define

$$\widetilde{\mathcal{P}}_G(H) = \{(\gamma, g) | \gamma : [0, 1] \to X \text{ is a Hamiltonian orbit and } \gamma(1) = g\gamma(0)\} = \bigsqcup_{g \in G} \mathcal{P}_g(H)$$

and moreover we write $\mathcal{P}_G(H) = \widetilde{\mathcal{P}}_G(H)/G$. Morally, the set $\mathcal{P}_G(H)$ is the set of Hamiltonian 1-periodic orbits in the orbifold $\mathcal{X}$. Indeed an element $[\gamma_g] \in \mathcal{P}_G(H)$ has a topological realization as a map $S^1 \to |\mathcal{X}| = X/G$ induced by the composition $[0, 1] \xrightarrow{\gamma} X \longrightarrow X/G = |\mathcal{X}|$. Once again we remark that different elements of $\mathcal{P}_G(H)$ might have the same topological realization; this happens for instance if the constant map $c^{x_0} : [0, 1] \to X$ equal to $x_0 \in X$ is a Hamiltonian orbit – that is, x_0 is a critical point of H_t for every t – and $1 \neq g \in G_{x_0}$ fixes x_0; then $[c^{x_0}_g] \neq [c^{x_0}_1]$ but they have the same topological realization. This fact will be crucial in chapter 6, where it will lead to the appearance of twisted sectors.

4.1.1 Floer complex in characteristic 2 and the orientability problem

We shall now explain how are we going to construct our Floer complex and see how in characteristic not 2 the naive approach to define the complex fails because of a problem with orientations. Because of this, some of the notation in this section is provisional and will be corrected in 4.4.

The naive approach to define the Floer complex over a ring Λ (for instance a Novikov ring) is to consider the free Λ-module with generators $\mathcal{P}_G(H)$, that is,

$$CF(\mathcal{X}, H; \Lambda) = \bigoplus_{[\gamma_g] \in \mathcal{P}_G(H)} \Lambda \cdot [\gamma_g].$$

An alternative description is the following: if we let

$$\widetilde{CF}(\mathcal{X}, H; \Lambda) = \bigoplus_{g \in G} CF(X, g, H; \Lambda) = \bigoplus_{\gamma_g \in \widetilde{\mathcal{P}}_G(H)} \Lambda \cdot \gamma_g \tag{4.1}$$

then the G-action on $\widetilde{\mathcal{P}}_G(H)$ extends Λ-linearly to a G-action on $\widetilde{CF}(\mathcal{X}, H; \Lambda)$ and we can regard $CF(\mathcal{X}, H; \Lambda)$ as the G-invariant part of $\widetilde{CF}(\mathcal{X}, H; \Lambda)$, which we denote by $\widetilde{CF}(\mathcal{X}, H; \Lambda)^G$. Indeed it's easy to check that we have an isomorphism given by

$$CF(\mathcal{X}, H; \Lambda) \ni [\gamma_g] \mapsto \sum_{\gamma'_{g'} \sim \gamma_g} \gamma'_{g'} \in \widetilde{CF}(\mathcal{X}, H; \Lambda)^G.$$

So from now on when we talk about $CF(\mathcal{X}, H; \Lambda)$ we actually mean the G-invariant part of $\widetilde{CF}(\mathcal{X}, H; \Lambda)$. We now turn to the definition of a differential on $CF(\mathcal{X}, H; \Lambda)$. As expected, for this one has to assume a regularity and no-bubbling condition for all the elements $g \in G$.

Definition 4.1.2. *From this section on (unless otherwise specified), we say that a pair $(H, J) \in C_G(\mathbb{R} \times X) \times \mathcal{J}_G(X, \omega)$ is regular if it's regular as defined in 3.2.8 for every $g \in G$.*

We say that (H, J) has no-bubbling if it has no-bubbling as defined in 3.3.3 for every $g \in G$.

Assume now that $\Lambda = \Lambda^{\mathrm{univ}}(R)$ is a Novikov ring. For each g we have a differential

$$\partial : CF(X, g, H; \Lambda) \to CF(X, g, H; \Lambda)$$

defined in 3.3.5. By assembling such differentials we get a differential

$$\partial = \partial_{H,J} : \widetilde{CF}(\mathcal{X}, H; \Lambda) = \bigoplus_{g \in G} CF(X, g, H; \Lambda) \to \bigoplus_{g \in G} CF(X, g, H; \Lambda) = \widetilde{CF}(\mathcal{X}, H; \Lambda).$$

We need the differential to preserve the G-invariant part. Unfortunately that is not true and this is the problem of our naive approach. Before we see how this fails in general, let's prove that in characteristic 2 it's true.

Lemma 4.1.3. *Suppose that (H, J) is regular and has no-bubbling. Let $\Lambda = \Lambda^{\mathrm{univ}}(R)$ be a Novikov ring over a ring R of characteristic 2. Then the differential $\partial : \widetilde{CF}(\mathcal{X}, H; \Lambda) \to \widetilde{CF}(\mathcal{X}, H; \Lambda)$ is equivariant with respect to the G-action on $\widetilde{CF}(\mathcal{X}, H; \Lambda)$. In particular ∂ restricts to a differential on the G-invariant part*

$$\partial : CF(\mathcal{X}, H; \Lambda) \to CF(\mathcal{X}, H; \Lambda).$$

Proof. We want to prove that $h\partial = \partial h$. For $\gamma^+ \in \mathcal{P}_g(H)$ we have

$$(h\partial)\gamma^+ = \sum_{\gamma^- \in \mathcal{P}_g(H)} \left(\sum_{\substack{u \in \mathcal{M}_g(\gamma^-, \gamma^+; H, J) \\ \mu(u)=1}} \nu(u) T^{\omega(u)} \right) h\gamma^-$$

and

$$(\partial h)\gamma^+ = \sum_{\hat{\gamma}^- \in \mathcal{P}_{hgh^{-1}}(H)} \left(\sum_{\substack{v \in \mathcal{M}_{hgh^{-1}}(\hat{\gamma}^-, h\gamma^+; H, J) \\ \mu(v)=1}} \nu(v) T^{\omega(v)} \right) \hat{\gamma}^-.$$

But it's easy to see that there is a bijection

$$\mathcal{P}_g(H) \ni \gamma^- \mapsto \hat{\gamma}^- = h\gamma^- \in \mathcal{P}_{hgh^{-1}}(H)$$

and also a bijection

$$\mathcal{M}_g(\gamma^-, \gamma^+) \ni u \mapsto v = hu \in \mathcal{M}_{hgh^{-1}}(h\gamma^-, h\gamma^+).$$

We have $\nu(u) = \nu(v) = 1$ in Λ because R has characteristic 2 and $\omega(u) = \omega(hu) = \omega(v)$ because h is a symplectomorphism. Thus $(\partial h)\gamma^+ = (h\partial)\gamma^+$ for every generator γ^+ and we are done. $\square$

Hence we can define right now the Floer complex (and thus Floer homology) over Novikov rings of characteristic 2.

Definition 4.1.4. *Let (X, ω) be a symplectic manifold, G a finite group acting on (X, ω) by symplectomorphisms.*

Suppose that $(H, J) \in C_G(\mathbb{R} \times X) \times \mathcal{J}_G(X, \omega)$ is a regular pair and has no-bubbling (in the sense of 4.1.2). Let $\Lambda = \Lambda^{\mathrm{univ}}(R)$ be a Novikov ring over a ring R of characteristic 2.

Let $CF(\mathcal{X}, H; \Lambda)$ be the G-invariant part of $\widetilde{CF}(\mathcal{X}, H; \Lambda)$ defined in (4.1). By lemma 4.1.3 we have a restricted differential $\partial = \partial_{H,J} : CF(\mathcal{X}, H; \Lambda) \to CF(\mathcal{X}, H; \Lambda)$. Then $(CF(\mathcal{X}, H; \Lambda), \partial_{H,J})$ is called the Floer complex of $\mathcal{X}$.

Assuming 3.3.6 we define the Floer homology of the orbifold $\mathcal{X}$ to be the Λ-module

$$HF(\mathcal{X}, H, J; \Lambda) = \frac{\ker \partial}{\operatorname{im} \partial}.$$

The role of characteristic 2 is quite clear from the proof of lemma 4.1.3: for ∂ to be G-equivariant we need that, in our assignment of coherent orientations, we have $\nu(hu) = \nu(u)$ for every Floer trajectory u with $\mu(u) = 1$ and $h \in G$. In general it is not possible to ensure this. For instance we may have a Floer trajectory u between $\gamma^-, \gamma^+ \in \mathcal{P}_g(H)$ and $h \in G$ fixing γ^-, γ^+ such that $\nu(u) = -\nu(hu)$; when this is the case there is no hope in getting equivariance of ∂, even if we change our coherent orientations.

This phenomenon is already well known in the context of orbifold Morse homology, as explained in [CH14] and in section 2.4. In Morse homology, each critical point is assigned an orientation space (which is the determinant bundle of the unstable manifold at x) and one has to exclude the critical points x for which there is some $g \in G_x$ reversing the orientation. Indeed attaching cells corresponding to these critical points that reverse the orientation doesn't change the topology.

Essentially the problem with the naive approach is that the G-action we defined on $\widetilde{CF}(\mathcal{X}, H; \Lambda)$ is not considering the action of G in certain orientation spaces corresponding to the orbits. Roughly speaking, there is some sign we should introduce when the action reverses these orientations. To understand this issue we'll have to discuss orientations and the signs $\nu(u)$ better, which we will do in 4.3. To avoid having to consider a cover of $\widetilde{\mathcal{P}}_G(H)$ we will only do so when $\mathcal{X}$ is Calabi-Yau; in this case each orbit has a canonical trivialization and this makes it possible to define coherent orientations by picking orientations on certain determinant line bundles δ_γ corresponding to each $\gamma \in \widetilde{\mathcal{P}}_G(H)$. This is also the case in which we can define a natural grading on the Floer complex, and this is done in the next section. We come back to the definition of the Floer complex of an orbifold in section 4.4.

4.2 Absolute index using trivialization of $\Lambda^n_{\mathbb{C}} TX$

In section 3.2 we showed how to assign a relative index $\mu(u) = \operatorname{ind} D_u$ to any Floer trajectory $u \in \widehat{\mathcal{M}}_g(\gamma^-, \gamma^+)$. To define a grading on the Floer homology we would like to be able to write this relative index in terms on an absolute index assigned to Hamiltonian orbits $\gamma_g \in \widetilde{\mathcal{P}}_G(H)$.

Even in the smooth case $G = \{1\}$ this is not possible in general. If $u, v \in \widehat{\mathcal{M}}(\gamma^-, \gamma^+)$ then

$$\operatorname{ind} D_u = \operatorname{ind} D_v + 2\langle c_1(X), u\#(-v)\rangle$$

where $c_1(X) = c_1(TX)$.

So the index doesn't depend only on the endpoints $\gamma^\pm$ but also on the homology class of u relative to the boundary. There are several ways to avoid this problem.

1. The first is simply to only consider the case $c_1(X) = 0$ or some slightly weaker condition such as $\langle c_1(X), T^2\rangle = 0$ for any torus $T^2 \to X$ (or $\langle c_1(X), S^2\rangle = 0$ for spheres $S^2 \to X$ if we restrict ourselves to contractible orbits)

2. When this doesn't happen, we can still give a mod $2N$ grading to our Floer complex where $N \in \mathbb{Z}^+$ is the (homological) minimal Chern number defined by

$$\langle c_1(X), H_2(X;\mathbb{Z})\rangle = N\mathbb{Z}.$$

3. To get a $\mathbb{Z}$-graded index in general conditions we can use a graded Novikov ring; this approach is followed in [BO09]. A conceptual way to say what this means is the following: instead of considering Morse homology of the action functional defined on the loop space of X, we define it in an appropriate covering of the loop space with fibers $H_2(X;\mathbb{Z})$.

Here we explain how we can use a trivialization of the determinant line bundle to assign a canonical absolute index to Hamiltonian orbits when $\mathcal{X}$ is Calabi-Yau.

Definition 4.2.1. *We say that a global quotient orbifold $\mathcal{X} = [X/G]$ is Calabi-Yau if the equivariant first Chern class (see definition B.1.2) $c_1^G(X) = c_1^G(TX)$ vanishes.*

The orbifold $\mathcal{X}$ is Calabi-Yau if and only if the G-equivariant line bundle (see appendix B.1) $\Lambda^n_{\mathbb{C}}TX \to X$ has trivial (equivariant) first Chern class

$$c_1^G(\Lambda^n_{\mathbb{C}}TX) = c_1^G(TX) = 0,$$

and by theorem B.1.3 this happens if and only if $\Lambda^n_{\mathbb{C}}TX$ is a trivial G-bundle.

Remark 4.2.2. Saying that $\Lambda^n_{\mathbb{C}}TX$ is trivial as a G-equivariant bundle is the same as saying that the orbibundle $\Lambda^n_{\mathbb{C}}T\mathcal{X} \to \mathcal{X}$ is trivial. In particular $\Lambda^n_{\mathbb{C}}T\mathcal{X}$ is an honest bundle, so any Calabi-Yau orbifold is Gorenstein.

Conversely, if $\mathcal{X}$ is Gorenstein, $c_1(X) = 0$, $H^1(X;\mathbb{Z}) = 0$ and for every $g \in G$ the fixed point set X^g is non-empty then by proposition B.1.6, applied to $E = \Lambda^n_{\mathbb{C}}TX$, $\mathcal{X}$ is Calabi-Yau.

We now fix a G-equivariant non-vanishing section $\mathfrak{s} : X \to \Lambda^n_{\mathbb{C}}TX$. Such a section determines a trivialization

$$X \times \mathbb{C} \ni (x, z) \mapsto (x, z\mathfrak{s}(x)) \in \Lambda^n_{\mathbb{C}}TX.$$

Remark 4.2.3. Suppose we have a G-equivariant trivial line bundle over X, meaning that it is isomorphic to $X \times \mathbb{C} \to X$. A choice of a G-equivariant non-vanishing section in this bundle is the same as a choice of a G-invariant map $X \to \mathbb{C}^\times$. So there is a bijection between homotopy classes of sections $\mathfrak{s}$ and

$$[X/G, \mathbb{C}^*] \cong [X/G, S^1] \cong H^1(X/G;\mathbb{Z})$$

since $S^1 = K(\mathbb{Z}, 1)$.

Recall from 3.2 that the relative indices of Floer trajectories can be computed as a difference of Conley-Zehnder indices of two paths of symplectic matrices obtained by choosing trivializations of the limit orbits. The problem is that we must choose the trivializations of the two limit orbits in a compatible way: they must extend to a trivialization of the whole Floer trajectory. What happens when $\Lambda^n_{\mathbb{C}}TX$ is a trivial bundle and we fix $\mathfrak{s}$ is that we now have a way to choose for each Hamiltonian orbit a canonical trivialization, and with this choice we get trivializations that always extend to Floer trajectories.

Let now $\gamma_g \in \widetilde{\mathcal{P}}_G(H)$ or, equivalently, $\gamma \in \mathcal{P}_g(H)$. Similarly to what we did in 3.2 consider the S^1-bundle $\gamma^*TX/\sim \to S^1$ where $\sim$ identifies $T_{\gamma(0)}X$ and $T_{\gamma(1)}X$ via $(dg)_{\gamma(0)}$. The point is that choosing a trivialization for this bundle is the same as choosing a trivialization for its highest exterior power.

Lemma 4.2.4. *Let $E \to B$ be a symplectic or unitary bundle of rank n over B and suppose that B is homotopically equivalent to S^1. Then E is trivial as a symplectic/unitary bundle. Moreover there are natural bijections between homotopy classes of trivializations of E, homotopy classes of trivializations of $\Lambda^n_{\mathbb{C}}E$ and $\mathbb{Z}$.*

Proof. First, since every matrix in $Sp(2n;\mathbb{R})$ admits a polar decomposition as a product of a matrix in $U(n) \hookrightarrow Sp(2n;\mathbb{R})$ and a symplectic positive definite matrix (the space of such matrices forms a contractible set) the inclusion $U(n) \hookrightarrow Sp(2n;\mathbb{R})$ is a homotopy equivalence (see also [dS18, Section 2.7]), so there is no difference between symplectic and unitary bundles.

By the classification of bundles B.0.2 the set of isomorphism classes of $U(n)$ bundles is in bijection with

$$[B, BU(n)] \cong [S^1, BU(n)] \cong 0$$

since $U(n)$ is connected. Hence every unitary bundle over B is trivial.

Now if $E = B \times \mathbb{C}^n \to B$ is a trivial bundle, then a trivialization (as an unitary bundle) of E is the same as a continuous map $B \to U(n)$, so the set of homotopy classes of trivializations of E is

$$[B, U(n)] \cong [S^1, U(n)] \cong \pi_1(U(n))$$

where the last isomorphism follows from $U(n)$ being a connected Lie group. If a map $B \to U(n)$ gives a trivialization of $E = B \times \mathbb{C}^n$, it is clear that $B \to U(n) \xrightarrow{\det} U(1)$ gives a trivialization on $\Lambda^n_{\mathbb{C}}E \cong B \times \mathbb{C}$. But it's well known that the determinant $\det : U(n) \to U(1)$ induces an isomorphism on π_1, so

$$\pi_1(U(n)) \cong \pi_1(U(1)) \cong \mathbb{Z}$$

and we get the claimed natural bijections. $\square$

Now our fixed G-equivariant section $\mathfrak{s} : X \to \Lambda^n_{\mathbb{C}}TX$ determines a trivialization of $\Lambda^n_{\mathbb{C}}(\gamma^*TX/\sim) \to S^1$ as follows: we first get a trivialization

$$[0,1] \times \mathbb{C} \ni (t, z) \mapsto z\mathfrak{s}(\gamma(t)) \in \gamma^*\Lambda^n_{\mathbb{C}}TX = \Lambda^n_{\mathbb{C}}(\gamma^*TX).$$

Since $\mathfrak{s}$ is equivariant $\mathfrak{s}(\gamma(1))z = \mathfrak{s}(g\gamma(0))z = g_*\mathfrak{s}(\gamma(0))z$, so we get an induced trivialization

$$S^1 \times \mathbb{C}^n \to \Lambda^n_{\mathbb{C}}(\gamma^*TX/\sim).$$

Now by lemma 4.2.4 there is a unique up to homotopy trivialization Ψ of $\gamma^*TX/\sim$ that induces the above one. We write the trivialization as $\Psi(t) : \mathbb{R}^{2n} \to T_{\gamma(t)}X$ with the condition that

$$\Psi(1) = (dg)_{\gamma(0)} \circ \Psi(0).$$

We say that such a trivialization is compatible with $\mathfrak{s}$.

Remark 4.2.5. In [McL16, Lemma 4.3] it's shown that if $H^1(X/G;\mathbb{Q}) = 0$ then the trivialization Ψ does not depend on the section $\mathfrak{s}$ up to homotopy. This is quite expected as this means that the set $H^1(X/G;\mathbb{Z})$ parametrizing the choices of $\mathfrak{s}$ (see remark 4.2.3) is torsion.

Moreover in [McL16] it's explained how to relax the condition $c_1^G(X) = 0$ to $c_1^G(X)$ being torsion in $H^2_G(X;\mathbb{Z})$; in this case the bundle $(\Lambda^n_{\mathbb{C}}TX)^{\otimes N} \to X$ is trivial as a G-equivariant bundle for some $N \in \mathbb{Z}^+$.

Definition 4.2.6. *Let $\gamma \in \mathcal{P}_g(H) \subseteq \widetilde{\mathcal{P}}_G(H)$ be a non-degenerate (see 3.1.2) Hamiltonian orbit and consider a trivialization Ψ as discussed before. Let $\Phi : [0,1] \to Sp(2n;\mathbb{R})$ be defined by*

$$\Phi(t) = \Phi_\gamma(t) = \Psi(t)^{-1}(d\varphi_t)_{\gamma(0)}\Psi(0) \text{ for } t \in [0,1].$$

Then we define a grading on the set $\widetilde{\mathcal{P}}_G(H)$ by

$$|\gamma| = \mu_{CZ}(\Phi_\gamma).$$

Since Ψ is uniquely determined up to homotopy, by the homotopy invariance of the Conley-Zehnder index, the index $|\gamma|$ is well defined.

Now we finally get our desired result that the relative index of a Floer trajectory is the difference of the absolute indices.

Proposition 4.2.7. *Let $\gamma^-, \gamma^+ \in \mathcal{P}_g(H)$ be non-degenerate orbits and let $u \in \widehat{\mathcal{M}}_g(\gamma^-, \gamma^+)$. Then*

$$\mu(u) = \operatorname{ind} D_u = |\gamma^+| - |\gamma^-|.$$

Proof. Take a trivialization Ψ as in proposition 3.2.1 that is compatible with $\mathfrak{s}$ in the same sense as before: the trivialization it induces on the highest exterior power of $u^*TX/\sim$ is given by the trivialization of $\Lambda^n_{\mathbb{C}}TX$ as a G-bundle determined by $\mathfrak{s}$. Then clearly Ψ^-, Ψ^+ are also compatible with $\mathfrak{s}$, so theorem 3.2.4 gives

$$\mu(u) = \mu_{CZ}(\Phi^+) - \mu_{CZ}(\Phi^-) = |\gamma^+| - |\gamma^-|. \quad \square$$

4.3 Coherent orientations and orientable orbits

We will now explain how to assign coherent orientations by orienting certain determinant line bundles δ_γ for each Hamiltonian orbit $\gamma \in \widetilde{\mathcal{P}}_G(H)$ in the Calabi-Yau case. By doing this, we will define (non-)orientable Hamiltonian orbits in a way that mimics the definition of (non-)orientable critical points given in definition 2.4.1. As we explained earlier, to construct the Floer complex of $\mathcal{X}$ we will need to exclude non-orientable orbits from the generators. We think that the Calabi-Yau assumption can be avoided and we sketch how in remark 4.3.7, but since this is the case we're mainly interested in and it makes the exposition cleaner we'll stick to it. We will follow the approach in [Abo15].

