RIEMANN SURFACES AND MODULAR FUNCTIONS

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Contents

1. Introduction

This paper is divided in two parts: the first is an introduction to Riemann surfaces and some of their fundamental results and the second one is devoted to elliptic and modular functions.

For the first part, our exposition will follow closely [6]. We begin by defining a Riemann surface (which is a 1-dimensional complex manifold) and holomorphic functions between them. In 2.1 we prove some generalizations of classical complex analysis results, such as the identity theorem and the open mapping theorem. Section 2.2 is about differential forms and integration in a Riemann surface. In section 2.3 we introduce sheaves and sheaf cohomology, a very powerful tool in Riemann surfaces. After that, in 2.4 we present the exact sequence of cohomology of sheaves and Dolbeault lemma. Finally, 2.5 is dedicated to some central results in Riemann surfaces: the Riemann-Roch theorem, the uniformization theorem (which we will only prove in the compact case, as an easy consequence of Riemann-Roch theorem), Serre duality and the Hurwitz formula.

In the second part we talk about elliptic functions and modular functions/forms. Here we will combine the approach in [2] with the Riemann surface's tools we developed earlier. This will make some of the proofs and results much more elegant and motivated; we will usually skip the more computational proofs. We begin with the study of elliptic functions in 3.1, which can be seen as meromorphic functions on a torus; in particular we construct the Weierstrass elliptic function $\wp$. In section 3.2 we consider the modular group and modular functions, motivated by the problem of determining the complex tori up to biholomorphism. After that, in 3.3 we define Hecke operators and use them to prove that the Fourier coefficients of Δ satisfy a multiplicative property. In the last section 3.4 we study a class of subgroups of the modular group and study the Riemann surfaces associated to them using the Dedekind eta functions to construct explicitly isomorphisms with the Riemann sphere when possible; we use the results obtained and Hecke operators to show some remarkable congruences satisfied by the Fourier coefficients of J.

2. Riemann surfaces

2.1. **Riemann surfaces and holomorphic functions.** We begin by introducing Riemann surfaces. A Riemann surface is a 1-dimensional