Suppose that $\mathcal{X}$ is Calabi-Yau and fix a G-equivariant section $\mathfrak{s}$ as in 4.2. We saw that in this case there is a trivialization $\Psi(t) : \mathbb{R}^{2n} \to T_{\gamma(t)}X$ compatible with $\mathfrak{s}$ which is canonical up to homotopy. Given such trivialization we have a path of symplectic matrices $\Phi = \Phi_\gamma : [0,1] \to Sp(2n;\mathbb{R})$ defined in 4.2.6. Similarly to what we did in 3.2.3, we let $S : S^1 \to M_{2n\times 2n}(\mathbb{R})$ be the path of symmetric matrices defined by

$$\dot{\Phi}(t) = J_0 S(t)\Phi(t).$$

The idea is that we will associate to the path Φ a Fredholm operator D_Φ defined over $\mathbb{C}$ (instead of the cylinder $[0,1]\times\mathbb{R}$) that near the cylindrical end of $\mathbb{C}$ looks like $\partial_s + J_0\partial_t + S$. Such operators can be glued with the operators D_u defined on the cylinder, and using that we can orient $\det(D_u)$ (and in particular attribute the numbers $\nu(u)$) by orienting the determinant line bundles $\delta_\gamma = \det(D_\Phi)$. We now make this precise.

First, we give $\mathbb{C} \setminus \{0\}$ cylindrical coordinates $(s,t) \in \mathbb{R} \times [0,1]$ by $(s,t) \mapsto e^{-2\pi(s+it)}$. We also let x, y be the standard $\mathbb{C}$ coordinates, so $e^{-2\pi(s+it)} = x + iy$. We consider an area form μ that is cylindrical in the cylindrical end, i.e., $\mu = \mathrm{d}s\,\mathrm{d}t$ for $s \ll 0$, and is the usual area form near $0 \in \mathbb{C}$, i.e., $\mu = \mathrm{d}x\,\mathrm{d}y$ for $s \gg 0$. We also assume that the area form μ is radial, in the sense that $\mu = \gamma(s)\mathrm{d}x\,\mathrm{d}y$ for some $\gamma : \mathbb{R} \to \mathbb{R}^+$.

Before we proceed, let's compute $\partial_s + J_0\partial_t$ in the Cartesian x, y coordinates.

Proposition 4.3.1. *Consider the cylindrical coordinates (s,t) such that $e^{-2\pi(s+it)} = x + iy$. Then we have*

$$\partial_s + J_0\partial_t = 2\pi(-x + J_0 y)(\partial_x + J_0\partial_y).$$

Proof. This is a straightforward application of the chain rule. Since $x = e^{-2\pi s}\cos(2\pi t)$ and $y = -e^{-2\pi s}\sin(2\pi t)$ we have

$$\begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{bmatrix} = 2\pi \begin{bmatrix} -x & -y \\ y & -x \end{bmatrix}.$$

Hence

$$\partial_s + J_0\partial_t = 2\pi\left(-x\partial_x - y\partial_y + J_0 y\partial_x - J_0 x\partial_y\right) = 2\pi(-x + J_0 y)(\partial_x + J_0\partial_y). \qquad \square$$

We want our operator D_Φ to interpolate between $\partial_s + J_0\partial_t + S$ in the cylindrical end and a (perturbed) Cauchy-Riemann operator in the usual x, y coordinates near 0. We let $B : \mathbb{C} \to M_{2n\times 2n}(\mathbb{R})$ be a matrix valued continuous function defined on $\mathbb{C}$ that is constant and equal to S on the cylindrical end of $\mathbb{C}$, that is, we ask that

$$B\left(e^{-2\pi(s+it)}\right) = S(t) \text{ for s} \ll 0.$$

Moreover we pick $\alpha : \mathbb{C} \to \mathbb{R} \oplus J_0\mathbb{R} \cong \mathbb{C}$ such that

$$\alpha(x,y) = \alpha(x+iy) = \begin{cases} 1 & \text{if } s \gg 0 \\ 2\pi(-x + J_0 y) & \text{if } s \ll 0 \end{cases}$$

and α never vanishes. Finally we take the operator $D_\Phi : W^{1,p}_\mu(\mathbb{C}, \mathbb{R}^{2n}) \to L^p_\mu(\mathbb{C}, \mathbb{R}^{2n})$ defined by

$$D_\Phi Z = \alpha\left(\partial_x Z + J_0\partial_y Z\right) + BZ. \tag{4.2}$$

This operator is an admissible $\bar\partial$-operator defined on $\mathbb{C}$ with a negative cylindrical end (negative because of the minus sign in the cylindrical coordinates $(s,t) \mapsto e^{-2\pi(s+it)}$) in the sense of [Sch96, Definition 3.1.6]. In particular we have:

Theorem 4.3.2. *Assume that the symplectic path Φ is admissible (see A.0.1). Then the operator $D_\Phi : W^{1,p}_\mu(\mathbb{C}, \mathbb{R}^{2n}) \to L^p_\mu(\mathbb{C}, \mathbb{R}^{2n})$ defined in (4.2) is a Fredholm operator of Fredholm index $n - \mu_{CZ}(\Phi)$ for every $p \geq 2$. Moreover, its kernel does not depend on $p \geq 2$.*

Proof. This is contained in [Sch96, Theorems 3.1.9 and 3.3.8]. $\square$

With this in mind, we can now define the determinant line bundle associated to γ.

Definition 4.3.3. *Given a non-degenerate orbit $\gamma \in \widetilde{\mathcal{P}}_G(H)$ let $\Phi = \Phi_\gamma$ and D_Φ be an operator as before. We let δ_γ be the determinant line bundle of D_Φ, that is,*

$$\delta_\gamma = \det(D_\Phi) = \left(\Lambda^{\text{top}}\ker(D_\Phi)\right) \otimes \left(\Lambda^{\text{top}}\text{coker}(D_\Phi)\right)^\vee.$$

It should be noted that the definition of D_Φ depends on some extra data, namely the extension B of S and the interpolating function α. This extra data forms a contractible set (the possible operators form a convex set), so given any two such operators D_Φ and D'_Φ there are isomorphisms between $\det(D_\Phi)$ and $\det(D'_\Phi)$ which are canonical up to multiplication by a positive constant. Moreover, it's shown in [Abo15, Proposition 1.4.10] that if Φ_1 and Φ_2 are given by two different trivializations both compatible with $\mathfrak{s}$ (hence homotopic) then there are isomorphisms $\det(D_{\Phi_1}) \cong \det(D_{\Phi_2})$ which are canonical up multiplication by a positive constant. Note that this last part doesn't follow from a simple topological argument as before, since the space of possible trivializations is

$$\mathrm{Maps}(\mathbb{C}^*, U(n))^\circ \simeq \mathrm{Maps}_*(S^1, U(n))^\circ \times U(n)$$

(see the proof of lemma 4.2.4), which has fundamental group $\mathbb{Z}$. Here $\circ$ means the connected component including the constant maps and Maps_* means based maps.

There is a more conceptual way to understand δ_γ (which we won't use). The determinant line bundle gives an actual line bundle over the set of possible operators. The canonical isomorphisms we discussed between the fibers give a trivialization of the bundle. For instance, in this sense an orientation of δ_γ means an orientation of this bundle.

Remark 4.3.4. In [Abo15] the author considers an operator globally defined as

$$D_\Phi = \partial_s + J_0\partial_t + B = 2\pi(-x + J_0 y)(\partial_x + J_0\partial_y) + B.$$

This operator in not a $\bar{\partial}$-operator because of the vanishing of $-x + J_0 y$ at the origin. It seems that, because of the failure of the uniform ellipticity condition (again because of the vanishing of $-x + J_0 y$) such operators might not be Fredholm, and this is why we introduce the interpolating function α. I thank Miguel Santos for very helpful discussions in this regard.

The next theorem is a result from the gluing theory of operators explained in [FH93], [Abo15] or [Sch96]. This is what enables us to define an orientation on $\det(D_u)$ given orientations on δ_γ for each γ.

Theorem 4.3.5. *Let $\gamma^-, \gamma^+ \in \mathcal{P}_g(H) \subseteq \widetilde{\mathcal{P}}_G(H)$ for some $g \in G$ be non-degenerate Hamiltonian orbits and $u \in \widehat{\mathcal{M}}_g(\gamma^-, \gamma^+)$ be a connecting orbit. Assume that $\mathcal{X}$ is Calabi-Yau. Then we have an isomorphism*

$$\delta_{\gamma^-} \cong \det(D_u) \otimes \delta_{\gamma^+} \tag{4.3}$$

which is canonical up to multiplication by a positive constant.

Proof. This is essentially [Abo15, Theorem 1.5.1]. If we take Ψ to be a trivialization of $u^*TX/\sim$ compatible with $\mathfrak{s}$ then it restricts along the ends to trivializations of $(\gamma^\pm)^*TX/\sim$ compatible with $\mathfrak{s}$, so the same proof applies: we can apply [Abo15, Lemma 1.4.5] (or [FH93, Proposition 9]) and [Abo15, Proposition 1.4.6] to get the result. $\square$

Now in order to get coherent orientations on the moduli spaces we fix for each $\gamma \in \widetilde{\mathcal{P}}_G(H)$ an orientation of δ_γ. Equation (4.3) then induces an orientation in the determinant line bundle $\det(D_u)$. Suppose now that (H, J) is regular, and thus D_u is surjective. Then

$$\det(D_u) = \Lambda^{\mathrm{top}} \ker(D_u) = \Lambda^{\mathrm{top}} \left(T_u\widehat{\mathcal{M}}_g(\gamma^-, \gamma^+)\right).$$

So we get an orientation on the manifold $\widehat{\mathcal{M}}_g(\gamma^-, \gamma^+)$. Recall that $\mathcal{M}_g(\gamma^-, \gamma^+)$ is defined as the quotient $\widehat{\mathcal{M}}_g(\gamma^-, \gamma^+)/\mathbb{R}$ where $\mathbb{R}$ acts on $\widehat{\mathcal{M}}_g(\gamma^-, \gamma^+)$ by $(\sigma \cdot u)(t, s) = u(t, s + \sigma)$ for $\sigma \in \mathbb{R}$. In particular we get signs for the rigid Floer trajectories.

Definition 4.3.6. *Assume that $\mathcal{X}$ is Calabi-Yau and that (H, J) is a regular pair. Let $\gamma^-, \gamma^+ \in \mathcal{P}_g(H)$ be non-degenerate orbits and let $u \in \widehat{\mathcal{M}}_g(\gamma^-, \gamma^+)$ be a rigid Floer trajectory, that is, $\mu(u) = |\gamma^+| - |\gamma^-| = 1$.*

Then $\ker D_u$ is one dimensional and generated by the translation in the s-variable which we denote by $\partial_s \in \ker D_u = \det(D_u)$. If the orientation of $\ker D_u$ induced by equation (4.3) makes ∂_s positive we assign the number $\nu(u) = 1$ and otherwise we assign $\nu(u) = -1$.

It can be shown that the orientations assigned in this way are coherent in the sense discussed in section 3.3 by considering the natural isomorphism we get from gluing theory

$$\det(D_{u\#v}) \cong \det(D_u) \otimes \det(D_v).$$

Remark 4.3.7. It should be possible to adapt this point of view for coherent orientations even if we drop the Calabi-Yau condition. To do this we can consider the covering of Hamiltonian orbits whose elements are pairs (γ, Ψ) where Ψ is a trivialization up to homotopy (or some other appropriate covering) and define determinant line bundles $\delta_{(\gamma,\Psi)}$. Then we get an isomorphism

$$\delta_{(\gamma^-,\Psi^-)} \cong \det(D_u) \otimes \delta_{(\gamma^+,\Psi^+)}$$

as in 4.3.5 as long as the trivializations $\Psi^\pm$ extend to a trivialization over the cylinder u.

4.3.1 Maps induced by G on determinant line bundles

Recall that we have a G-action on the set $\widetilde{\mathcal{P}}_G(H)$ of Hamiltonian orbits. Indeed given $\gamma \in \mathcal{P}_g(H)$ and $h \in G$ we have $h\gamma \in \mathcal{P}_{hgh^{-1}}(H)$. Similarly if $u \in \widehat{\mathcal{M}}_g(\gamma^-, \gamma^+)$ then $hu \in \widehat{\mathcal{M}}_{hgh^{-1}}(h\gamma^-, h\gamma^+)$. We'll see in this section that we have induced maps

$$h_* : \delta_\gamma \to \delta_{h\gamma} \text{ and } h_* : \det(D_u) \to \det(D_{hu}).$$

Given a trivialization $\Psi(t) : \mathbb{R}^{2n} \to T_{\gamma(t)}X$ of $\gamma^*TX/\sim$ compatible with $\mathfrak{s}$, let the trivialization $h_*\Psi$ be given by the composition

$$h_*\Psi(t) = \mathbb{R}^{2n} \xrightarrow{\Psi(t)} T_{\gamma(t)}X \xrightarrow{(dh)_{\gamma(t)}} T_{h\gamma(t)}X.$$

This is a trivialization of the bundle $(h\gamma)^*TX/\sim$; indeed note that

$$h_*\Psi(0) = (dh)_{\gamma(0)}(dg)_{\gamma(0)}\Psi(1) = d(hgh^{-1})_{h\gamma(0)}h_*\Psi(1).$$

Moreover since since $\mathfrak{s}$ is G-equivariant and $h \in G$ it's clear that if Ψ is compatible with $\mathfrak{s}$ then $h_*\Psi$ is also compatible with $\mathfrak{s}$. It follows that the path of symplectic matrices $\Phi_{h\gamma}$ associated to $h\gamma$ using the trivialization $h_*\Psi$ is

$$\Phi_{h\gamma}(t) = \Psi(t)^{-1}(dh)^{-1}_{\gamma(t)}(d\varphi_t)_{h\gamma(0)}(dh)_{\gamma(0)}\Psi(0).$$

Since H is h-invariant it's easy to see that $h_*X_t^H = X_t^H$, and thus the differentials $(\varphi_t)_*$ and h_* commute. Hence

$$\Phi_{h\gamma}(t) = \Psi(t)^{-1}(d\varphi_t)_{\gamma(0)}\Psi(0) = \Phi_\gamma(t).$$

Thus we get a canonical isomorphism (up to multiplication by a positive constant)

$$\delta_\gamma \cong \det(D_{\Phi_\gamma}) = \det(D_{\Phi_{h\gamma}}) \cong \delta_{h\gamma}.$$

We denote by $h_* : \delta_\gamma \to \delta_{h\gamma}$ this isomorphism. Similarly, we get an isomorphism $h_* : \det(D_u) \to \det(D_{hu})$. This last isomorphism can be also understood as follows: assume that (H, J) is regular. Then composition with h is a diffeomorphism

$$\widehat{\mathcal{M}}_g(\gamma^-, \gamma^+) \to \widehat{\mathcal{M}}_{hgh^{-1}}(h\gamma^-, h\gamma^+)$$

given by $u \mapsto hu$. The differential at u then gives an isomorphism

$$h_* : \ker(D_u) = T_u\widehat{\mathcal{M}}_g(\gamma^-, \gamma^+) \to T_{hu}\widehat{\mathcal{M}}_{hgh^{-1}}(h\gamma^-, h\gamma^+) = \ker(D_{hu}).$$

Finally, since D_u is surjective, we have $\det(D_u) = \Lambda^{\mathrm{top}} \ker D_u$, so we get the induced map $h_* : \det(D_u) \to \det(D_{hu})$. It should be noted that $h_* : \det(D_u) \to \det(D_{hu})$ sends the translation in the s-variable $\partial_s \in \ker(D_u)$ to $\partial_s \in \ker(D_{hu})$.

With this in mind we can now say what we mean with an orientable Hamiltonian orbit. This is the Floer analogue of orientable critical points in [CH14].

Definition 4.3.8. *Let $\gamma \in \mathcal{P}_g(H) \subseteq \widetilde{\mathcal{P}}_G(H)$ and assume that $\mathcal{X}$ is Calabi-Yau. We say that γ is an orientable Hamiltonian orbit if for every $h \in C(g)$ such that $h\gamma = \gamma$ the isomorphism $h_* : \delta_\gamma \to \delta_\gamma$ is orientation preserving. Otherwise we say that γ is non-orientable. We denote by $\widetilde{\mathcal{P}}_G(H)^+$ the set of orientable Hamiltonian orbits.*

Note that $h \in C(g)$ and $h\gamma = \gamma$ means that h is in the isotropy group of γ_g with respect to the action of G on $\widetilde{\mathcal{P}}_G(H)$.

4.4 Orbifold Floer complex

We are now ready to give a correct definition of orbifold Floer homology in characteristic not 2. Once again, in this section we will always assume that $c_1^G(X) = 0$. We will begin by giving the construction from a more abstract point of view using orientation lines instead of fixing coherent orientations, as in [Abo15]; in this approach the problems we described earlier are solved naturally. After that we fix orientations to give a more concrete description of the complex; in doing so we have to exclude non-orientable Hamiltonian orbits.

4.4.1 Abstract orbifold Floer complex

For each $\gamma \in \widetilde{\mathcal{P}}_G(H)$ we have the determinant line bundle δ_γ. Letting Λ be a Novikov ring, we have an orientation line Λ_γ which is defined as the Λ-module

$$\Lambda_\gamma = \Lambda\langle o_1, o_2 \rangle / (o_1 + o_2 = 0)$$

where o_1, o_2 are the two orientations of the determinant line bundle δ_γ. Note that picking an orientation o_1 or o_2 determines an isomorphism $\Lambda_\gamma \cong \Lambda \cdot \gamma$ (here $\Lambda \cdot \gamma$ is simply the Λ-module generated by Λ, which is isomorphic to Λ as a module over itself). Moreover, any map on determinant line bundles which is canonical up to multiplication by a positive factor induces a canonical map on orientation lines. For instance $h_* : \delta_\gamma \to \delta_{h\gamma}$ induces a map also denoted by $h_* : \Lambda_\gamma \to \Lambda_{h\gamma}$; if we fix orientations and identify $\Lambda_\gamma \cong \Lambda \cong \Lambda_{h\gamma}$ such a map is either the identity or multiplication by -1, depending on whether $h_* : \delta_\gamma \to \delta_{h\gamma}$ respects the given orientations or not.

We let

$$\widetilde{CF}_k(\mathcal{X}, H; \Lambda) = \bigoplus_{\substack{\gamma \in \widetilde{\mathcal{P}}_G(H) \\ |\gamma| = k}} \Lambda_\gamma \text{ and } \widetilde{CF}(\mathcal{X}, H; \Lambda) = \bigoplus_{k \in \mathbb{Z}} \widetilde{CF}_k(\mathcal{X}, H; \Lambda) \tag{4.4}$$

where $|\gamma|$ is the grading defined in 4.2.6. If we pick orientations for each δ_γ we get an identification between this description and the one in (4.1).

Assume that (H, J) is regular. To describe the differential in this terms, recall the isomorphism in lemma 4.3.5. If $u \in \widehat{\mathcal{M}}_g(\gamma^-, \gamma^+)$ is rigid, that is, $\mu(u) = 1$ then $\det(D_u) = \ker(D_u) = \mathbb{R} \cdot \partial_s$ where ∂_s corresponds to translation in the s-variable. By picking the orientation on $\det(D_u)$ corresponding to the generator ∂_s the isomorphism in 4.3.5 induces a map on the orientation lines $\partial_u : \Lambda_{\gamma^+} \to \Lambda_{\gamma^-}$. More precisely, ∂_u is the map induced on the orientation lines by

$$\delta_{\gamma^+} \xrightarrow{\partial_s \otimes \cdot} \det D_u \otimes \delta_{\gamma^+} \xrightarrow{4.3} \delta_{\gamma^-}.$$

Finally we define $\partial : \widetilde{CF}_k(\mathcal{X}, H; \Lambda) \to \widetilde{CF}_{k-1}(\mathcal{X}, H; \Lambda)$ as follows: for $\gamma^+ \in \mathcal{P}_g(H) \subseteq \widetilde{\mathcal{P}}_G(H)$ with $|\gamma^+| = k$ the differential restricted to Λ_{γ^+} is given by

$$\partial_{|\Lambda_{\gamma^+}} = \sum_{\substack{\gamma^- \in \mathcal{P}_g(H) \\ |\gamma^-| = k-1}} \sum_{u \in \mathcal{M}_g(\gamma^-, \gamma^+)} T^{\omega(u)} \partial_u. \tag{4.5}$$

Indeed this differential is nothing else than the differential we already defined in 3.3.5. First, by proposition 4.2.7, the sum runs through the rigid trajectories $u \in \mathcal{M}_g(\gamma^-, \gamma^+)$, since $\mu(u) = |\gamma^+| - |\gamma^-| = 1$. Moreover, when we fix orientations on every δ_γ the maps δ_u are just multiplication by $\nu(u) \in \{+1, -1\}$, that is, the following square commutes:

$$\begin{tikzcd} \Lambda_{\gamma^+} \arrow[r, "\partial_u"] \arrow[d] & \Lambda_{\gamma^-} \arrow[d] \\ \Lambda \cdot \gamma^+ \arrow[r, "\times \nu(u)"] & \Lambda \cdot \gamma^- \end{tikzcd}$$

The vertical arrows are just the identifications given by fixing orientations. This square commuting is just a restatement of the definition of $\nu(u)$ in 4.3.6. In particular if (H, J) has no-bubbling and we assume 3.3.6 then $\partial^2 = 0$.

The group G acts on the complex $\widetilde{CF}_k(\mathcal{X}, H; \Lambda)$ via the maps $h_* : \Lambda_\gamma \to \Lambda_{h\gamma}$ for each $h \in G$. It's very important that if we fix orientations this action does not correspond to the G action in the beginning of this chapter: it differs by some signs that are introduced when $h_* : \delta_\gamma \to \delta_{h\gamma}$ doesn't respect the given orientations. Indeed, this action now makes the differential G-equivariant.

Lemma 4.4.1. *Suppose that (H, J) is regular and has no-bubbling. Let $\Lambda = \Lambda^{\text{univ}}(R)$ be a Novikov. Then the differential $\partial : \widetilde{CF}(\mathcal{X}, H; \Lambda) \to \widetilde{CF}(\mathcal{X}, H; \Lambda)$ is equivariant with respect to the G-action on $\widetilde{CF}(\mathcal{X}, H; \Lambda)$. In particular ∂ restricts to a differential on the G-invariant part*

$$\partial : CF(\mathcal{X}, H; \Lambda) \to CF(\mathcal{X}, H; \Lambda)$$

where $CF(\mathcal{X}, H; \Lambda) = \widetilde{CF}(\mathcal{X}, H; \Lambda)^G$ is the G-invariant part.

Proof. Let $u \in \widehat{\mathcal{M}}_g(\gamma^-, \gamma^+)$. First note that the following diagram commutes:

$$\begin{tikzcd}
\delta_{\gamma^+} \arrow[r, "\partial_s \otimes"] \arrow[d, "h_*"] & \det(D_u) \otimes \delta_{\gamma^+} \arrow[r] \arrow[d, "h_*"] & \delta_{\gamma^-} \arrow[d, "h_*"] \\
\delta_{h\gamma^+} \arrow[r, "\partial_s \otimes"] & \det(D_{hu}) \otimes \delta_{h\gamma^+} \arrow[r] & \delta_{h\gamma^-}
\end{tikzcd}$$

The first square commutes since the map $h_* : \det(D_u) \to \det(D_{hu})$ sends $\partial_s \in \det(D_u)$ to $\partial_s \in \det(D_{hu})$. The second square commuting is straightforward from the definitions of the maps involved.

Thus it follows that $h_* \partial_u = \partial_{hu} h_*$. After this observation, the proof follows a formal argument as in the proof of 4.1.3. $\square$

With this in mind, we can now give a definition of Floer homology for the global quotient orbifold $\mathcal{X}$.

Definition 4.4.2. *Let (X, ω) be a symplectic manifold and G a finite group acting on (X, ω) by symplectomorphisms such that $\mathcal{X} = [X/G]$ is Calabi-Yau (equivalently, $c_1^G(X) = 0$). Suppose that $(H, J) \in C_G(\mathbb{R} \times X) \times \mathcal{J}_G(X, \omega)$ is a regular pair and has no-bubbling (in the sense of 4.1.2). Let $\Lambda = \Lambda^{\mathrm{univ}}(R)$ be a Novikov ring.*

Let $CF_k(\mathcal{X}, H; \Lambda)$ be the G-invariant part of $\widetilde{CF}_k(\mathcal{X}, H; \Lambda)$ defined in (4.4). By lemma 4.4.1 we have a restricted differential $\partial = \partial_{H,J} : CF_k(\mathcal{X}, H; \Lambda) \to CF_{k-1}(\mathcal{X}, H; \Lambda)$. Then $(CF_*(\mathcal{X}, H; \Lambda), \partial_{H,J})$ is called the Floer complex of $\mathcal{X}$.

Assuming 3.3.6 we define the k-th Floer homology group of the orbifold $\mathcal{X}$ to be the Λ-module

$$HF_k(\mathcal{X}, H, J; \Lambda) = \frac{\ker(\partial : CF_k \to CF_{k-1})}{\operatorname{im}(\partial : CF_{k+1} \to CF_k)}.$$

4.4.2 Orbifold Floer complex with fixed orientations

We will now explain how does the Floer complex that we defined in the previous section look like when we fix orientations.

The first observation is that if $\gamma \in \mathcal{P}_g(H) \subseteq \widetilde{\mathcal{P}}_G(H)$ is non-orientable (see 4.3.8) then there is $h \in C(g)$ such that $h\gamma = \gamma$ and $h_* : \delta_\gamma \to \delta_\gamma$ is orientation-reversing. Hence $h_* : \Lambda_\gamma \to \Lambda_\gamma$ is multiplication by -1. But then an element of $\widetilde{CF}(\mathcal{X}, H; \Lambda) = \bigoplus_{\gamma \in \widetilde{\mathcal{P}}_G(H)} \Lambda_\gamma$ which is G-invariant must have 0 in each component Λ_γ relative to a non-orientable orbit γ, so $\widetilde{CF}(\mathcal{X}, H; \Lambda)^G$ is contained in the sub-complex generated by orientable orbits

$$\widetilde{CF}(\mathcal{X}, H; \Lambda)^+ = \bigoplus_{\gamma \in \widetilde{\mathcal{P}}_G(H)^+} \Lambda_\gamma.$$

We can choose orientations of δ_γ for $\gamma \in \widetilde{\mathcal{P}}_G(H)^+$ so that $h_* : \delta_\gamma \to \delta_{h\gamma}$ preserves these orientations for every $\gamma \in \widetilde{\mathcal{P}}_G(H)^+$. We can do this as follows: pick representatives $\gamma_1, \dots, \gamma_\ell \in \widetilde{\mathcal{P}}_G(H)^+$ for each equivalence class in $\mathcal{P}_G(H)^+ = \widetilde{\mathcal{P}}_G(H)^+/G$ and fix an orientation of δ_{γ_j} for $j = 1, \dots, \ell$. For each $\gamma \in \widetilde{\mathcal{P}}_G(H)^+$ there are j and $h \in G$ such that $\gamma = h\gamma_j$. Then we define the orientation on δ_γ so that the isomorphism $h_* : \delta_{\gamma_j} \to \delta_\gamma$ is orientation preserving; this is well defined because, if $h, h' \in G$ are such that $\gamma = h\gamma_j = h'\gamma_j$, then we have a commutative diagram

$$\begin{tikzcd}
\delta_{\gamma_j} \arrow[r, "h'_*"] \arrow[d, "(h^{-1}h')_*"'] & \delta_\gamma \\
\delta_{\gamma_j} \arrow[ur, "h_*"'] &
\end{tikzcd}$$

and the vertical arrow is orientation preserving since γ_j is an orientable orbit. With such a choice of orientations we get identifications $\Lambda_\gamma \cong \Lambda \cdot \gamma$ for $\gamma \in \widetilde{\mathcal{P}}_G(H)^+$ such that for $h \in G$ the map $h_* : \Lambda \cdot \gamma = \Lambda_\gamma \to \Lambda_{h\gamma} = \Lambda \cdot (h\gamma)$ sends the generator γ to the generator $h\gamma$. Thus the Floer complex can be described as

$$\begin{aligned} CF(\mathcal{X}, H; \Lambda) = \widetilde{CF}(\mathcal{X}, H; \Lambda)^G &= \left(\widetilde{CF}(\mathcal{X}, H; \Lambda)^+\right)^G \\ &\cong \left(\bigoplus_{\gamma \in \widetilde{\mathcal{P}}_G(H)^+} \Lambda \cdot \gamma\right)^G \end{aligned}$$

and the G-action on $\bigoplus_{\gamma \in \widetilde{\mathcal{P}}_G(H)^+} \Lambda \cdot \gamma$ is just the Λ-linear extension of the action of G on $\widetilde{\mathcal{P}}_G(H)^+$. Equivalently, this is isomorphic to the Λ-module generated by $\mathcal{P}_G(H)^+$.

In this description, $CF(\mathcal{X}, H; \Lambda)$ is a sub-complex of the naive version $\widetilde{CF}(\mathcal{X}, H; \Lambda) \cong \bigoplus_{\gamma \in \widetilde{\mathcal{P}}_G(H)} \Lambda \cdot \gamma$, which we discussed in 4.1.1, and inherits its differential. The point is that to define the orbifold Floer complex we can't just take the G-invariant part as we tried in 4.1.1 but first we must exclude non-orientable orbits.

Remark 4.4.3. It seems that the need to exclude non-orientable critical points (Hamiltonian orbits) from the Morse (Floer) homology is related to the need to exclude bad Reeb orbits in contact homology. In [HHM17] a discrete approach to (local) contact homology is considered; this approach uses a $\mathbb{Z}/k$-equivariant Morse homology. In lemma 2.22 it's shown that good Reeb orbits correspond precisely to an orientation preserving condition.

Chapter 5

Equivariant transversality

In this chapter we will discuss the problem of transversality in our construction of Floer homology for global quotient orbifolds. Recall that in order to define the Floer differential ∂ we need that, for certain equivariant Hamiltonian function H and equivariant almost complex structure J, the moduli spaces $\widehat{\mathcal{M}}_g(\gamma^-, \gamma^+; H, J)$ are manifolds of the correct dimension. In theorem 3.2.5 we saw that this is a consequence of the regular value theorem for Banach manifolds if the operator $D_u : W^{1,p}_g(u^*TX) \to L^p_g(u^*TX)$ is surjective for every $u \in \widehat{\mathcal{M}}_g(H, J)$ and $g \in G$. Recall that we call a pair

$$(H, J) \in C^\infty_G(\mathbb{R} \times X) \times \mathcal{J}_G(X, \omega)$$

regular if this happens (definition 4.1.2).

The approach usually taken is to perturb H or J so that the pair becomes regular. In our work we chose to consider perturbations of J, as these are slightly easier to work out for technical reasons. Hence, given some fixed H we would like to show that the set of $J \in \mathcal{J}_G(X, \omega)$ such that (H, J) is regular is dense in $\mathcal{J}_G(X, \omega)$. Unfortunately the equivariant restriction poses a severe obstruction to transversality, as we shall explain in the next section. However, we will prove a weaker result in theorem 5.2.7. This is enough to get transversality in the usual sense for instance when the singularities of the orbifold $[X/G]$ are isolated.

5.1 Obstruction to equivariant transversality

We now briefly explain the simplest possible obstruction to equivariant transversality. Suppose that for some pair (H, J) we have a Floer trajectory $u \in \widehat{\mathcal{M}}_1(H, J)$ whose image is contained in X^g for some $g \in G$. Then the spaces $W^{1,p}(u^*TX), L^p(u^*TX)$ admit an obvious $\langle g \rangle$-action since $(dg)_{u(s,t)}$ maps $T_{u(s,t)}X$ to itself. Moreover if (H, J) are equivariant then clearly the operator D_u is also equivariant with respect to this action. Hence D_u restricts to

$$D^g_u : W^{1,p}(u^*TX)^g \to L^p(u^*TX)^g.$$

By $W^{1,p}(u^*TX)^g, L^p(u^*TX)^g$ we mean the subspaces of $W^{1,p}(u^*TX), L^p(u^*TX)$, respectively, fixed by the $\langle g \rangle$-action described above. The operator D^g_u is the operator that would appear if we were trying to define Floer homology in X^g associated to the Floer trajectory $u : [0, 1] \times \mathbb{R} \to X^g$. So geometrically it's expected that, if transversality holds, then $\operatorname{ind} D_u \geq \operatorname{ind} D^g_u$, as these are the dimensions of moduli spaces of Floer trajectories in X and X^g, respectively. Indeed if D_u is surjective

$$\operatorname{ind} D_u = \dim\ker D_u \geq \dim\ker D_u^g \geq \operatorname{ind} D_u^g$$

since $\ker D_u^g = \ker D_u \cap W^{1,p}(u^*TX)^g$.

Thus, if $\operatorname{ind} D_u^g > \operatorname{ind} D_u$ then D_u is not surjective. Moreover this situation is stable under (equivariant) perturbation of H or J: if (H', J') is close to (H, J) and still equivariant then there is a Floer trajectory $u' \in \widehat{\mathcal{M}}_1(H', J')$ near u, and by stability of the Fredholm index we still have $\operatorname{ind} D_{u'}^g > \operatorname{ind} D_{u'}$, and hence $D_{u'}$ is not surjective. So we conclude that there is a neighbourhood of (H, J) without regular pairs, which is a very unpleasant situation.

A concrete example in which this situation happens in Lagrangian Floer homology may be found in [SS10]. But an easy example can even be found in Morse homology. Consider the torus $\mathbb{T}^2$ with the classical embedding in $\mathbb{R}^3$ and the height function $H : \mathbb{T}^2 \to \mathbb{R}$; H has 4 critical points of indices $0, 1, 1, 2$ and there are two trajectories between the two points of indices 1. Take a $\mathbb{Z}/2$ action given by reflection in a plane containing the trajectories between the two points of indices 1. Then what we described before happens for these trajectories: the Fredholm index relative to these trajectories in $\mathbb{T}^2$ is 0 but the Fredholm index of these trajectories in $(\mathbb{T}^2)^{\mathbb{Z}/2}$ is 1. Geometrically it's also clear that an equivariant perturbation of H cannot destroy these trajectories between the points of index 1, which would not exist if we had transversality.

5.2 Proof of transversality when $\operatorname{im} u \not\subseteq X^g$

We saw before an obstruction to obtain equivariant transversality when there are Floer trajectories fixed by some $g \in G \setminus \{1\}$. We now will prove that when this does not happen then transversality can be achieved by perturbing the (time dependent) almost complex structure. Our proof is an adaptation of [KS02, Proposition 5.13] (which is in the context of Lagrangian Floer homology and $G = \mathbb{Z}/2$). As in the usual proof, ours will use the existence of "many" injective points, in an appropriate sense.

Definition 5.2.1. *Given a Floer trajectory $u \in \widehat{\mathcal{M}}_g(\gamma^-, \gamma^+; H, J)$ we say that a point $(t_0, s_0) \in [0,1] \times \mathbb{R}$ is G-injective if*

$$\partial_s u(t_0, s_0) \neq 0 \text{ and } u(t_0, s_0) \notin u(t_0, \overline{\mathbb{R}} \setminus \{s_0\}) \cup \bigcup_{g \in G \setminus \{1\}} g\, u(t_0, \overline{\mathbb{R}}).$$

We denote by $R(u)$ the set of G-injective points of u.

Recall that we're using the notation $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ and $u(t, \pm\infty) = \gamma^\pm(t)$. We now proceed to prove an abundance of G-injective points for Floer trajectories that are not fixed by some non-trivial element of G.

Lemma 5.2.2. *Let $u \in \widehat{\mathcal{M}}_g(\gamma^-, \gamma^+; H, J)$ be a non-constant Floer trajectory (i.e., with $\partial_s u \neq 0$) and assume that* $\operatorname{im}(u)$ *is not contained in X^g for any $g \in G \setminus \{1\}$. Then the set of G-injective points $R(u)$ is dense in* $[0,1] \times \mathbb{R}$.

Remark 5.2.3. Note that the condition that u does not lie in any fixed point set is necessary, otherwise $u(t_0, s_0) = gu(t_0, s_0)$ for every (t_0, s_0) and there are no injective points.

It's expected, given the obstruction discussed before, that our proof fails to address solutions with image in X^g.

We proceed now to state a few facts (which are fairly standard) that will be useful in the proof of lemma 5.2.2.

Claim 5.2.4. *Let $u : [0,1] \times \mathbb{R} \to X$ be a solution of Floer equation (3.2) such that $\partial_s u \neq 0$. Then the set*

$$\{(t,s) \in [0,1] \times \mathbb{R} : \partial_s u(t,s) \neq 0,\, u(t,s) \notin u(t, \overline{\mathbb{R}} \setminus \{s\}\}$$

is open and dense.

Proof. This is the usual abundance of injective points, see for instance [FHS95, Theorem 4.3]. □

Claim 5.2.5. *Let $u, v : [0,1] \times \mathbb{R} \to X$ be solutions of Floer equation (3.2), and assume that u is not obtained from v by a translation in the s variable. Then the set*

$$\{(t,s) \in [0,1] \times \mathbb{R} : u(t,s) \notin v(t, \mathbb{R})\}$$

is of Baire second category.

Proof. By 3.4.1 we can assume that $H = 0$ and u, v are holomorphic curves. But in this case this follows from claim $(J5)$ in the proof of [KS02, Lemma 5.12] as the set is the intersection

$$\bigcap_{n>0} \{(t,s) \in [0,1] \times \mathbb{R} : u(t,s) \notin v(t \times [-n,n])\}$$

of open and dense sets. □

Claim 5.2.6. *Let $u : [0,1] \times \mathbb{R} \to X$ be a solution of Floer equation (3.2) such that $\partial_s u \neq 0$ and $\gamma : [0,1] \to X$ a Hamiltonian orbit. Then the set*

$$\{(t,s) \in [0,1] \times \mathbb{R} : u(t,s) \neq \gamma(t)\}$$

is of Baire second category.

Proof. Apply claim 5.2.5 to u and $v(t,s) = \gamma(t)$, which is also a solution of Floer equation since γ is Hamiltonian. □

Proof of 5.2.2. Write

$$R(u) = A(u) \cup \bigcup_{g \in G} B_g(u) \cup C_g^+(u) \cup C_g^-(u)$$

where

$$\begin{aligned}
A(u) &= \{(t,s) \in [0,1] \times \mathbb{R} : \partial_s u(t,s) \neq 0\} \\
B_1(u) &= \{(t,s) \in [0,1] \times \mathbb{R} : u(t,s) \notin u(t, \mathbb{R} \setminus \{s\})\} \\
B_g(u) &= \{(t,s) \in [0,1] \times \mathbb{R} : u(t,s) \notin gu(t, \mathbb{R})\} \text{ for } g \in G \setminus \{1\} \\
C_g^{\pm}(u) &= \{(t,s) \in [0,1] \times \mathbb{R} : u(t,s) \notin gu(t, \pm\infty)\} \text{ for } g \in G
\end{aligned}$$

To prove denseness it's enough to prove that each of these sets is of Baire second category. Note that

$$A(u) \cup B_1(u) \cup C_1^+(u) \cup C_1^-(u)$$

is the set of injective points of u in the usual sense, so it's open and dense by claim 5.2.4. By claim 5.2.6 the sets $C_g^{\pm}(u)$ are of Baire second category. It remains to show that $B_g(u)$ is also dense for $g \neq 1$. Let $v(t,s) = g(u(t,s))$. We prove that v is not a translation of u in the s-variable. Assume otherwise that

$$u(t, s+\sigma) = v(t,s) = gu(t,s)$$

and let $m = |G|$. Since $g^m = 1$ it follows that

$$u(t, s+m\sigma) = g^m u(t,s) = u(t,s)$$

so u is periodic in the s-variable. But this contradicts the fact that the limit $\lim_{s\to+\infty} u(t,s)$ exists, unless $u(t,s) = u(t,+\infty)$ and u is constant. Thus it follows that $B_g(u)$ is of Baire second category by 5.2.5.

We're left with proving that $R(u)$ is open. Since $A(u) \cup B_1(u) \cup C_1^+(u) \cup C_1^-(u)$ is already open by claim 5.2.4, it's enough to prove that $B_g(u) \cup C_g^+(u) \cup C_g^-(u)$ is open for $g \neq 1$. Assume it's not. Then there is a sequence $(t_k, s_k) \to (t,s)$ such that $(t,s) \in B_g(u) \cap C_g^+(u) \cap C_g^-(u)$ and $(t_k, s_k) \notin B_g(u) \cap C_g^+(u) \cap C_g^-(u)$; pick a sequence $s_k' \in \overline{\mathbb{R}}$ such that $u(t_k, s_k) = gu(t_k, s_k')$. Since $\overline{\mathbb{R}}$ is compact we may assume by taking a subsequence that $s_k' \to s' \in \overline{\mathbb{R}}$. But then taking limits we have $u(t,s) = gu(t,s')$, contradiction. ☐

Theorem 5.2.7 (Weak equivariant transversality). *Let (X,ω) be a symplectic manifold, G a finite group acting on (X,ω) and $H : X \to \mathbb{R}$ a non-degenerate Hamiltonian. For $\ell \geq 1$ there is a C^ℓ dense subset $\mathcal{J}_G^{reg}(X,\omega)$ of $\mathcal{J}_G(X,\omega)$ such that if $J \in \mathcal{J}_G^{reg}(X,\omega)$ then, for every Floer trajectory $u \in \widehat{\mathcal{M}}_g(H,J)$ whose image is not contained in X^h for any $h \in G \setminus \{1\}$, D_u is surjective.*

Proof. As in the proof of 3.2.5 we consider the space $\mathcal{B}_g^{1,p}(\gamma^-,\gamma^+)$ of "$W^{1,p}$-maps" (with $p > 2$) $u : [0,1] \times \mathbb{R} \to X$ such that

$$u(1,s) = gu(0,s)$$

and

$$u(t,s) \xrightarrow{C^\infty} \gamma^{\pm}(t) \text{ when } s \to \pm\infty,$$

defined in (3.9). We also introduce the notation

$$\mathcal{B}_g^{1,p}(\gamma^-,\gamma^+)^* = \{u \in \mathcal{B}_g^{1,p}(\gamma^-,\gamma^+) : \operatorname{im}(u) \not\subseteq X^h \; \forall h \in G \setminus \{1\}\}.$$

The space $\mathcal{B}_g^{1,p}(\gamma^-,\gamma^+)$ is a Banach manifold locally modelled by $W_g^{1,p}(u^*TX)$ and $\mathcal{B}_g^{1,p}(\gamma^-,\gamma^+)^*$ is an open subset, hence also a Banach manifold. Denote by $\mathcal{J}_G^\ell$ the completion of $\mathcal{J}_G(X,\omega)$ with respect to the C^ℓ topology, which is also a Banach manifold. We let $\mathcal{E}$ be the bundle over $\mathcal{B}_g^{1,p}(\gamma^-,\gamma^+)$ whose fiber over $u \in \mathcal{B}_g^{1,p}(\gamma^-,\gamma^+)$ is $L_g^p(u^*TX)$. Then the parametrized Floer operator is the map

$$\mathcal{F} : \mathcal{B}_g^{1,p}(\gamma^-,\gamma^+) \times \mathcal{J}_G^\ell \to \mathcal{E}$$

defined by

$$\mathcal{F}(u,J) = \partial_s u + J_t(u)\left(\partial_t u - X_H^t(u)\right) \in \mathcal{E}_u.$$

We now consider the universal moduli space

$$\widehat{\mathcal{M}}_{\text{univ}} = \widehat{\mathcal{M}}_{\text{univ},\,g}(\gamma^-,\gamma^+;H) = \{(u,J) \in \mathcal{B}_g^{1,p}(\gamma^-,\gamma^+)^* \times \mathcal{J}_G^\ell : \mathcal{F}(u,J) = 0\}.$$

In order to apply the Sard-Smale theorem (see [Sma65, Theorem 1965] or [dS18, Theorem 2.10.6]) to get our result we'll need to show that $\widehat{\mathcal{M}}_{\text{univ}}$ is a Banach manifold, which is the difficult part of transversality. Since $\widehat{\mathcal{M}}_{\text{univ}}$ is the pre-image of the zero section of $\mathcal{E}$ by $\mathcal{F}$, we want to show that the differential

$$D_{u,J} : W^{1,p}_g(u^*TX) \times T_J\mathcal{J}^\ell_G \xrightarrow{(d\mathcal{F})_{(u,J)}} T_{(u,0)}\mathcal{E} \xrightarrow{\pi_2} L^p_g(u^*TX)$$

is surjective whenever $\mathcal{F}(u, J) = 0$ – compare with the proof of theorem 3.2.5 for a similar argument. The tangent space $T_J\mathcal{J}^\ell_G$ consists of paths of sections Y_t of $\mathrm{End}(TX)$ such that

$$J_tY_t + Y_tJ_t = 0,\ \omega(Y_tv, w) + \omega(v, Y_tw) = 0 \text{ and } g^*Y_{t+1} = Y_t\ \forall g \in G.$$

The tangent space $T_J\mathcal{J}^\ell$ of not-necessarily equivariant (time dependent) almost complex structures is given by the first two conditions, and consists of sections of $\mathrm{End}(TX, J_t, \omega)$.

We have

$$D_{u,J}(\xi, Y) = D_u\xi + Y_t(u)\left(\partial_tu - X^H_t(u)\right) = D_u\xi - Y_t(u)\partial_su.$$

Since D_u is Fredholm by 3.2.4 and $\operatorname{im} D_{u,J} \supseteq \operatorname{im} D_u$ we have that $\operatorname{im} D_{u,J}$ is closed, so it's enough to prove that it's dense. Suppose not; then there is $\eta \in L^q_g(u^*TX) \setminus \{0\}$ such that $\langle\eta, D_{u,J}(\xi, Y)\rangle = 0$ for every $(\xi, Y) \in W^{1,p}_g(u^*TX) \times T_J\mathcal{J}^\ell$, where $1/p + 1/q = 1$ (making $L^q_g(u^*TX)$ the dual of $L^p_g(u^*TX)$. Then we have

$$\int_{[0,1]\times\mathbb{R}} \langle\eta, D_u\xi\rangle = 0 \quad \text{and} \quad \int_{[0,1]\times\mathbb{R}} \langle\eta, Y_t\partial_su\rangle = 0 \tag{5.1}$$

for every (ξ, Y). From the first equation η is a weak solution of $D^*_u\eta = 0$, and by elliptic regularity it follows that η is C^ℓ. We now use the second equation to arrive at a contradiction. Assume $\eta(t_0, s_0) \neq 0$ and $(t_0, s_0) \notin R(u)$. Since $\partial_su(t_0, s_0) \neq 0$ we can construct $Y_0 \in \mathrm{End}(T_{u(t_0,s_0)}X, J_{t_0}, \omega)$ such that (see [SZ92, Section 8])

$$\langle\eta(t_0, s_0), Y_0\partial_su(t_0, s_0)\rangle > 0.$$

We extend Y_0 arbitrarily to a section of $\widehat{Y} \in T_J\mathcal{J}^\ell$ (not necessarily G-equivariant) such that $\widehat{Y}_{t_0}(u(t_0, s_0)) = Y_0$. Multiplying $\widehat{Y}$ by a cut-off function as in [FHS95, Remark 4.4] we get a new section $\widetilde{Y} \in T_J\mathcal{J}^\ell$ such that we still have $\widetilde{Y}_{t_0}(u(t_0, s_0)) = Y_0$ and $\widetilde{Y}_t(x)$ is supported in a small neighbourhood of $(t, x) = (t_0, u(t_0, s_0))$. To be precise, by taking the support of the cut-off function to be arbitrarily small we can assume that:

1. There is a neighbourhood U of $u(s_0, t_0)$ and $\varepsilon > 0$ such that $Y_t(x) = 0$ if $t \notin]t_0 - \varepsilon, t_0 + \varepsilon[$ or $x \notin U$.

2. We have $\langle\eta(t, s), \widetilde{Y}_t(u(t, s))\partial_su(t, s)\rangle \geq 0$ for every $(t, s) \in [0, 1] \times \mathbb{R}$.

3. Since $u(t_0, s_0) \notin gu(t_0, \overline{\mathbb{R}})$ by shrinking U and ε we assume that

$$U \cap gu(]t_0 - \varepsilon, t_0 - \varepsilon[, \overline{\mathbb{R}}) = \emptyset.$$

Now we average to get an equivariant section

$$Y_t = \sum_{g\in G} g^*\widetilde{Y}_t \in T_J\mathcal{J}^\ell_G.$$

Finally we see that this Y contradicts (5.1). Indeed we have

$$\int_{[0,1]\times\mathbb{R}} \langle\eta, Y_t\partial_su\rangle = \sum_{g\in G} \int_{[0,1]\times\mathbb{R}} \langle\eta, (g^*\widetilde{Y}_t)\partial_su\rangle$$

$$= \sum_{g\in G} \int_{[0,1]\times\mathbb{R}} \langle \eta(t,s), g_*^{-1}\widetilde{Y}_t(g(u(t,s)))g_*\partial_s u(s,t)\rangle \mathrm{d}t\mathrm{d}s. \tag{5.2}$$

By 1. and 3. above we have $\widetilde{Y}_t(g(u(t,s))) = 0$ for $g \neq 1$ and $(t,s) \in [0,1]\times\mathbb{R}$, so the respective integral for each $g \neq 1$ is 0. For $g = 1$ the integral is positive by 2., thus we get a contradiction with (5.1). Hence $\eta(t_0,s_0) = 0$ for every $(t_0,s_0) \in R(u)$, and since $R(u)$ is dense by lemma 5.2.2 it follows that $\eta = 0$. Thus we conclude that $D_{u,J}$ is surjective and $\widehat{\mathcal{M}}_{\text{univ}}$ is a Banach manifold.

We now prove that the projection $\pi : \widehat{\mathcal{M}}_{\text{univ}} \to \mathcal{J}_G^\ell$ is Fredholm. Its differential is the projection

$$(d\pi)_{(u,J)} : T_{(u,J)}\widehat{\mathcal{M}}_{\text{univ}} \to T_J\mathcal{J}_G^\ell$$

where

$$T_{(u,J)}\widehat{\mathcal{M}}_{\text{univ}} = \{(\xi, Y) \in W_g^{1,p}(u^*TX) \times T_J\mathcal{J}_G^\ell \colon D_u\xi - Y_t(u)\partial_s u = 0\}.$$

So $\ker(d\pi)_{(u,J)} = \ker D_u$ is finite dimensional since D_u is Fredholm by 3.2.4. For the cokernel, we denote by $\rho : T_J\mathcal{J}_G^\ell \to L^p(u^*TX)$ the map $\rho(Y) = Y_t(u)\partial_s u$. It's then clear that $Y \in \operatorname{im}(d\pi)_{(u,J)}$ if and only if $\rho(Y) \in \operatorname{im} D_u$, and thus it follows that ρ induces an injection $\operatorname{coker}(d\pi)_{(u,J)} \to \operatorname{coker} D_u$, and again by the fact that D_u is Fredholm we get that $\operatorname{coker}(d\pi)_{(u,J)}$ is finite dimensional and hence $(d\pi)_{(u,J)}$ is Fredholm.

Finally by Sard-Smale theorem (see [Sma65, Theorem 1965] or [dS18, Theorem 2.10.6]) there is a second Baire category subset $\mathcal{J}_G^{reg,\,g,\,\gamma^-,\gamma^+}(X,\omega) \subseteq \mathcal{J}_G(X,\omega)$ of regular points of the projection π. We claim that for $J \in \mathcal{J}_G^{reg,\,g,\,\gamma^-,\gamma^+}(X,\omega)$ and every $u \in \widehat{\mathcal{M}}_g(\gamma^-,\gamma^+;H,J) \cap \mathcal{B}_g^{1,p}(\gamma^-,\gamma^+)^*$ the operator D_u is surjective. Indeed, since J is a regular point of π the operator $(d\pi)_{(u,J)}$ is surjective for $u \in \widehat{\mathcal{M}}_g(\gamma^-,\gamma^+;H,J)$; by our previous observation it follows that $\rho(Y) \in \operatorname{im} D_u$ for every $Y \in T_J\mathcal{J}_G^\ell$. But then

$$L^p(u^*TX) = \operatorname{im} D_{u,J} = \operatorname{im} D_u + \operatorname{im} \rho = \operatorname{im} D_u$$

so D_u is also surjective.

Now by letting $\mathcal{J}_G^{reg}$ be the intersection of $\mathcal{J}_G^{reg,\,g,\,\gamma^-,\gamma^+}$ running through $g \in G$ and $\gamma^-,\gamma^+ \in \mathcal{P}_g(H)$ we get the desired second Baire category (in particular dense) set. □

Translating this to the orbifold language, we're saying that we only have transversality problems for Floer trajectories contained in the singularity set. Note that we showed that there is no problem even if a Floer trajectory intersects the singularity set as long as it's not contained in it; because of this, we would not be able to prove theorem 5.2 by simply looking for transversality in the non-singular set.

As we discussed before, full transversality can't be achieved in general. But by requiring some condition on the orbifold $[X/G]$ this is possible. The simplest such condition is that its singularities are isolated.

Corollary 5.2.8. *Assume that X^g is a discrete set for $g \in G \setminus \{1\}$. Then there is a C^ℓ dense subset $\mathcal{J}_G^{reg}(X,\omega)$ of $\mathcal{J}_G(X,\omega)$ such that if $J \in \mathcal{J}_G^{reg}(X,\omega)$ then (H,J) is regular.*

Proof. Since X^g is discrete, the only Floer trajectories contained in some X^g are constant maps (in t and s), and for those D_u is automatically surjective. Hence this is a direct consequence of theorem 5.2.7. □

We suspect that in general the moduli spaces $\widehat{\mathcal{M}}_g(\gamma^-,\gamma^+;H,J)$ have reasonable ramified singularities at the trajectories u fixed by some $g \neq 1$. It would be interesting to understand this and to try to define Floer homology from these singular moduli spaces.

Remark 5.2.9. In [HHM17] it's shown that when $G = \mathbb{Z}/k$ is cyclic and acts on a manifold X there is always a G-invariant Morse-Smale pair $(H, \mathfrak{g})$, even though the set of such pairs is not dense! This gives some hope that there is always some regular pair $(H, J) \in C_G(\mathbb{R} \times X) \times \mathcal{J}_G(X, \omega)$. Indeed by proposition 6.0.2 it's enough to find a pair (H, J) such that $H_{|X^g}$ is Morse-Smale for every g.

5.3 Another approach to equivariant transversality

We would now like to describe a way to go around this inevitable equivariant transversality problem by defining Floer homology in a different way. We'll only give a rough idea of this approach and we won't use it again, but it really is a better way to define Floer homology of principal quotient orbifolds as it avoids completely equivariant transversality and only needs the usual transversality theory for Floer homology. This is an adaptation of the solution explained in the last section of [CH14] to the analogous problem in Morse homology.

The Floer homology, as defined earlier, admits a decomposition indexed by conjugacy classes (g) of G

$$HF(\mathcal{X}, H, J; \Lambda) = \bigoplus_{(g)} HF^{(g)}(\mathcal{X}, H, J; \Lambda)$$

where $HF^{(g)}(\mathcal{X}, H, J; \Lambda)$ is the homology of the complex

$$CF^{(g)}(\mathcal{X}, H; \Lambda) \cong \left(\bigoplus_{\gamma \in \mathcal{P}_g(H)^+} \Lambda \cdot \gamma \right)^{C(g)}.$$

This will be explained in the beginning of section 6.1. For simplicity of the discussion let's pretend that there are no non-orientable orbits, i.e. $\mathcal{P}_g(H)^+ = \mathcal{P}_g(H)$.

What we will do is redefining $HF^{(g)}(\mathcal{X}, H, J; \Lambda)$ by using not G-equivariant Hamiltonian and almost complex structure, but only an Hamiltonian and an almost complex structure that satisfy the conditions we asked for in chapter 3. For each conjugacy class (g) we choose a representative g and pick a pair

$$(H, J) \in C_g(\mathbb{R} \times X) \times \mathcal{J}_g(X, \omega)$$

that is regular in the sense of definition 3.2.8. This condition is generic (both in H and J) by [FHS95, Theorem 5.1]. So we can define the complex $CF(X, g, H; \Lambda)$ as in 3.3.5; note that if $\mathcal{X}$ is Calabi-Yau then $c_1(X) = 0$, so we have no-bubbling – see section 3.3.2.

The tricky part is that if H is not G-invariant we don't have an action of $C(g)$ on $CF(X, g, H; \Lambda)$. Indeed, if γ is a Hamiltonian orbit then $h\gamma$ is a Hamiltonian orbit of $H \circ h^{-1}$, and not H. So for each $h \in C(g)$ what we get is a chain complex isomorphism

$$h\cdot : CF(X, g, H; \Lambda) \to CF(X, g, H \circ h^{-1}; \Lambda).$$

But the standard theory of invariance in Floer homology (see for instance [AD14, Part 11]) allows us to construct a chain map

$$\phi_h : CF(X, g, H \circ h^{-1}; \Lambda) \to CF(X, g, H; \Lambda)$$

which induces an isomorphism on homology. This is done by picking a regular homotopy of pairs between $(H \circ h^{-1}, (h^{-1})^* J)$ and (H, J) and counting solutions of a parametrized Floer equation. Then we get a weak action in the sense of [CH14, Definition 7.2] of $C(g)$ on $CF(X, g, H; \Lambda)$ by letting $h \in C(g)$ act through the composition

$$CF(X, g, H; \Lambda) \xrightarrow{h_*} CF(X, g, H \circ h^{-1}; \Lambda) \xrightarrow{\phi_h} CF(X, g, H; \Lambda).$$

A weak action induces an action on homology.

Remark 5.3.1. Suppose C is a chain complex over a field Λ of characteristic 0 and K is a finite group acting on C. Then the K-invariant part of the homology of C is the homology of the K-invariant part of C, that is, $H_*(C^K) = H_*(C)^K$.

By the remark above, it's now clear that if we're working over a field of characteristic 0 we can define $HF^{(g)}(\mathcal{X}, H, J; \Lambda)$ as the $C(g)$-invariant part of $HF(X, g, H, J; \Lambda)$.

Note that if (H, J) is already a regular pair and we then $H \circ h^{-1} = H$ and $(h^{-1})^* J = J$, so we can choose a constant homotopy of pairs making $\phi_h = \mathrm{id}$. Then this alternative approach leads to the original definition of $HF^{(g)}(\mathcal{X}, H, J; \Lambda)$.

Finally, we observe that from this perspective we should be able to prove invariance of Floer homology of orbifolds as done in [CH14, Section 7.3] and using the standard techniques in Floer homology. Even more, it should be true that the decomposition $HF^{(g)}(\mathcal{X}, H, J; \Lambda)$ is also invariant of the pair (H, J).

Chapter 6

Floer homology for “small” autonomous Hamiltonians

This chapter will consist essentially in a proof that for “small” and autonomous Hamiltonians H the Floer homology of an orbifold with rational Novikov coefficients is isomorphic to the Chen-Ruan cohomology. This generalizes the well known fact that the Floer homology of a smooth manifold is isomorphic to its singular homology. We state the result here:

Theorem 6.0.1. *Let $\mathcal{X} = [X/G]$ be a global quotient compact Calabi-Yau orbifold with symplectic form ω. Let $H \in C_G(\mathbb{R} \times X)$ and $J \in \mathcal{J}_G(X, \omega)$ be an autonomous Hamiltonian and an autonomous almost complex structure, respectively, and denote $H_\tau = \tau H$ for $\tau > 0$. Assume that for sufficiently small τ the pair (H_τ, J) is regular in the sense of 4.1.2. Let $\Lambda = \Lambda^{\text{univ}}(\mathbb{Q})$ be the rational universal Novikov ring. Then for sufficiently small $\tau > 0$ we have*

$$HF_*(\mathcal{X}, H_\tau, J; \Lambda) \cong H^{n-*}_{CR}(\mathcal{X}; \Lambda).$$

A few comments about the hypothesis are in order. The first is that we don't need to ask for the no-bubbling condition to hold because it's automatic for such a Hamiltonian. We'll see in proposition 6.1.3 that every Floer trajectory is constant in the s variable, and hence has zero energy; thus if we have bubbling from some holomorphic sphere such sphere must also have zero energy, and thus would be constant. The condition that $\mathcal{X}$ is Calabi-Yau is needed so that we can define a grading – otherwise the same isomorphism holds without gradings (or with grading modulo 2) but we would have to adapt slightly our discussion of orientations. We think that the fact that the indices agree is actually great part of the interest of this result as we can see the degree shifting numbers for the Chen-Ruan cohomology arising naturally from the Conley-Zehnder index.

Asking for the Novikov ring to be defined over the rationals (or other field of characteristic 0) is crucial. We will use the isomorphism between singular cohomology of an orbifold and its Morse homology given by theorem 2.4.4, and this only holds with coefficients in $\mathbb{Q}$. This is perhaps not too surprising as Chen-Ruan cohomology is much “nicer” with rational coefficients: for instance we can only define a product over $\mathbb{Q}$ and the crepant resolution conjecture fails with $\mathbb{Z}$ coefficients.

In the smooth case such an isomorphism is known to hold for every regular pair (H, J). Indeed taking a pair of the form (H_τ, J) with H autonomous and $\tau > 0$ small is needed only in order to have an isomorphism at the complex level, and two more ingredients are needed to extend to a general

Hamiltonian: first we need to prove that Floer homology doesn't depend on the pair (H, J) and second we need that we can actually find some regular pair (H_τ, J) with τ sufficiently small. For now we can't conclude this. We didn't prove invariance of Floer homology and the regularity condition is not in general guaranteed as we can't achieve equivariant transversality if we have non-isolated singularities (see chapter 5).

The regularity condition, that (H_τ, J) is regular for every small τ, might seem a bit odd and stronger than necessary. However the next proposition clarifies that this is actually automatic as soon as it's true just for some small τ.

Proposition 6.0.2. *Let $H \in C_G(\mathbb{R} \times X)$ and $J \in \mathcal{J}_G(X, \omega)$ be an autonomous Hamiltonian and an autonomous almost complex structure, respectively, and assume that H is C^2-small. Then D_u is surjective for every Floer trajectory u which is t-independent if and only if $(H_{|X^g}, \mathfrak{g})$ is a Morse-Smale pair for every $g \in G$ where $\mathfrak{g}(u, v) = \omega(u, Jv)$. In particular if $(H_{|X^g}, \mathfrak{g})$ is Morse-Smale for every $g \in G$ then (H_τ, J) is regular for every sufficiently small τ.*

Proof. If $u \in \widehat{\mathcal{M}}_g(H, J)$ is t-independent then $u(s) = u(t, s)$ is a Morse trajectory of $H_{|X^g}$. Thus we can proceed precisely as in the proofs of [AD14, Theorem 10.1.5, Corollary 10.1.8].

For the last statement, note that (with a fixed Riemannian metric) H_τ is Morse-Smale if and only if H is Morse-Smale, as the stable and unstable manifolds are the same, and that for sufficiently small τ every Floer trajectory $u \in \widehat{\mathcal{M}}_g(H_\tau, J)$ is t-independent by proposition 6.1.3. $\square$

Corollary 6.0.3. *Let $\mathcal{X} = [X/G]$ be a global quotient compact Calabi-Yau orbifold with symplectic form ω. Assume that $\mathcal{X}$ has isolated singularities. Then there is a regular pair $(H, J) \in C_G(\mathbb{R} \times X) \times \mathcal{J}_G(X, \omega)$ such that*

$$HF_*(\mathcal{X}, H_\tau, J; \Lambda) \cong H^{n-*}_{CR}(\mathcal{X}; \Lambda)$$

where $\Lambda = \Lambda^{\text{univ}}(\mathbb{Q})$.

Proof. Take a C^2-small autonomous G-equivariant Hamiltonian H. By perturbing H we may assume that $H_{|X^g}$ is a Morse function in X^g for every g (see [Was69, Lemma 4.8]). Since every orbit of H is constant (see proposition 6.1.1) the Morse condition translates to H being a non-degenerate Hamiltonian.

By corollary 5.2.8 we can find an almost complex structure J such that (H, J) is regular, but if we apply it directly we get a non-autonomous J. However, it's easy to modify the proof of theorem 5.2.7 to show that if H is autonomous then we can find J also autonomous such that D_u is surjective for every trajectory $u \in \widehat{\mathcal{M}}_g(H, J)$ that is t-independent.

But then theorem 6.0.1 and proposition 6.0.2 show that replacing H by H_τ for sufficiently small $\tau > 0$ we get the result. $\square$

Note that this corollary (which is already very heavy on hypothesis) applies to the Kummer surface (example 2.1.5) and many other high dimensional examples which we didn't discuss.

6.1 Proof of theorem 6.0.1

We begin by introducing some notation. What follows is general for any regular pair (H, J), with H not-necessarily small. We can decompose the set of Hamiltonian orbits according to the conjugacy class of the group element associated to them, that is,

$$\widetilde{\mathcal{P}}_G(H) = \bigsqcup_{(g)} \widetilde{\mathcal{P}}_{(g)}(H)$$

where, given a conjugacy class (g) of G, we write

$$\widetilde{\mathcal{P}}_{(g)}(H) = \bigsqcup_{g' \in (g)} \widetilde{\mathcal{P}}_{g'}(H).$$

Recall that the G-action on $\widetilde{\mathcal{P}}_G(H)$ is given by $h \cdot \gamma_g = (h\gamma)_{hgh^{-1}}$, so $\widetilde{\mathcal{P}}_{(g)}(H)$ are invariant subsets with respect to this action. We also set

$$\mathcal{P}_{(g)}(H) = \widetilde{\mathcal{P}}_{(g)}(H)/G, \ \widetilde{\mathcal{P}}_{(g)}(H)^+ = \widetilde{\mathcal{P}}_G(H)^+ \cap \widetilde{\mathcal{P}}_{(g)}(H) \text{ and } \mathcal{P}_{(g)}(H)^+ = \mathcal{P}_G(H)^+ \cap \mathcal{P}_{(g)}(H).$$

This decomposition of $\mathcal{P}_G(H)$ also gives a decomposition of the Floer complex. If we let

$$CF_k^{(g)}(\mathcal{X}, H; \Lambda) = \left(\bigoplus_{\substack{\gamma \in \widetilde{\mathcal{P}}_{(g)}(H)^+ \\ |\gamma| = k}} \Lambda \cdot \gamma \right)^G \tag{6.1}$$

then

$$CF_k(\mathcal{X}, H; \Lambda) = \bigoplus_{(g)} CF_k^{(g)}(\mathcal{X}, H; \Lambda).$$

Note that $CF_k^{(g)}(\mathcal{X}, H; \Lambda)$ are invariant with respect to the differential ∂, so we also have a decomposition of the Floer homology indexed by conjugacy classes:

$$HF_k(\mathcal{X}, H, J; \Lambda) = \bigoplus_{(g)} HF_k^{(g)}(\mathcal{X}, H, J; \Lambda)$$

where $HF_k^{(g)}(\mathcal{X}, H, J; \Lambda)$ is the homology of $\left(CF_k^{(g)}(\mathcal{X}, H; \Lambda), \partial_{H,J} \right)$. Finally, we have an isomorphism

$$CF_k^{(g)}(\mathcal{X}, H; \Lambda) \cong \left(\bigoplus_{\gamma \in \mathcal{P}_g(H)^+} \Lambda \cdot \gamma \right)^{C(g)}.$$

As discussed before the left hand side can be seen as the Λ-module generated by $\mathcal{P}_{(g)}(H)^+$ and the right hand side as the Λ-module generated by $\mathcal{P}_g(H)^+/C(g)$. But these are naturally identified: consider the map $\widetilde{\mathcal{P}}_g(H)^+ \to \mathcal{P}_{(g)}(H)^+$ given by taking the equivalence class. This is a surjection and two elements in $\widetilde{\mathcal{P}}_g(H)^+$ have the same image if and only if they are in the same orbit with respect to the $C(g)$-action.

6.1.1 Hamiltonian orbits and Floer trajectories are constant

The first step to identify the Floer complex with the Morse complex computing the Chen-Ruan cohomology is to see that every Hamiltonian orbit of a Hamiltonian as in theorem 6.0.1 is constant. We do so by reducing the proof to the 1-periodic case.

Proposition 6.1.1. *Assume that H is a sufficiently C^2-small autonomous Hamiltonian. Then every Hamiltonian orbit $\gamma \in \widetilde{\mathcal{P}}_G(H)$ is constant.*

Proof. Let $m = |G|$. As γ is a Hamiltonian orbit and H is G-invariant it's easy to see that the extension $\widehat{\gamma} : [0, m] \to X$ given by

$$\widehat{\gamma}(t) = g^k \gamma(t - k) \text{ for } t \in [k, k+1]$$

is also a solution of Hamilton's equation; note that it's continuous because $\gamma(1) = g\gamma(0)$. Moreover $\widehat{\gamma}(m) = g^m \gamma(0) = \gamma(0)$. Hence $t \mapsto \widehat{\gamma}(t/m)$ is a 1-periodic Hamiltonian orbit of mH. It follows from [AD14, Proposition 6.1.5] that for sufficiently C^2-small and autonomous H (and thus sufficiently small mH) such orbits are constant. $\square$

As usual we denote by $c^x : [0, 1] \to X$ the constant map equal to x. Since every Hamiltonian orbit is constant, it's equal to c^x_g for some $x \in X$ and $g \in G$. The constant c^x is a Hamiltonian orbit if and only if x is a critical point of H and $gc^x(0) = c^x(1)$ if and only if $g \in G_x$. Hence

$$\widetilde{\mathcal{P}}_G(H) = \{c^x_g : (dH)_x = 0 \text{ and } gx = x\}.$$

We can say a bit more about this set. It's a consequence of the slice theorem (and the fact that $\langle g \rangle \subseteq G$ is finite, and hence compact) that the fixed point set

$$X^g = \{x \in X : gx = x\}$$

is a manifold and its tangent space at $x \in X$ is

$$T_x X^g = (T_x X)^g = \{v \in T_x X : (dg)_x v = v\}.$$

Now if $c^x_g \in \widetilde{\mathcal{P}}_G(H)$ then $x \in X^g$ and x is a critical point of the function $H_{|X^g}$. Conversely, if x is a critical point of $H_{|X^g}$ then $c^x_g \in \widetilde{\mathcal{P}}_G(H)$, thanks to the following elementary lemma:

Lemma 6.1.2. *Let X be a manifold with an action of a finite group G and suppose that $H : X \to \mathbb{R}$ is G-invariant; let $g \in G$. Then $x \in X^g$ is a critical point of $H_{|X^g}$ if and only if x is a critical point of H.*

Proof. The if part is immediate. Suppose that $\left(dH_{|X^g}\right)_x = 0$. This means that for every $w \in T_x X^g$ we have $(dH)_x w = 0$. Let $v \in T_x X$ be any vector and define

$$w = \frac{1}{m} \sum_{k=0}^{m-1} (dg)^k_x v \in T_x X$$

where m is the order of g. Clearly $w \in T_x X^g$ and since H is G-invariant $(dH)_x v = (dH)_x w = 0$, so $(dH)_x = 0$. $\square$

This means that there is a bijection between $\widetilde{\mathcal{P}}_G(H)$ and

$$\bigsqcup_{g \in G} \operatorname{Crit}\left(H_{|X^g}\right).$$

The next step in establishing the connection between Floer homology and Morse homology is to prove that all the Floer trajectories are constant in the t-variable. This will identify the Floer trajectories with gradient flow trajectories of H.

Proposition 6.1.3. *Assume that H is an autonomous Hamiltonian and let $H_\tau = \tau H$ for $\tau > 0$. Then for sufficiently small $\tau > 0$ every Floer trajectory $u \in \widehat{\mathcal{M}}_g(\gamma^-, \gamma^+; H_\tau, J)$ does not depend on t, that is, $u(t, s) = u(s)$ is a Morse trajectory.*

Proof. We can use a similar reduction to the 1-periodic case as the one we did in the proof of 6.1.1. We let

$$\widehat{u}(t,s) = g^k u(t-k,s) \text{ for } t \in [k,k+1].$$

Then $(t,s) \mapsto \widehat{u}(t/m,s)$ is a "1-periodic" Floer trajectory of $(mH_\tau, J) = (H_{m\tau}, J)$. Applying [SZ92, Theorem 7.3] gives that $\widehat{u}$ is independent of t for small enough τ, and hence so is u. $\square$

Remark 6.1.4. This proposition is the only place in which we really have to use a Hamiltonian of the form H_τ with sufficiently small $\tau > 0$ and not only a C^2-small autonomous Hamiltonian. It seems to be believed that actually it's true that Floer trajectories of a C^2-small autonomous Hamiltonian must be independent of t, but the author could not find either a proof or a precise reference in the literature.

From now on we'll assume that $H = H_\tau$ is C^2-small and such that every Floer trajectory doesn't depend on t.

Suppose that x^-, x^+ are critical points of $H_{|X^g}$ and let $u \in \widehat{\mathcal{M}}_g(c^{x^-}, c^{x^+}; H, J)$. Since $\partial_t u = 0$, and using the relation between the gradient and the Hamiltonian vector field (3.1), Floer equation becomes

$$0 = \partial_s u - J(u) X^H(u) = \partial_s u - \nabla H(u)$$

so Floer trajectories are in bijection with gradient flow lines of H contained in X^g, that is, gradient flow lines of $H_{|X^g}$.

6.1.2 Index computation

We now proceed to compute the index of the constant orbits $c_g^x \in \mathcal{P}_g(H) \subseteq \widetilde{\mathcal{P}}_g(H)$ according to a trivialization of $\Lambda^n_{\mathbb{C}} TX$, as explained in 4.2. This is done by finding a trivialization

$$\Psi(t) : \mathbb{R}^{2n} \to T_x X$$

that obeys $\Psi(1) = (dg)_x \circ \Psi(0)$ and is compatible with $\mathfrak{s}$. First we note that when $g = 1$ there is not much to do: we can just pick a constant trivialization, meaning that $\Psi(t)$ doesn't depend on t. The trick to get a trivialization with $\Psi(1) = (dg)_x \circ \Psi(0)$ when $g \neq 1$ is to correct a fixed trivialization with a path of matrices in $SU(n)$; asking the path to be in $SU(n)$ ensures that the trivialization is compatible with $\mathfrak{s}$. This idea, although in a different context, already appeared in [AMM18].

Let's make this precise. First, we fix any isomorphism $\Psi(0) : \mathbb{R}^{2n} \to T_x X$ respecting the symplectic and almost complex structures. The canonical symplectic form on $\mathbb{R}^{2n}$ is denoted by ω_0 and the almost structure by J_0. We have an induced map on the determinant vector spaces

$$\Lambda^n_{\mathbb{C}} \Psi(0) : \mathbb{C} \to \Lambda^n_{\mathbb{C}} TX$$

(note that $(\mathbb{R}^{2n}, J_0)$ identifies with $(\mathbb{C}^n, i)$ so $\Lambda^n_{\mathbb{C}} \mathbb{R}^{2n} \cong \mathbb{C}$ naturally). By multiplying the section $\mathfrak{s}$ by some constant in $\mathbb{C}^*$ we may assume that

$$\mathfrak{s}(x) = \left(\Lambda^n_{\mathbb{C}} \Psi(0)\right)(1) \in \Lambda^n_{\mathbb{C}} TX. \tag{6.2}$$

Multiplication by such a constant doesn't change the homotopy class of $\mathfrak{s}$, since $\mathbb{C}^*$ is connected, so doesn't change whether our constructed trivialization is compatible or not. Essentially this makes the trivialization compatible with $\mathfrak{s}$ at $t = 0$.

Since ω and J are G-invariant, we have $(dg)_x \in U(T_xX)$. Moreover, since $\mathcal{X}$ is Calabi-Yau then $\mathcal{X}$ is Gorenstein (see remark 4.2.2) and, as $gx = x$, actually $(dg)_x \in SU(T_xX)$ (see characterization 2. in 2.5.3). Hence the matrix G_1 corresponding to $(dg)_x$ via the identification $\Psi(0) : \mathbb{R}^{2n} \to T_xX$, i.e., such that

$$\begin{tikzcd} \mathbb{R}^{2n} \arrow[r, "G_1"] \arrow[d, "\Psi(0)"] & \mathbb{R}^{2n} \arrow[d, "\Psi(0)"] \\ T_xX \arrow[r, "(dg)_x"] & T_xX \end{tikzcd}$$

is in $SU(n) \subseteq GL(2n;\mathbb{R})$. As $SU(n)$ is connected, there is a path of matrices $[0,1] \ni t \mapsto G_t \in SU(n)$ starting from $G_0 = \mathrm{Id}$ to G_1. Now we set

$$\Psi(t) = G_t \circ \Psi(0) \tag{6.3}$$

and we prove that such a trivialization has the required properties. First,

$$\Psi(1) = G_1 \circ \Psi(0) = \Psi(0) \circ (dg)_x$$

as required. Moreover, to see that it's compatible with $\mathfrak{s}$ note that $\Lambda^n_{\mathbb{C}}\Psi(t)$ is the composition

$$\Lambda^n_{\mathbb{C}}\mathbb{R}^{2n} \xrightarrow{\Lambda^n_{\mathbb{C}} G_t} \Lambda^n_{\mathbb{C}}\mathbb{R}^{2n} \xrightarrow{\Lambda^n_{\mathbb{C}}\Psi(0)} \Lambda^n_{\mathbb{C}} T_xX.$$

Since $G_t \in SU(n) \subseteq SL(n,\mathbb{C})$ the map $\Lambda^n_{\mathbb{C}} G_t$ is the identity. By equation (6.2) the map $\Lambda^n_{\mathbb{C}}\Psi(0)$ is given by $z \mapsto \mathfrak{s}(x)z$, and this proves compatibility with $\mathfrak{s}$.

To perform the computation of the index we need appropriate local coordinates. These are given by an equivariant Darboux theorem.

Theorem 6.1.5 (Equivariant Darboux theorem)**.** *Let (X,ω,J) be a symplectic manifold with compatible almost complex structure J. Suppose that a finite group K acts on X preserving ω and J and let $x \in X$ be a fixed point of the action. Then there are open neighbourhoods $U \subseteq X$ (invariant with respect to the K-action) of x and $V \subseteq \mathbb{R}^{2n}$ of 0 and a symplectomorphism $f : V \to U$ sending 0 to x, such that $f^*J = J_0$ and such that the induced action on V is linear.*

Proof. First, we can identify the symplectic space (T_xX, ω_x) with $(\mathbb{R}^{2n}, \omega_0)$ where ω_0 is the canonical symplectic form. Since K fixes x, we have an infinitesimal linear action of K on $T_xX \cong \mathbb{R}^{2n}$. We can find an equivariant embedding $\iota : V \hookrightarrow X$, where $V \subseteq T_xX \cong \mathbb{R}^{2n}$ is an open neighbourhood of 0, such that the differential $(d\iota)_0 : T_0V = T_xX \to T_xX$ is the identity. Now the forms ω_0 and $\iota^*\omega$ are both K-invariant forms on V agreeing on 0. By [DM93, Corollary 2] and possibly by restricting V there is $j : V \to V$ such that $j^*\iota^*\omega = \omega_0$; hence $f = \iota \circ j$ has the desired property. □

Thanks to the equivariant Darboux theorem, in order to compute the index we may assume that $X = \mathbb{R}^{2n}$, $x = 0$ and $g = G_1$ is a linear transformation.

Proposition 6.1.6. *Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be an autonomous Hamiltonian with $(dH)_0 = 0$, let φ_t be the Hamiltonian flow of H and let S be the Hessian of $-H$ at 0. Then*

$$(d\varphi_t)_0 = \exp(tJ_0S).$$

Proof. First we should make clear that $(d\varphi_t)_0$ means the Jacobian matrix of $\varphi_t : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ at $x = 0$, which we can also write as $(D\varphi_t)(0)$. Since $X^H = -J_0 \nabla H$ where $\nabla H = \left(\frac{\partial H}{\partial x_1}, \ldots, \frac{\partial H}{\partial x_1}\right)$ is the gradient with respect to the usual Riemannian metric Hamilton's equation is

$$\frac{d}{dt}\varphi_t(x) = -J_0 \nabla H(\varphi_t(x)).$$

By taking the Jacobian on both sides we have

$$\frac{d}{dt}(d\varphi_t)_x = -J_0(d\nabla H)_{\varphi_t(x)}(d\varphi_t)_x.$$

Plugging $x = 0$ in the above equation we get

$$\frac{d}{dt}(d\varphi_t)_0 = J_0 S (d\varphi_t)_0.$$

since $\varphi_t(0) = 0$ because x is a critical point and $(d\nabla H)_0$ is the Hessian of H (and thus $-(d\nabla H)_0 = S$). The result now clearly follows. $\square$

Consider now local coordinates around x given by the equivariant Darboux theorem 6.1.5 applied to $K = \langle g \rangle$ (which fixes x if $gx = x$). Such coordinates induce an identification $\Psi(0) : \mathbb{R}^{2n} \cong T_0\mathbb{R}^{2n} \to T_x X$. Then the path of symplectic matrices $\Phi = \Phi_{c_g^x}$, computed using the trivialization Ψ in (6.3), used to calculate the index is

$$\Phi(t) = \Psi(t)^{-1}(d\varphi_t)_0\Psi(0) = G_t^{-1}\Psi(0)^{-1}(d\varphi_t)_0\Psi(0) = G_t^{-1}\exp(tJ_0S)$$

where S is the Hessian matrix of $-H$ at x calculated in the coordinates we're using. Note that we used proposition 6.1.6 in the last identity.

By doing a linear unitary change of variables in $\mathbb{R}^n$ we may assume that the matrix G_1 is diagonal as a complex matrix; this is possible by the spectral theorem. So

$$G_1 = \begin{bmatrix} e^{2\pi i\lambda_1} & & \\ & \ddots & \\ & & e^{2\pi i\lambda_n} \end{bmatrix} \in SU(n) \subseteq GL(2n; \mathbb{R}).$$

where $e^{2\pi i\lambda_j}$ are the complex eigenvalues of G_1. We assume that $\lambda_{k+1} = \lambda_{k+2} = \ldots = \lambda_n = 0$ and that $\lambda_j \notin \mathbb{Z}$ for $j = 1 \ldots, k$; this means that the eigenspace of 1 has (complex) dimension $d = n - k$. Since $G_1 \in SU(n)$ it follows that

$$\sum_{j=1}^{k}\lambda_j = \sum_{j=1}^{n}\lambda_j \in \mathbb{Z}$$

so by changing the value of (say) λ_1 by an integer value we assume that $\sum_{j=1}^{k}\lambda_j = 0$. As a real matrix, G_1 is

$$G_1 = \begin{bmatrix} \cos(2\pi i\lambda_1) & -\sin(2\pi i\lambda_1) & & & & & & \\ \sin(2\pi i\lambda_1) & \cos(2\pi i\lambda_1) & & & & & & \\ & & \ddots & & & & & \\ & & & \cos(2\pi i\lambda_k) & -\sin(2\pi i\lambda_k) & & & \\ & & & \sin(2\pi i\lambda_k) & \cos(2\pi i\lambda_k) & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{bmatrix}.$$

We will denote by $\widetilde{G}_1$ the upper diagonal $2k \times 2k$ block.

Note that by the equivariant Darboux theorem 6.1.5 the fixed point set X^g is a symplectic manifold. Indeed $X^g \cap U$ is identified with (the intersection of V with) the eigenspace associated with 1 of $(dg)_x$. Via the identification $\Psi(0) : \mathbb{R}^{2n} \to T_x X$ the tangent space $T_x X^g \subseteq T_x X$ corresponds to the eigenspace of G_1 associated with 1, which consists of vectors where the first $2k$ coordinates vanish. Thus

$$\dim X^g = \dim T_x X^g = 2n - 2k = 2d.$$

Here, by dimension of X^g we mean the dimension of the connected component of X^g containing x. The invariance of H with respect to g imposes a strong restriction on the Hessian matrix S: it has to be a block diagonal matrix.

Claim 6.1.7. *The Hessian S of $-H$ is a block diagonal matrix*

$$S = \begin{bmatrix} S_1 & \\ & S_2 \end{bmatrix}$$

where S_1 is a $2k \times 2k$ block and S_2 is a $2d \times 2d$ block. Moreover S_2 is the Hessian of $-H_{|X^g}$.

Proof. The geometric idea is that this follows from the fact that the flow of φ_t must preserve X^g. To be precise, since $H \circ g = H$ we have $g \circ \varphi_t = \varphi_t \circ g$ and taking the derivative at x we get $(dg)_x \circ (d\varphi_t)_x = (d\varphi_t)_x \circ (dg)_x$. In local coordinates, by proposition 6.1.6, this means that $\exp(tJ_0S)G_1 = G_1 \exp(tJ_0S)$. Taking the derivative at $t = 0$ in both sides it follows that $J_0SG_1 = G_1J_0S$, so J_0S commutes with G_1. Hence J_0S preserves the eigenspaces of G_1. Since the span of the first $2k$ coordinate vectors is the union of the eigenspaces associated with eigenvalues different from 1 and the span of the last $2d$ coordinate vectors is the eigenspace associated to 1 it follows that J_0S is block diagonal, and thus so is S. ☐

To compute the index, we finally fix our choice of the path G_t. Recall that this is any path in $SU(n)$ with $G_0 = \mathrm{Id}$ and G_1 the already fixed matrix. We use the obvious choice

$$G_t = \begin{bmatrix} \cos(2\pi i\lambda_1 t) & -\sin(2\pi i\lambda_1 t) & & & & & & \\ \sin(2\pi i\lambda_1 t) & \cos(2\pi i\lambda_1 t) & & & & & & \\ & & \ddots & & & & & \\ & & & \cos(2\pi i\lambda_k t) & -\sin(2\pi i\lambda_k t) & & & \\ & & & \sin(2\pi i\lambda_k t) & \cos(2\pi i\lambda_k t) & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{bmatrix}$$

and denote by $\widetilde{G}_t$ the upper $2k \times 2k$ diagonal block. Note that $G_t \in SU(n)$ because the sum of the λ_j's is 0. We have the expression

$$\Phi(t) = \begin{bmatrix} \widetilde{G}_t^{-1} \exp(tJ_0S_1) & \\ & \exp(tJ_0S_2) \end{bmatrix} \in Sp(2n).$$

For this path to be admissible one needs that the matrices $\exp(J_0S_2)$ and $\widetilde{G}_1^{-1} \exp(J_0S_1)$ don't have 1 as an eigenvalue. Since S_2 is small, $\exp(J_0S_2)$ has 1 as eigenvalue if and only if S_2 is singular. Moreover for small S_1 the eigenvalues of $\widetilde{G}_1^{-1} \exp(J_0S_1)$ are close to the eigenvalues of $\widetilde{G}_1^{-1}$, none of which is 1, so $\widetilde{G}_t^{-1}\exp(tJ_0S_1)$ is automatically admissible. Given that H is C^2-small we've shown that H is non-degenerate as a Hamiltonian if and only if $H_{|X^g}$ is a Morse function for every g. We suppose that this holds.

Finally we turn to the computation of the index $|c_g^x| = \mu_{CZ}(\Phi)$. By the same argument as before, if we consider the homotopy

$$\Phi^s(t) = \begin{bmatrix} \widetilde{G}_t^{-1}\exp(stJ_0S_1) & \\ & \exp(tJ_0S_2) \end{bmatrix} \in Sp(2n)$$

then each Φ^s, for $s \in [0,1]$, is an admissible path of symplectic matrices. By the homotopy and the direct sum properties of the Conley-Zehnder index, in appendix A we then have

$$|c_g^x| = \mu_{CZ}(\Phi^1) = \mu_{CZ}(\Phi^0) = \mu_{CZ}(\widetilde{G}_t^{-1}) + \mu_{CZ}(\exp(tJ_0S_2)).$$

We compute each of these.

$$\begin{aligned} \mu_{CZ}(\widetilde{G}_t^{-1}) &= \sum_{j=1}^{k}(2\lfloor -\lambda_j \rfloor + 1) = \sum_{j=1}^{k}(-2\lambda_j + 2\{\lambda_j\} - 1) \\ &= 2\sum_{j=1}^{n-k}\{\lambda_j\} - k = 2\iota_{(g)} - k \end{aligned} \tag{6.4}$$

In the first equality we used the direct sum property of the Conley-Zehnder index and proposition A.0.4. In the second equality we used the identity

$$\lfloor -\lambda \rfloor = -\lambda + \{\lambda\} - 1 \text{ for } \lambda \notin \mathbb{Z}$$

and in the third we used that the sum of the λ_j's is 0. On the other hand, by the signature property of the Conley-Zehnder index we have

$$\mu_{CZ}(\exp(tJ_0S)) = d - \mathrm{ind}(S) = d - \mathrm{ind}_x(-H_{|X^g}) = \mathrm{ind}_x(H_{|X^g}) - d. \tag{6.5}$$

Summing (6.4) and 6.5 and using that $k + d = n$ we finally have

$$|c_g^x| = \mathrm{ind}_x\left(H_{|X^g}\right) + 2\iota_{(g)} - n. \tag{6.6}$$

6.1.3 Orientations

In this section we discuss the compatibility of orientations in the Floer setting and in the Morse setting; this is important so that orientable critical points correspond to orientable Floer trajectories and so that the signs $\nu(u)$ appearing in the definitions of the Morse and Floer complexes agree. The Floer complex orientations are determined by orientations of the determinant line bundles $\delta_{c_g^x} = \det(D_\Phi)$ (see 4.3) and the orientations of the Morse complex of each twisted sector $\mathcal{X}^{(g)}$ are determined by orientations of the tangent space of the unstable manifolds $T_xW^u_{H_{|X^g}}(x) = (T_xW^u_H(x))^g$. The latter notation means the subspace of $T_xW^u_H(x)$ fixed by $(dg)_x$.

First, recall that in the last section we saw that the path of symplectic matrices associated to c_g^x can be taken to be $\Phi = \Phi_1 \oplus \Phi_2$ where $\Phi_1(t) = \widetilde{G}_1^{-1}\exp(tJ_0S_1)$ and $\Phi_2(t) = \exp(tJ_0S_2)$. Therefore

$$\delta_{c_g^x} = \det(D_\Phi) \cong \det(D_{\Phi_1}) \otimes \det(D_{\Phi_2}) \cong \det\left(D_{\widetilde{G}_t^{-1}}\right) \otimes \det\left(D_{\Phi_2}\right). \tag{6.7}$$

The determinant line bundle $\det\left(D_{\widetilde{G}_t^{-1}}\right)$ only depends on g, so we can fix an orientation for it that won't affect anything. We're only interested in $\det(D_{\Phi_2})$. Since $\Phi_2(t) = \exp(tJ_0S_2)$ we have

$$\dot{\Phi}_2 = J_0S_2\Phi_2$$

so in this case the path of symplectic matrices $S : S^1 \to M_{2n\times 2n}(\mathbb{R})$ described in 4.3 can be taken to be constant $S(t) = S_2$. Now we consider an operator $D_{\Phi_2} : W^{1,p}_\mu(\mathbb{C}, \mathbb{R}^{2d}) \to L^p_\mu(\mathbb{C}, \mathbb{R}^{2d})$ given by

$$D_{\Phi_2} = \alpha(\partial_x + J_0\partial_y) + \beta S_2$$

where $\alpha : \mathbb{C} \to \mathbb{R} \oplus \mathbb{R}J_0$ and $\beta : \mathbb{R} \to [0, 1]$ are such that α never vanishes,

$$\alpha(x, y) = \begin{cases} 1 \text{ if } s \geq 2 \\ -x + J_0y \text{ if } s \leq 1 \end{cases}$$

and

$$\beta(s) = \begin{cases} 0 \text{ if } s \geq 0 \\ 1 \text{ if } s \leq -1 \end{cases} .$$

We will now compute the kernel of this operator. By the last part of theorem 4.3.2 we may assume that $p = 2$. Suppose that $Y \in \ker(D_{\Phi_2})$. By proposition 4.3.1 we have

$$\partial_sY(t, s) + J_0\partial_tY(t, s) + \beta(s)S_2Y(t, s) = 0 \text{ for } s \leq 1. \tag{6.8}$$

Moreover, for $s \geq 0$ we have $\beta(s) = 0$ so

$$0 = D_{\Phi_2}Y = \alpha(\partial_x + J_0\partial_y)Y \Rightarrow (\partial_s + J_0\partial_t)Y = 2\pi(-x + J_0y)\frac{\bar{\alpha}}{|\alpha|^2}\alpha(\partial_x + J_0\partial_y)Y = 0$$

since α never vanishes. So actually we have the equation

$$\partial_sY(t, s) + J_0\partial_tY(t, s) + \beta(s)S_2Y(t, s) = 0 \text{ for every } s \in \mathbb{R}.$$

Recalling that S_2 is small, this equation implies that Y does not depend on t.

Lemma 6.1.8. *Suppose that $Y \in W^{1,2}_\mu(\mathbb{C}, \mathbb{R}^{2d})$ satisfies equation (6.8) and that S_2 is sufficiently small. Then Y does not depend on t.*

Proof. This is almost the same as [dS18, Lemma 7.9.9], except that we don't necessarily have $Y \in W^{1,2}(S^1 \times \mathbb{R}, \mathbb{R}^{2n})$ because near 0 the volume form μ is not cylindrical; in particular $Y(t, s = +\infty) = Y(0)$ doesn't have to vanish. However, the proof still holds.

Defining

$$\bar{Y}(t, s) = \bar{Y}(s) = \int_0^1 Y(\tau, s)\mathrm{d}\tau \;\text{ for } s \in \mathbb{R}$$

$\bar{Y}$ is also a solution of (6.8) which is independent of t. Replacing Y by $Y - \bar{Y}$ we assume that $\int_0^1 Y(t, s)\mathrm{d}t = 0$ for every $s \in \mathbb{R}$ and we want to prove that $Y = 0$.

By [dS18, Lemma 7.9.7] applied to $t \mapsto Y(t, s)$ we have

$$\int_0^1 \|Y(t, s)\|^2\mathrm{d}t \leq \int_0^1 \|\partial_tY(t, s)\|^2\mathrm{d}t \text{ for all } s \leq 1.$$

Integrating in the s variable from $-\infty$ to 1 gives $\|\partial_tY\|_2 \geq \|Y\|_2$.

We have

$$\begin{aligned}\|\partial_s Y + J_0\partial_t Y\|_2^2 &= \langle \partial_s Y + J_0\partial_t Y, \partial_s Y + J_0\partial_t Y\rangle = -\langle Y, (\partial_s - J_0\partial_t)(\partial_s + J_0\partial_t)Y\rangle \\ &= -\langle Y, \partial_s^2 Y\rangle - \langle Y, \partial_t^2 Y\rangle = \|\partial_s Y\|_2^2 + \|\partial_t Y\|_2^2.\end{aligned} \tag{6.9}$$

Note that we used integration by parts in the second and last equalities. In neither case is there a boundary term because $\partial_s Y$ and $\partial_t Y$ vanish on the cylindrical ends: they vanish at $s = -\infty$ because we're assuming that $Y \in W^{1,2}(\mathbb{C}, \mathbb{R}^{2n})$ and they vanish at $s = +\infty$ by the computation in the proof of 4.3.1. Now we finish the proof:

$$\|Y\|_2^2 \leq \|\partial_t Y\|_2^2 \leq \|\partial_s Y\|_2^2 + \|\partial_t Y\|_2^2 = \|\partial_s Y + J_0\partial_t Y\|_2^2 \leq \|S_2\|\|Y\|_2^2$$

and since S_2 is small enough it follows that $Y = 0$. □

Taking the lemma into account, equation (6.8) simplifies to

$$\partial_s Y + \beta(s) S_2 Y = 0 \tag{6.10}$$

This is a standard linear ordinary differential equation and has solution

$$Y(s) = \exp\left(\left(-\int_0^s \beta(\sigma)\,\mathrm{d}\sigma\right) S_2\right) Y_0 \tag{6.11}$$

for some $Y_0 = Y(0)$. In particular, since $\beta(\sigma) = 0$ for $\sigma \geq 0$ it follows that $Y(s) = Y_0$ is constant for $s \geq 0$. Moreover, for $s \leq -1$, and since $\beta(\sigma) = 1$ we have

$$Y(s) = \exp(-(s + c_0)S_0)Y_0 \text{ for } s \leq -1$$

where $c_0 = 1 - \int_{-1}^0 \beta(\sigma)\,\mathrm{d}\sigma$ is a constant.

From this we see that $Y \in W^{1,p}_\mu(\mathbb{C}, \mathbb{R}^{2d})$ if and only if Y_0 is in the subspace of $\mathbb{R}^{2d}$ in which the Hessian $S_2 = \operatorname{Hess}\left(-H_{|X^g}\right)$ is negative definite. Indeed this is clear after we diagonalize S_2 and consider the 1-dimensional case:

Example 6.1.9. Suppose that $S_2 = [\mu]$ is a 1×1 matrix and we have

$$Y(s,t) = \exp(-(s + c_0)\mu)Y_0 \text{ for } s \leq -1.$$

The $Y \in W^{1,p}_\mu(\mathbb{C}, \mathbb{R}^1)$ if and only if $Y(s) \to 0$ when $s \to -\infty$. If $\mu > 0$ this happens if and only if $Y_0 = 0$, and if $\mu < 0$ this happens independently of Y_0.

Now the subspace of $\mathbb{R}^{2d}$ in which $S_2 = \operatorname{Hess}\left(-H_{|X^g}\right)$ is negative definite is the subspace of $\mathbb{R}^{2d}$ in which $\operatorname{Hess}\left(H_{|X^g}\right)$ is positive definite, which is identified (via the identification $\Psi(0) : \mathbb{R}^{2d} \to T_x X^g$) with $T_x W^u_{H_{|X^g}}(x)$. Thus we have an identification

$$\ker\left(D_{\Phi_2}\right) \cong T_x W^u_{H_{|X^g}}(x).$$

Moreover, now that we know explicitly the kernel, we can prove that $\operatorname{coker}(D_{\Phi_2}) = 0$ by looking at the Fredholm index. Indeed we know from theorem 4.3.2 and from equation (6.5) that

$$\operatorname{ind} D_{\Phi_2} = d - \mu_{CZ}(\Phi_2) = 2d - \operatorname{ind}_x\left(H_{|X^g}\right) = \dim T_x W^u_{H_{|X^g}}(x) = \dim\ker D_{\Phi_2}.$$

Thus $\dim \operatorname{coker} D_{\Phi_2} = 0$. In particular we know that $\det(D_{\Phi_2}) \cong \Lambda^{\text{top}} T_x W^u_{H_{|X^g}}(x)$. Combining this with (6.7) we get an identification

$$\delta_{c_g^x} \cong \det\left(D_{\widetilde{G}_t^{-1}}\right) \otimes \Lambda^{\text{top}} T_x W^u_{H_{|X^g}}(x).$$

Now since we have a fixed orientation in $\det\left(D_{\widetilde{G}_t^{-1}}\right)$, by picking orientations of the unstable manifolds $W^u_{H_{|X^g}}(x)$ (needed to define Morse homology of the twisted sectors) we get orientations of the determinant line bundles $\delta_{c_g^x}$ (needed to define Floer homology).

Now that we can define orientations in the Morse and Floer contexts in a compatible way, there are two details we need to verify in order to establish theorem 6.0.1. The first is that orientable critical points $x \in X^g$ correspond to orientable orbits c_g^x. But this follows from the fact that the following diagram commutes (up to multiplication by a positive constant) for any $h \in C(g)$ such that $hx = x$:

$$\begin{tikzcd}
\delta_{c_g^x} \arrow[r, "h_*"] \arrow[d, "\cong"] & \delta_{c_g^x} \arrow[d, "\cong"] \\
\det\left(D_{\widetilde{G}_t^{-1}}\right) \otimes \Lambda^{\text{top}} T_x W^u_{H_{|X^g}}(x) \arrow[r, "\mathrm{id}\otimes(dh)_x"] & \det\left(D_{\widetilde{G}_t^{-1}}\right) \otimes \Lambda^{\text{top}} T_x W^u_{H_{|X^g}}(x)
\end{tikzcd}$$

The second detail is that by choosing orientations in this compatible way the numbers $\nu^{\text{Floer}}(u)$ and $\nu^{\text{Morse}}(u)$ appearing in the definitions of the Morse and Floer complexes agree.

Lemma 6.1.10. *Let H be a C^2-small autonomous Hamiltonian and let $u \in \widehat{\mathcal{M}}_g(c^{x^-}, c^{x^+}; H, J)$ be a Floer trajectory of index $\mu(u) = 1$ that does not depend on t, that is, a Morse trajectory. Assume that $H_{|X^g}$ is Morse-Smale. Then*

$$\nu^{\text{Morse}}(u) = \nu^{\text{Floer}}(u).$$

Sketch. To keep a less heavy notation let's consider only the case $g = 1$; the general one is essentially the same. Write $\gamma^{\pm} = c^{x^{\pm}}$.

We recall how $\nu^{\text{Floer}}(u)$ is defined. There is an isomorphism 4.3.5 of determinant line bundles $\delta_{\gamma^-} \cong \det(D_u) \otimes \delta_{\gamma^+}$. Since in this case the operators involved are all surjective, this isomorphism is induced by the isomorphism coming from gluing theory

$$\ker(D_{\Phi^-}) \cong \ker(D_u \# D_{\Phi^+}) \cong \ker(D_u) \oplus \ker(D_{\Phi^+}).$$

By fixing the orientations of $\delta_{\gamma^{\pm}}$ we have an induced orientation on $\ker(D_u)$. If $\mu(u) = 1$ we know that $\ker(D_u)$ is one dimensional and generated by $\partial_s u$; then we assign $\nu^{\text{Floer}}(u) = 1$ if the orientation given as explained makes $\partial_s u$ positive and -1 otherwise.

On the other hand, a way to define $\nu^{\text{Morse}}(u)$ is the following. Letting $p = u(0) \in W^u_H(x^-) \cap W^s_H(x^+)$ we have a map

$$T_{x^-} W^u_H(x^-) \cong T_p W^u_H(x^-) \to \frac{T_p X}{T_p W^s_H(x^+)} \cong \frac{T_{x^+} X}{T_{x^+} W^s_H(x^+)} \cong T_{x^+} W^u_H(x^+)$$

The first and second isomorphisms are obtained by choosing trivializations of $u^* T W^u_H(x^-)$, $u^* T W^s_H(x^+)$ and $u^* TX$. Note that this map is canonical up to homotopy. By the Morse-Smale (see proposition 6.0.2) condition at p we know that the map is surjective and has kernel identified with

$$T_p W^u_H(x^-) \cap T_p W^s_H(x^+) = T_p \left(W^u_H(x^-) \cap W^s_H(x^+)\right).$$

If we orient $T_p W^u_H(x^\pm)$ we then get an orientation of $T_p\left(W^u_H(x^-)\cap W^s_H(x^+)\right)$. Note that

$$T_p\left(W^u_H(x^-)\cap W^s_H(x^+)\right)$$

is one dimensional since H is Morse-Smale and u is also a rigid Morse trajectory (by the fact that the relative Morse and Floer indices agree). If the orientation given makes $\partial_s u(0) \in T_p(W^u_H(x^-)\cap W^s_H(x^+))$ positive then we assign $\nu^{\mathrm{Morse}}(u) = 1$, and otherwise we assign -1.

We already saw that we can identify $\ker(D_{\Phi^-}) \cong T_{x^-}W^u_H(x^-)$ and $\ker(D_{\Phi^+}) \cong T_{x^+}W^u_H(x^+)$ via some "explicit" maps. But moreover we can identify $\ker(D_u)$ with $T_p\left(W^u_H(x^-)\cap W^s_H(x^+)\right)$ by sending $\xi \in \ker(D_u) \subseteq C^\infty(u^*TX)$ to

$$ev_0(\xi) = \xi(0) \in T_pX.$$

Indeed this is a bijection onto $T_p\left(W^u_H(x^-)\cap W^s_H(x^+)\right)$ because $\partial_s u \in \ker(D_u)$ is sent to $\partial_s u(0) \in T_p\left(W^u_H(x^-)\cap W^s_H(x^+)\right)$ which is a generator. All this fits in a (non-strictly commutative) diagram

$$\begin{tikzcd}
0 \arrow[r] & \ker(D_u) \arrow[r] \arrow[d, "ev_0"', "\cong"] & \ker(D_{\Phi_-}) \arrow[r] \arrow[d, "\cong"] & \ker(D_{\Phi_+}) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & T_p\left(W^u_H(x^-)\cap W^s_H(x^+)\right) \arrow[r] & T_{x^-}W^u_H(x^-) \arrow[r] & T_{x^+}W^u_H(x^+) \arrow[r] & 0
\end{tikzcd}$$

and it's enough to see that if we choose compatible orientations for Morse and Floer homologies, meaning that the two last isomorphisms preserve said orientations, then ev_0 also preserves the orientations induced by the exact sequences.

Showing this needs more careful considerations about the gluing map, in which we won't enter. We refer to the proof of [Sch93, Theorem 13]. The idea is that by picking a trivialization that makes the operators $D_u, D_{\Phi^\pm}$ particularly simple (as in [Sch93, Lemma B.2]) we can identify explicitly all the kernels, vertical maps and maps in the bottom row. Finally, considering a homotopy of operators D^τ_u, $D^\tau_{\Phi^+}$, $D^\tau_{\Phi^-} = D^\tau_u \# D^\tau_{\Phi^+}$, with $\tau \in [0,1]$, as in [Sch93, B.18] we also understand explicitly the gluing map – it reduces to pre-gluing – for $\tau = 1$ and from that we can prove what we need. $\square$

6.1.4 Concluding the proof

We are now ready to finish the proof of theorem 6.0.1. For simplicity of the notation we assume that X^g are connected and thus the twisted sectors of $\mathcal{X}$ are $\mathcal{X}^{(g)} = X^g/C(g)$ (see proposition 2.2.7).

First, proposition 6.1.1 (together with lemma 6.1.2) shows that for each $g \in G$ there is a correspondence between $\mathcal{P}_g(H)$ and $\mathrm{Crit}\left(H_{|X^g}\right)$. This correspondence is given by sending $x \in \mathrm{Crit}\left(H_{|X^g}\right)$ to $c^x_g \in \mathcal{P}_g(H)$. By our discussion of orientations, a critical point x is orientable if and only if the corresponding Hamiltonian orbit c^x_g is orientable, and thus the correspondence between $\mathcal{P}_g(H)$ and $\mathrm{Crit}\left(H_{|X^g}\right)$ restricts to a correspondence between $\mathcal{P}_g(H)^+$ and $\mathrm{Crit}\left(H_{|X^g}\right)^+$. Hence, comparing the construction of the Morse complex (2.1) and the decomposition of Floer homology in conjugacy classes (6.1) we have

$$CF^{(g)}_k(\mathcal{X}, H; \Lambda) \cong CM_{k-2\iota_{(g)}+n}\left(\mathcal{X}^{(g)}, H_{|X^g}; \Lambda\right) \tag{6.12}$$

where $\mathcal{X}^{(g)} = [X^g/C(g)]$ is the twisted sector associated with the conjugacy class (g). The index part follows from our computation of the index, see equation (6.6); indeed if c^x_g contributes in degree k to CF then x contributes in degree $\mathrm{ind}_x\left(H_{|X^g}\right) = k - 2\iota_{(g)} + n$ to CM.

It now remains to compare the differentials defined on each of these complexes, see (2.3) and (3.11). First, proposition 6.1.3 (and the discussion after its proof) shows that there is an identification between the moduli space of Floer trajectories $\widehat{\mathcal{M}}_g^{\mathrm{Floer}}\left(c^{x^-}, c^{x^+}\right)$ (see (3.3)) and the moduli space $\widehat{\mathcal{M}}^{\mathrm{Morse}}\left(x^-, x^+; H_{|X^g}\right)$ of flow trajectories of the gradient $\nabla H_{|X^g}$ (see (2.2)). The correspondence is the following: given $u \in \widehat{\mathcal{M}}^{\mathrm{Morse}}\left(x^-, x^+; H_{|X^g}\right)$, which is a map $u : \mathbb{R} \to X^g \subseteq X$, we get a Floer trajectory, also denoted by $u : [0,1] \times \mathbb{R} \to X$, given by $u(t,s) = u(s)$.

To see that the differentials really agree, first note that

$$\mu(u) = |c_g^{x^+}| - |c_g^{x^-}| = |x^+| - |x^-|,$$

so in both cases the relevant trajectories needed to define the differential are the ones between critical points with index difference 1. Clearly $\omega(u) = 0$ for any Floer trajectory that doesn't depend on t. By lemma 6.1.10 the signs $\nu(u)$ appearing in the definitions of the Morse and the Floer differentials agree, and thus we have a complete identification of the two differentials. Hence by theorem 2.4.4, it follows that

$$HF_k^{(g)}(\mathcal{X}, H; \Lambda) \cong H_{k-2\iota_{(g)}+n}(\mathcal{X}^{(g)}; \Lambda).$$

Note that here we're using that $\Lambda = \Lambda^{\mathrm{univ}}(\mathbb{Q})$ is a field of characteristic 0 to apply theorem 2.4.4. By the Poincaré duality for orbifold singular cohomology (see proposition 2.3.4) we have

$$H_{k-2\iota_{(g)}+n}(\mathcal{X}^{(g)}; \Lambda) \cong H^{k-2\iota_{(g)}+n}(\mathcal{X}^{(g)}; \Lambda).$$

Hence we get an isomorphism

$$HF_k(\mathcal{X}, H; \Lambda) = \bigoplus_{(g)} HF_k^{(g)}(\mathcal{X}, H; \Lambda) \cong \bigoplus_{(g)} H^{k-2\iota_{(g)}+n}(\mathcal{X}^{(g)}; \Lambda) = H_{CR}^{k+n}(\mathcal{X}; \Lambda).$$

Applying now Poincaré duality for orbifold Chen-Ruan cohomology (see 2.5.5) we get

$$HF_k(\mathcal{X}, H; \Lambda) \cong H_{CR}^{n-k}(\mathcal{X}; \Lambda)$$

and this finishes the proof of theorem 6.0.1.

Chapter 7

Final remarks and future work

There are a few directions in which we think the present work can be improved and extended. We discuss them informally in the subsequent sections. This entire chapter is mostly speculative and non-rigorous.

7.1 Invariance

One of the important features of Floer homology is that it is independent of the admissible pair (H, J). In this dissertation we didn't prove such a result, and this certainly would be nice to have. The proof of invariance usually goes by picking a homotopy of pairs and using the associated parametrized Floer equation to define a quasi-isomorphism between Floer complexes relative to these two pairs. We believe that the only real obstruction to reproduce this proof is that we need a transversality result to show that we can choose a homotopy in a way that the relevant moduli spaces of the parametrized Floer equation are smooth manifolds (see the discussion in chapter 5). When $\mathcal{X}$ has isolated singularities we can adapt the proof of theorem 5.2 to show a transversality result for the parametrized Floer equation and in principle it can be used to prove invariance in that case.

In section 5.3 we sketched a way to redefine Floer homology in a way that bypasses the problem of equivariant transversality. As we mentioned already, this idea can probably also be used to bypass the transversality problems in the proof of invariance, even with non-isolated singularities. It would also be nice to have a proof of 6.0.1 using the redefinition of Floer homology of 5.3. Together, these would imply the stronger result that Floer homology of orbifolds, when defined (either as in chapter 4 or section 5.3), is always isomorphic to Chen-Ruan cohomology.

7.2 Arbitrary orbifolds

A clear limitation of the present work is that it only applies to global quotient orbifolds. For us it's not clear how to work out the general case. For general orbifolds we don't have an easy description of the orbifold loop space as 2.2.5. We suspect that the idea of configuration spaces in [LU04] might be used to represent loops on orbifolds, but defining Floer homology using such representations looks very messy.

An interesting idea that might adapt to arbitrary orbifolds, perhaps in a way that we can even use existing literature, is to interpret our construction in terms of immersed Lagrangian Floer homology (see for instance [AJ10]). We discussed the relation between Floer homology with g-periodic boundary conditions on X and Lagrangian Floer homology on $X \times X$ (with symplectic form $(-\omega) \oplus \omega$) in section 3.4, in particular in proposition 3.4.2. Thanks to this, the complex $\widetilde{CF}(\mathcal{X}, H; \Lambda)$ might be interpreted as the Lagrangian Floer complex relative to the pair of immersed Lagrangians

$$j : L_{\phi_1} = \{(x, \phi_1(x)) : x \in X\} \hookrightarrow X \times X$$

and

$$\iota : \bigsqcup_{g \in G} L_g \hookrightarrow X \times X.$$

Note that j is an embedding but ι is just an immersion as it's not injective in the presence of isotropy.

Interestingly, if we have a general orbifold presented by a groupoid $\mathcal{G}$ with symplectic form $\omega \in \Omega^2(G_0)$ (see definition 2.3.7) we can recover this. We consider Lagrangian Floer homology on $G_0 \times G_0$ and replace ι by the immersion

$$\iota = (s, t) : G_1 \to G_0 \times G_0.$$

It's straightforward to check that this is an immersed Lagrangian and it's also clear that if $\mathcal{G} = G \ltimes X$ then ι is precisely what we already described. In the global quotient case the G-action on $\widetilde{CF}(\mathcal{X}, H; \Lambda)$ comes from the G-action on the immersed Lagrangian. In general, we have a groupoid $\mathcal{G}$-action on the immersed Lagrangian.

7.3 Product structure

Floer homology of a smooth manifold admits a ring structure. The product structure is defined by counting pairs-of-pants satisfying Floer equation with certain boundary conditions; details can be found in [Sch96]. It turns out that Floer homology is isomorphic, as a ring, to quantum cohomology.

It looks very reasonable that we can still define a product on the Floer homology $HF(\mathcal{X})$ of an orbifold. This product should come from the Donaldson product defined on g-periodic Hamiltonian Floer homology

$$HF(X, g) \otimes HF(X, h) \to HF(X, hg).$$

We refer to [MS12, Section 10.4]. In the perspective of Lagrangian Floer homology (see proposition 3.4.2) this product is a map

$$HLF(L_1, L_g) \otimes HLF(L_1, L_h) = HLF(L_1, L_g) \otimes HLF(L_g, L_{hg}) \to HLF(L_1, L_{hg}).$$

Such map can be defined by counting holomorphic triangles in $X \times X$ with certain boundary conditions involving L_1, L_g and L_{gh}. This product is part of a much larger structure called the Fukaya category of $X \times X$. The Fukaya category plays a central role in the homological mirror symmetry conjecture.

On the other hand, Chen-Ruan cohomology also admits a product structure, as we already mentioned. This product, as in the smooth case, is obtained as the classical limit of a quantum product on $H^*_{CR}(\mathcal{X}; \Lambda)$ where Λ is an appropriate Novikov ring.

Conjecture 7.3.1. *We can endow $HF(\mathcal{X}; \Lambda)$ with a pair-of-pants product that makes $HF(\mathcal{X}; \Lambda)$ isomorphic as a ring to the Chen-Ruan cohomology $H^*_{CR}(\mathcal{X}; \Lambda)$ with the (small) quantum product.*

7.4 Symplectic homology

As we mentioned in the introduction, our study of Floer homology of orbifolds was motivated by the possibility of also defining symplectic homology in a non-compact setting and adapting existing tools to the orbifold case. Here we mention a couple of applications we would get if we had such tools. We begin by describing very briefly symplectic homology of smooth contact type boundary manifolds; further details can be found in [Vit99, BO09, BO13, BO17].

Definition 7.4.1. *A compact symplectic manifold (W, ω) with boundary $\partial W = M$ is said to be of contact type boundary if there is a vector field X defined in a neighbourhood of M, transverse to M and pointing outwards along M, such that $\mathcal{L}_X\omega = \omega$. In this case the 1-form $\lambda = (\iota(X)\omega)_{|M}$ is a contact form in M.*

The contact type boundary hypothesis means that we can find a symplectic neighbourhood U of M that is symplectomorphic to $]-\delta, 0] \times M$ with symplectic form $d(e^r\lambda)$; the symplectomorphism is defined using the flow of X. This means that we can glue W and (part of) the symplectization of M along the boundary M. That is, we define the completion of W as

$$\widehat{W} = W \underset{M}{\cup} [0, +\infty[\times M.$$

We endow $\widehat{W}$ with a symplectic form $\widehat{\omega}$ defined as ω in W and $d(e^r\lambda)$ in $[0, +\infty[\times M$.

To define symplectic homology we consider Hamiltonians $H : S^1 \times \widehat{W} \to \mathbb{R}$ which are "linear" at infinity, that is, Hamiltonians for which there are constants $\alpha \in \mathbb{R}^+, \beta \in \mathbb{R}$ and $r_0 \in [0, +\infty[$ large enough such that

$$H(t, r, p) = \alpha e^r + \beta \text{ for } t \in S^1,\ r \in [r_0, +\infty[,\ p \in M.$$

The constant α is called the slope of H. If we assume that α is not a period of the Reeb flow R_λ then there are no periodic orbits in the region $[r_0, +\infty[\times M$. Moreover a maximum principle (see [Vit99, Lemma 1.8]) assures that Floer solutions stay in the compact set $W \cup [0, r_0] \times M$ (assuming also some behaviour of the almost complex form J at infinity). Thanks to this, we can define Floer homology $HF\left(\widehat{W}, H, J; \Lambda\right)$ with coefficients in a Novikov ring. If we assume that $c_1(\widehat{W}) = 0$ we can also give a grading to the Floer homology.

However, contrary to what happens when we consider Floer homology in a compact manifold, Floer homology depends on H, more specifically on the slope of H. Indeed if H_1, H_2 are Hamiltonians of slopes α_1, α_2 we can define continuation maps

$$HF\left(\widehat{W}, H_1, J_1; \Lambda\right) \to HF\left(\widehat{W}, H_2, J_2; \Lambda\right)$$

only when $\alpha_1 \leq \alpha_2$. Essentially this is because otherwise the maximum principle of Viterbo doesn't apply and there may be an infinite number of trajectories which should be counted to define the continuation maps. Symplectic homology is defined to be the direct limit of Floer homology groups when the slope tends to infinity, that is,

$$SH(W; \Lambda) = \operatorname*{colim}_{\alpha \to +\infty} HF\left(\widehat{W}, H_\alpha, J_\alpha; \Lambda\right)$$

where H_α is a Hamiltonian of slope α and J_α is an almost complex structure making the pair (H_α, J_α) regular.

Remark 7.4.2. The celebrated Viterbo theorem asserts that if Q is a spin manifold then the symplectic homology of its cotangent space is isomorphic to the homology of its loop space:

$$SH_k(T^*Q) \cong H_k(\Lambda Q).$$

There are a few important versions of symplectic homology. First, there are positive and negative versions $SH^+(W)$ and $SH^-(W)$. The negative version can be defined to be $FH\left(\widehat{W}, H_\varepsilon, J; \Lambda\right)$ where H_ε is a Hamiltonian of slope $\varepsilon > 0$ sufficiently small. We can compare this Floer homology with Morse homology to prove that

$$SH_*^-(W;\Lambda) \cong H_{*+n}(W, M; \Lambda) \cong H^{n-*}(W;\Lambda) \tag{7.1}$$

The positive and negative parts are related by a long exact sequence

$$\dots \to SH_{*+1}^+(W) \to SH_*^-(W) \to SH_*(W) \to SH_*^+(W) \to \dots \tag{7.2}$$

Informally speaking, $SH_*^+(W)$ retains information about the dynamics of the Reeb flow of M so in a way it's related to contact homology. However, while contact homology is intrinsically S^1-equivariant (it considers Reeb orbits up to reparametrization) $SH^+(W)$ is not. However, we can define S^1-equivariant versions of symplectic homology, namely $SH_*^{S^1}(W)$, $SH_*^{-,S^1}(W)$ and $SH_*^{+,S^1}(W)$. The positive S^1-equivariant symplectic homology $SH_*^{+,S^1}(W)$ is isomorphic to the (linearized) contact homology $HC_*(M)$ of $M = \partial W$; in particular it should be true that $SH_*^{+,S^1}(W)$ doesn't depend on the choice of the filling W but only on M.

A S^1-equivariant version of the long exact sequence (7.2) also exists. Moreover Oancea-Bourgeois proved a Gysin-type long exact sequence, relating S^1-equivariant and non-equivariant symplectic homology:

$$\dots \to SH_*^\dagger(W) \to SH_*^{\dagger,S^1}(W) \to SH_{*-2}^{\dagger,S^1}(W) \to SH_{*-1}^\dagger(W) \to \dots \tag{7.3}$$

for $\dagger = \emptyset, -, +$.

7.4.1 Applications of symplectic homology for orbifolds

Suppose now that we could define symplectic homology for a contact type boundary orbifold W with smooth boundary $M = \partial W$. It's reasonable to expect that in this situation the isomorphism (7.1) would be replaced by

$$SH_*^-(W;\Lambda) \cong H_{CR}^{n-*}(W;\Lambda) \tag{7.4}$$

and would have the same proof as theorem 6.0.1. There are several situations in which we are lead to consider an orbifold filling of a smooth contact manifold.

Example 7.4.3. Let $G \subseteq SL(n;\mathbb{C})$ be a finite group whose non-trivial elements don't have 1 as an eigenvalue. The group G acts on $\mathbb{C}^n$ and on $S^{2n-1} \subseteq \mathbb{C}^n$ and S^{2n-1}/G is a smooth contact manifold. This manifold admits an obvious crepant (meaning $c_1 = 0$) orbifold symplectic filling, namely $\mathbb{C}^n/G$. In some cases, for instance $\mathbb{R}P^7 = S^7/(\mathbb{Z}/2)$ there are no known crepant smooth fillings.

Example 7.4.4. An interesting family of examples studied in [AMM] are toric Gorenstein contact manifolds. These are classified by the so called toric diagrams which are integral polytopes in $\mathbb{R}^{n-1}$ where $2n-1$ is the dimension of the contact manifold. Crepant orbifold fillings of such contact manifolds can be obtained in a very combinatorial way from triangulations of the toric diagram. Very rarely (only when the triangulation is unimodal) these fillings are actually smooth.

The first example is very much related to the McKay correspondence. Indeed if $W \to \mathbb{C}^n/G$ is a crepant resolution then W is a crepant filling of S^{2n-1}/G. This idea was used in [MR18] to interpret the McKay correspondence using symplectic homology, but avoiding the symplectic homology of $\mathbb{C}^n/G$.

Regarding the second example, we proved in [AMM] an isomorphism of the form

$$HC_*(M) \cong \bigoplus_{j\geq 0} H_{CR}^{n-*+2j+1}(W) \tag{7.5}$$

for these toric crepant fillings W which are determined from a triangulation of the toric diagram (note that we're using a different grading convention). Our proof was essentially combinatorial, but if we had the tools of symplectic homology for orbifolds we could give a nice alternative proof as we now proceed to sketch.

First, one has to prove that $SH_*(W)$ vanishes. For this we have to choose Hamiltonians H_α of arbitrarily large slope α for which the Floer complex is concentrated in arbitrarily high degree. By the long exact sequence (7.2) we then get that

$$SH_*^+(W) \cong SH_{*-1}^-(W) \cong H_{CR}^{n-*+1}(W).$$

Now we would need to shown that the Gysin sequence (7.3) with $\dagger = +$ splits in short exact sequences

$$0 \to SH_*^+(W) \to SH_*^{+,S^1}(W) \to SH_{*-2}^{+,S^1}(W) \to 0.$$

This is true by naturality of the Gysin sequence with respect to the long exact sequence (7.2) and because the negative version of the Gysin sequence splits. From that and the isomorphism $HC_*(M) \cong SH_*^{+,S^1}(W)$ the isomorphism (7.5) follows.

These arguments can be made rigorous when W is smooth, but to make them work when W is an orbifold requires a development of symplectic homology for orbifolds.

Appendix A

The Conley-Zehnder index for paths of symplectic matrices

In this appendix we explain the main properties of the Conley-Zehnder index, introduced in [CZ84], for admissible paths of symplectic matrices. The Conley-Zehnder index is essential to Floer homology as the its grading is based on it (see 4.2.6). For a detailed account of the Conley-Zehnder index, including its construction and proof of the properties we'll state, see [Gut12].

Definition A.0.1. *We say that a path $\Phi : [0,1] \to Sp(2n)$ is admissible if $\Phi(0) = \mathrm{Id}$ and $\Phi(1)$ doesn't have 1 as an eigenvalue. We denote by $SP(n)$ the set of admissible paths.*

Note that admissible paths are associated to non-degenerate orbits. Indeed a Hamiltonian orbit γ is non-degenerate (see 3.1.2) if and only if the path Φ_γ (in 4.2.6) is admissible.

Before we introduce the Conley-Zehnder index we recall the Maslov index for loops of symplectic matrices. Since the inclusion $U(n) \to Sp(2n)$ induces a homotopy equivalence, we have an isomorphism

$$[S^1, Sp(2n)] \cong \pi_1(Sp(2n)) \xrightarrow{\cong} \pi_1(U(n)) \xrightarrow{\cong} \mathbb{Z}.$$

The last isomorphism is induced by the determinant map $U(n) \to U(1)$ and the isomorphism $\pi_1(U(1)) \cong \mathbb{Z}$ that sends the class $[t \mapsto e^{2\pi i t}] \in \pi_1(U(1))$ to $1 \in \mathbb{Z}$. Now the Maslov index $\mu(\alpha)$ of a loop $\alpha : S^1 \to Sp(2n)$ is the image of its class $[\alpha]$ in $\mathbb{Z}$ through this isomorphism.

Theorem A.0.2. *There is a unique map $\mu_{CZ} : SP(n) \to \mathbb{Z}$ with the following properties:*

1. (Homotopy) *μ_{CZ} is locally constant, that is, if $\{\Phi^s\}_{s\in[0,1]} \in SP(n)$ is a homotopy of admissible paths then $\mu_{CZ}(\Phi^0) = \mu_{CZ}(\Phi^1)$.*

2. (Loop) *If $\alpha : [0,1] \to Sp(2n)$ is a loop with $\alpha(0) = \alpha(1) = \mathrm{Id}$ then*

$$\mu_{CZ}(\alpha\Phi) = \mu_{CZ}(\Phi) + 2\mu(\alpha).$$

3. (Signature) *Let $S \in M_{2n\times 2n}(\mathbb{R})$ be a symmetric non-singular matrix with $\|S\| < 2\pi$ and let $\Phi(t) = \exp(tJ_0 S)$ for $t \in [0,1]$, where J_0 is the standard complex structure on $\mathbb{R}^{2n}$. Then*

$$\mu_{CZ}(\Phi) = n - \mathrm{ind}(S)$$

where $\mathrm{ind}(S)$ is the number of negative eigenvalues of S.

Proof. See [Gut12, Theorems 35, 37]. ☐

We call $\mu_{CZ}(\Phi)$ the Conley-Zehnder index of an admissible path Φ. The Conley-Zehnder index obeys certain other properties that we now proceed to state.

Proposition A.0.3. *Let $\Phi \in SP(n)$. The Conley-Zehnder index satisfies the following properties:*

4. (Naturality) *If $\psi \in Sp(2n)$ then*
$$\mu_{CZ}(\Phi) = \mu_{CZ}(\psi\Phi\psi^{-1}).$$
5. (Zero) *If $\Phi(t)$ has no eigenvalue on the unit circle for every $t > 0$ then $\mu_{CZ}(\Phi) = 0$.*
6. (Direct sum) *If $\Phi_1 \in SP(n_1)$ and $\Phi_2 \in SP(n_2)$ then $\Phi_1 \oplus \Phi_2 \in SP(n)$ where $n = n_1 + n_2$ and*
$$\mu_{CZ}(\Phi_1 \oplus \Phi_2) = \mu_{CZ}(\Phi_1) + \mu_{CZ}(\Phi_2).$$
7. (Parity)
$$\operatorname{sign}\det(\mathrm{Id} - \Phi(1)) = (-1)^{n-\mu_{CZ}(\Phi)}.$$
8. (Inverse) *Denoting by $\Phi^{-1}(t) = \Phi(t)^{-1}$ the inverse path we have*
$$\mu_{CZ}(\Phi^{-1}) = -\mu_{CZ}(\Phi).$$

Proof. See [Gut12, Theorem 35]. ☐

We will need the computation of the Conley-Zehnder index of certain paths of symplectic matrices.

Proposition A.0.4. *Let $\lambda \in \mathbb{R} \setminus \mathbb{Z}$ and let $\Phi \in SP(1)$ be defined by*
$$\Phi(t) = [e^{2\pi i\lambda t}] = \begin{bmatrix} \cos(2\pi\lambda t) & -\sin(2\pi\lambda t) \\ \sin(2\pi\lambda t) & \cos(2\pi\lambda t) \end{bmatrix} \in U(n) \subseteq Sp(2n).$$
Then
$$\mu_{CZ}(\Phi) = 2\lfloor\lambda\rfloor + 1.$$

Proof. Write $\lambda = N + \ell$ where $N = \lfloor\lambda\rfloor$ and $\ell = \{\lambda\} \in]0,1[$. We can write $\Phi(t) = \alpha(t)\Psi(t)$ where $\alpha(t) = e^{2\pi iNt}$ and
$$\Psi(t) = e^{2\pi i\ell t} = \exp(tJ_0S)$$
where
$$S = \begin{bmatrix} 2\pi\ell & 0 \\ 0 & 2\pi\ell \end{bmatrix}.$$
Note that $\|S\| < 2\pi$ and S has index 0 (because $\ell > 0$). By the loop and the signature properties we have
$$\mu_{CZ}(\Phi) = \mu_{CZ}(\Psi) + 2\mu(\alpha) = 1 - \operatorname{ind}(S) + 2N = 2\lfloor\lambda\rfloor + 1. \quad ☐$$

Remark A.0.5. The Conley-Zehnder index admits an extension to the set of symplectic paths Φ with $\Phi(0) = \mathrm{Id}$. This is sometimes called the Robbin-Salamon index of a path, and it takes values in $\frac{1}{2}\mathbb{Z}$. This generalization was introduced in [RS93].

Appendix B

Equivariant cohomology and equivariant bundles

In this appendix we fix a compact Lie group G acting on a space X. The equivariant cohomology of the G-space X is constructed using the universal G-principal bundle.

Theorem B.0.1. *Let G be a compact Lie group. Then there exists a (unique up to weak homotopy equivalence) contractible space EG with a free G-action. If we denote $BG = EG/G$ then the projection $EG \to BG$ is called the universal principal G-bundle.*

Proof. See [Coh02, Theorem 2.21]. □

The reason for this to be called "universal" is the following theorem:

Theorem B.0.2. *Let X be a manifold (or a CW-complex). If $P \to X$ is a principal G-bundle, then there exists continuous map $f : X \to BG$ uniquely defined up to homotopy such that $P = f^*EG$. Hence there is a bijective correspondence between equivalence classes of principal G-bundles over X and $[X, BG]$.*

Proof. See [Coh02, Theorem 2.8]. □

The point of the Borel construction and of equivariant cohomology is the following: when the action of G on X is not free, the quotient X/G is not very well behaved. For example there is no information about G if X is just a point, and $H^*(X/G)$ is not homotopy invariant. The way to correct this problem is to substitute the G-space X by a homotopy equivalent space in which G acts freely.

Definition B.0.3. *Let G be a compact Lie group acting on a topological space X. Its Borel construction is the space*

$$X \times_G EG = (X \times EG)/G$$

where G acts diagonally on $X \times EG$.

We define the equivariant cohomology of the G-space X to be

$$H^*_G(X) = H^*\left(X \times_G EG\right).$$

Associated to the Borel construction, and since G acts freely on EG, the projection in the second component gives a fibration

$$\begin{tikzcd} X \arrow[r] & X \times_G EG \arrow[d] \\ & BG. \end{tikzcd}$$

On the other hand, if G also acts freely on X we get a fibration

$$\begin{tikzcd} EG \arrow[r] & X \times_G EG \arrow[d] \\ & X/G \end{tikzcd}$$

and since the fiber EG is contractible it follows that when G acts freely on X then $X \times_G EG \simeq X/G$, and in particular $H^*_G(X) \cong H^*(X/G)$. This property and invariance of H^*_G up to G-equivariant homotopy equivalences are enough to characterize the functor H_G. In the general situation of a non-free action it's not true that $H^*_G(X) \cong H^*(X/G)$. However, when the action is almost free, this is still true with rational coefficients.

Proposition B.0.4. *Let X be a manifold and G a compact Lie group acting on X. Suppose that the action is almost free, that is, the isotropy groups G_x are finite for every $x \in X$. Then the projection $X \times_G EG \to X/G$ induces an isomorphism on rational cohomology*

$$H^*_G(X;\mathbb{Q}) \xrightarrow{\cong} H^*(X/G;\mathbb{Q}).$$

Proof. This is a particular case of 2.3.3 with $\mathcal{G} = G \ltimes X$. □

In particular when G is a finite group proposition B.0.4 implies that $H^*(BG;\mathbb{Q}) = 0$.

Example B.0.5. When $G = S^1$ we can take $EG = S^\infty = \operatorname*{colim}_{N\to+\infty} S^{2N+1}$ and $BG = \mathbb{C}P^\infty$; note that S^1 acts on $S^{2N+1} \subseteq \mathbb{C}^{N+1}$ with the diagonal action. In particular it follows that

$$H^*_{S^1}(\mathrm{pt}) = H^*(\mathbb{C}P^\infty) \cong \mathbb{Z}[u]$$

where u is a generator in degree 1.

Similarly, if $G = \mathbb{Z}/p$ then $EG = S^\infty$ and $BG = L^\infty_p$ is the infinite dimensional Lens space. Then

$$H^*_{\mathbb{Z}/p}(\mathrm{pt}) = H^*(L^\infty_p) = \begin{cases} \mathbb{Z} & \text{if } * = 0 \\ \mathbb{Z}/p & \text{if } * = 2j,\ j > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Example B.0.6. When G is finite, $EG \to BG$ is the universal cover of BG with automorphism group G. Since EG is contractible it follows that $BG = K(G,1)$.

B.1 Equivariant vector bundles

Definition B.1.1. *A G-equivariant vector bundle over a G-space X is a vector bundle $\pi : E \to X$ together with an action of G on E by fiberwise linear transformations which lifts the action on X. That is, for each $x \in X, g \in G$ we have a linear transformation*

$$\rho_x(g) : E_x \to E_{gx}.$$

An (iso)morphism of G-equivariant bundles is an (iso)morphism of bundles which is G-equivariant with respect to the actions on the total spaces.

Given a G-equivariant vector bundle $E \to X$ we have an associated vector bundle over the Borel construction $X \times_G EG$

$$E \times_G EG \to X \times_G EG.$$

that pulls-back to $\pi \times \mathrm{id}_{EG} : E \times EG \to X \times EG$ via the projection map $X \times EG \to X \times_G EG$. Using this construction we can define characteristic classes of an equivariant bundle (Chern classes, Pontrjagin classes, Stiefel-Whitney classes, etc.) as the characteristic classes of this new bundle over the Borel construction. We are mainly interested in Chern classes, and even more specifically in the first Chern class.

Definition B.1.2. *Let $E \to X$ be a G-equivariant complex vector bundle. We define its equivariant Chern classes $c_k^G(E) \in H_G^{2k}(X)$ to be*

$$c_k^G(E) = c_k(E \times_G EG) \in H^{2k}(X \times_G EG) = H_G^{2k}(X).$$

A remarkable fact is that equivariant line bundles are still classified by the (equivariant) Chern class.

Theorem B.1.3. *Let G be a compact Lie group acting on a connected manifold X. Then the first equivariant Chern class c_1^G gives a rise to a one-to-one correspondence between G-equivariant line bundles over X and elements of $H_G^2(X;\mathbb{Z})$.*

Proof. The proof can be found in [GGK02, Theorem C.47]. We'd just like to remark that the proof goes by showing that the above construction gives a one-to-one correspondence between (equivalence classes of) G-equivariant line bundles over X and (equivalence classes of) line bundles over $EG \times_G X$, and then the result follows from its well known non-equivariant version. □

We shall now discuss a little bit conditions for a G-equivariant bundle to be trivial; by the previous theorem, if E is a line bundle this is equivalent to $c_1^G(E) = 0$. Clearly if E is trivial as a G-equivariant bundle then it's also trivial as a vector bundle, so $c_1(E) = 0$. Another restriction that we have in a trivial G-equivariant bundle is that it must be an honest bundle.

Definition B.1.4. *We say that a G-equivariant vector bundle $E \to X$ is honest if for every $x \in X$ the representation of the isotropy group G_x on E_x is trivial.*

Equivalently, this means that for every $g \in G, x \in X$ such that $gx = x$ the map $\rho_x(g) : E_x \to E_x$ is the identity.

We can think of this definition as follows: the fiber of the induced map $E/G \to X/G$ at a point $[x] \in X/G$ is identified with $\mathbb{C}^r/G_x$ where G_x acts on $\mathbb{C}^r \cong E_x$ as explained above. So an honest bundle is such that after we quotient by G we can still identify the fibers with $\mathbb{C}^r$ and not with a quotient of $\mathbb{C}^r$. Note that this condition is vacuous when the action is free.

Example B.1.5. Consider the S^1-equivariant vector bundle $S^{2n+1} \times \mathbb{C} \to S^{2n+1}$ where S^1 acts diagonally on $S^{2n+1} \times \mathbb{C}$. This is trivial as a vector bundle and is a honest bundle since the action is free. However, it's not trivial as a S^1-equivariant vector bundle. Indeed after quotienting by S^1 we get the tautological bundle $\mathcal{O}(-1)$ over $\mathbb{C}P^n = S^{2n+1}/S^1$, which is non-trivial.

However, in certain circumstances (in which the action is far from being free) a honest trivial vector bundle is automatically trivial as a G-equivariant vector bundle.

Proposition B.1.6. *Let G be a compact Lie group acting on a connected manifold X. Assume that $H^1(X;\mathbb{Z}) = 0$ (or, more generally, that $H^1(BG; H^1(X)) = 0$) and that every $g \in G$ fixes some point, that is, $X^g \neq \emptyset$. Let E be a G-equivariant line bundle over X. Then $c_1^G(E) = 0$ if and only if $c_1(E) = 0$ and E is a honest G-equivariant bundle.*

Proof. The "only if" direction is clear without any of the conditions asked, as discussed before. For the non-trivial direction, we consider the Leray-Serre spectral sequence for the fibration

$$X \xrightarrow{\pi} X \times_G EG \xrightarrow{\sigma} BG$$

which has 2-page

$$E_2^{p,q} = H^p(BG; H^q(X)) \Rightarrow H_G^{p+q}(X).$$

We are assuming that

$$E_2^{1,1} = H^1(BG; H^1(X)) = 0$$

and in this case the sequence

$$E_2^{2,0} = H^2(BG) \xrightarrow{\sigma^*} H_G^2(X) \xrightarrow{\pi^*} H^2(X)$$

is exact. Indeed by the convergence of the spectral sequence

$$0 \longrightarrow E_\infty^{2,0} \longrightarrow H_G^2(X) \longrightarrow E_\infty^{0,2} \longrightarrow 0$$

is short exact and $E_2^{2,0} \to E_\infty^{2,0}$ and $E_\infty^{0,2} \to E_2^{0,2} = H^2(X)^G \hookrightarrow H^2(X)$ are surjective and injective, respectively.

Thus if we assume that $c_1(E) = \pi^* c_1^G(E) = 0$ it follows that $c_1^G(E) = \sigma^*\beta$ for some $\beta \in H^2(BG)$. By theorem B.1.3 there is a G-equivariant line bundle over a point $*$ with first Chern class $\beta \in H_G^2(*) = H^2(BG)$; note that giving such a line bundle is the same as giving a G-action on the only fiber $\mathbb{C}$ or, equivalently, a homomorphism $G \to \mathbb{C}^\times$ (called a character). By naturality of the Chern classes, and again theorem B.1.3, it follows that $E \to X$ is the pullback of $\mathbb{C} \to *$ by the constant map $X \to *$:

$$\begin{array}{ccc} G & & G \\ \curvearrowright & & \curvearrowright \\ E & \longrightarrow & \mathbb{C} \\ \big\downarrow & & \big\downarrow \\ X & \longrightarrow & *. \end{array}$$

For any $g \in G$ pick $x \in X^g$; since g acts trivially on E_x (because the bundle is honest), g must also act trivially on $\mathbb{C}$. Hence the G-equivariant bundle $\mathbb{C} \to *$ is trivial as a G-equivariant bundle, so $\beta = 0$ and $c_1^G(E) = 0$. ∎

Bibliography

[Abo15] Mohammed Abouzaid. Symplectic cohomology and Viterbo's theorem. In *Free loop spaces in geometry and topology*, volume 24 of *IRMA Lect. Math. Theor. Phys.*, pages 271–485. Eur. Math. Soc., Zürich, 2015.

[AD14] Michèle Audin and Mihai Damian. *Morse theory and Floer homology*. Universitext. Springer, London; EDP Sciences, Les Ulis, 2014. Translated from the 2010 French original by Reinie Erné.

[AJ10] Manabu Akaho and Dominic Joyce. Immersed Lagrangian Floer theory. *J. Differential Geom.*, 86(3):381–500, 2010.

[ALR07] Alejandro Adem, Johann Leida, and Yongbin Ruan. *Orbifolds and stringy topology*, volume 171 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2007.

[AMM] Miguel Abreu, Leonardo Macarini, and Miguel Moreira. Contact invariants of Gorenstein toric contact manifolds, the Ehrhart polynomial and Chen-Ruan cohomology. In preparation.

[AMM18] Miguel Abreu, Leonardo Macarini, and Miguel Moreira. On contact invariants of non-simply connected Gorenstein toric contact manifolds. *arXiv e-prints*, page arXiv:1812.10361, Dec 2018. Available at https://arxiv.org/abs/1812.10361.

[BO09] Frédéric Bourgeois and Alexandru Oancea. Symplectic homology, autonomous Hamiltonians, and Morse-Bott moduli spaces. *Duke Math. J.*, 146(1):71–174, 2009.

[BO13] Frédéric Bourgeois and Alexandru Oancea. The Gysin exact sequence for S^1-equivariant symplectic homology. *J. Topol. Anal.*, 5(4):361–407, 2013.

[BO17] Frédéric Bourgeois and Alexandru Oancea. S^1-equivariant symplectic homology and linearized contact homology. *Int. Math. Res. Not. IMRN*, (13):3849–3937, 2017.

[CH14] Cheol-Hyun Cho and Hansol Hong. Orbifold Morse-Smale-Witten complexes. *Internat. J. Math.*, 25(5):1450040, 35, 2014.

[CH17] Cheol-Hyun Cho and Hansol Hong. Finite group actions on Lagrangian Floer theory. *J. Symplectic Geom.*, 15(2):307–420, 2017.

[CLS11] David A. Cox, John B. Little, and Henry K. Schenck. *Toric varieties*, volume 124 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2011.

[Coh02] Ralph Cohen. The topology fo fiber bundles, 2002. Lecture notes.

[CP14] Cheol-Hyun Cho and Mainak Poddar. Holomorphic orbi-discs and Lagrangian Floer cohomology of symplectic toric orbifolds. *J. Differential Geom.*, 98(1):21–116, 2014.

[CR02] Weimin Chen and Yongbin Ruan. Orbifold Gromov-Witten theory. In *Orbifolds in mathematics and physics (Madison, WI, 2001)*, volume 310 of *Contemp. Math.*, pages 25–85. Amer. Math. Soc., Providence, RI, 2002.

[CR04] Weimin Chen and Yongbin Ruan. A new cohomology theory of orbifold. *Communications in Mathematical Physics*, 248(1):1–31, 2004.

[CZ83] C. C. Conley and E. Zehnder. The Birkhoff-Lewis fixed point theorem and a conjecture of V. I. Arnol' d. *Invent. Math.*, 73(1):33–49, 1983.

[CZ84] Charles Conley and Eduard Zehnder. Morse-type index theory for flows and periodic solutions for Hamiltonian equations. *Comm. Pure Appl. Math.*, 37(2):207–253, 1984.

[DHVW85] L. Dixon, J. A. Harvey, C. Vafa, and E. Witten. Strings on orbifolds. *Nuclear Phys. B*, 261(4):678–686, 1985.

[DM93] Michael Dellnitz and Ian Melbourne. The equivariant Darboux theorem. In *Exploiting symmetry in applied and numerical analysis (Fort Collins, CO, 1992)*, volume 29 of *Lectures in Appl. Math.*, pages 163–169. Amer. Math. Soc., Providence, RI, 1993.

[DS94] Stamatis Dostoglou and Dietmar A. Salamon. Cauchy-Riemann operators, self-duality, and the spectral flow. In *First European Congress of Mathematics, Vol. I (Paris, 1992)*, volume 119 of *Progr. Math.*, pages 511–545. Birkhäuser, Basel, 1994.

[dS18] Miguel M. dos Santos. The Arnol'd conjecture and Floer homology. Master's thesis, Instituto Superior Técnico, 2018.

[FH93] A. Floer and H. Hofer. Coherent orientations for periodic orbit problems in symplectic geometry. *Math. Z.*, 212(1):13–38, 1993.

[FHS95] Andreas Floer, Helmut Hofer, and Dietmar Salamon. Transversality in elliptic Morse theory for the symplectic action. *Duke Math. J.*, 80(1):251–292, 1995.

[Flo87] Andreas Floer. Morse theory for fixed points of symplectic diffeomorphisms. *Bull. Amer. Math. Soc. (N.S.)*, 16(2):279–281, 1987.

[Flo89] Andreas Floer. Symplectic fixed points and holomorphic spheres. *Comm. Math. Phys.*, 120(4):575–611, 1989.

[FO99] Kenji Fukaya and Kaoru Ono. Arnold conjecture and Gromov-Witten invariant for general symplectic manifolds. In *The Arnoldfest (Toronto, ON, 1997)*, volume 24 of *Fields Inst. Commun.*, pages 173–190. Amer. Math. Soc., Providence, RI, 1999.

[FOOO09a] Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, and Kaoru Ono. *Lagrangian intersection Floer theory: anomaly and obstruction. Part I*, volume 46 of *AMS/IP Studies in Advanced Mathematics*. American Mathematical Society, Providence, RI; International Press, Somerville, MA, 2009.

[FOOO09b] Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, and Kaoru Ono. *Lagrangian intersection Floer theory: anomaly and obstruction. Part II*, volume 46 of *AMS/IP Studies in Advanced Mathematics*. American Mathematical Society, Providence, RI; International Press, Somerville, MA, 2009.

[GGK02] Victor Guillemin, Viktor Ginzburg, and Yael Karshon. *Moment maps, cobordisms, and Hamiltonian group actions*, volume 98 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2002. Appendix J by Maxim Braverman.

[Gro85] Mikhail Gromov. Pseudo holomorphic curves in symplectic manifolds. *Invent. Math.*, 82(2):307–347, 1985.

[Gut12] Jean Gutt. The Conley-Zehnder index for a path of symplectic matrices. *arXiv e-prints*, page arXiv:1201.3728, Jan 2012. Available at https://arxiv.org/abs/1201.3728.

[Hep09] Richard Hepworth. Morse inequalities for orbifold cohomology. *Algebr. Geom. Topol.*, 9(2):1105–1175, 2009.

[HHM17] Doris Hein, Umberto Hryniewicz, and Leonardo Macarini. Transversality for local morse homology with symmetries and applications. *Mathematische Zeitschrift*, pages 1–87, 2017. Available at https://arxiv.org/abs/1702.04609.

[HS95] Helmut. Hofer and Dietmar A. Salamon. Floer homology and Novikov rings. In *The Floer memorial volume*, volume 133 of *Progr. Math.*, pages 483–524. Birkhäuser, Basel, 1995.

[KS02] Mikhail Khovanov and Paul Seidel. Quivers, Floer cohomology, and braid group actions. *J. Amer. Math. Soc.*, 15(1):203–271, 2002.

[Lan99] Serge Lang. *Fundamentals of differential geometry*, volume 191 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1999.

[LS14] Yuan-Pin Lee and Mark Shoemaker. A mirror theorem for the mirror quintic. *Geom. Topol.*, 18(3):1437–1483, 2014.

[LT97] Eugene Lerman and Susan Tolman. Hamiltonian torus actions on symplectic orbifolds and toric varieties. *Trans. Amer. Math. Soc.*, 349(10):4201–4230, 1997.

[LU04] Ernesto Lupercio and Bernardo Uribe. Inertia orbifolds, configuration spaces and the ghost loop space. *Q. J. Math.*, 55(2):185–201, 2004.

[McL16] Mark McLean. Reeb orbits and the minimal discrepancy of an isolated singularity. *Inventiones mathematicae*, 204(2):505–594, 2016.

[MP97] Ieke Moerdijk and Dorette A. Pronk. Orbifolds, sheaves and groupoids. *K-Theory*, 12(1):3–21, 1997.

[MR18] Mark McLean and Alexander F. Ritter. The McKay correspondence via Floer theory. *arXiv preprint arXiv:1802.01534*, 2018. Available at https://arxiv.org/abs/1802.01534.

[MS12] Dusa McDuff and Dietmar A. Salamon. *J-holomorphic curves and symplectic topology*, volume 52 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, second edition, 2012.

[MS17] Dusa McDuff and Dietmar Salamon. *Introduction to symplectic topology*. Oxford Graduate Texts in Mathematics. Oxford University Press, Oxford, third edition, 2017.

[Oh93] Yong-Geun Oh. Floer cohomology of Lagrangian intersections and pseudo-holomorphic disks. I. *Comm. Pure Appl. Math.*, 46(7):949–993, 1993.

[Ped18] Andrés Pedroza. A quick view of Lagrangian Floer homology. In *Geometrical themes inspired by the N-body problem*, volume 2204 of *Lecture Notes in Math.*, pages 91–125. Springer, Cham, 2018.

[RS93] Joel Robbin and Dietmar Salamon. The Maslov index for paths. *Topology*, 32(4):827–844, 1993.

[RS01] Joel W. Robbin and Dietmar A. Salamon. Asymptotic behaviour of holomorphic strips. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 18(5):573–612, 2001.

[Rua06] Yongbin Ruan. The cohomology ring of crepant resolutions of orbifolds. In *Gromov-Witten theory of spin curves and orbifolds*, volume 403 of *Contemp. Math.*, pages 117–126. Amer. Math. Soc., Providence, RI, 2006.

[Sat56] Ichiro Satake. On a generalization of the notion of manifold. *Proc. Nat. Acad. Sci. U.S.A.*, 42:359–363, 1956.

[Sch93] Matthias Schwarz. *Morse homology*, volume 111 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1993.

[Sch96] Matthias Schwarz. *Cohomology operations from S^1-cobordism in Floer homology*. PhD thesis, ETH Zürich, 1996. Available at http://www.math.uni-leipzig.de/~schwarz/diss.pdf.

[Sma65] Stephen Smale. An infinite dimensional version of Sard's theorem. *Amer. J. Math.*, 87:861–866, 1965.

[SS10] Paul Seidel and Ivan Smith. Localization for involutions in Floer cohomology. *Geom. Funct. Anal.*, 20(6):1464–1501, 2010.

[SZ92] Dietmar Salamon and Eduard Zehnder. Morse theory for periodic solutions of Hamiltonian systems and the Maslov index. *Comm. Pure Appl. Math.*, 45(10):1303–1360, 1992.

[Vit99] Claude Viterbo. Functors and computations in Floer homology with applications. I. *Geom. Funct. Anal.*, 9(5):985–1033, 1999.

[Was69] Arthur G. Wasserman. Equivariant differential topology. *Topology*, 8:127–150, 1969.

[Yas04] Takehiko Yasuda. Twisted jets, motivic measures and orbifold cohomology. *Compos. Math.*, 140(2):396–422, 2004.

[Zil02] Fabian Ziltener. Floer-Gromov-compactness and stable connecting orbits. Master's thesis, ETH Zürich, 2002.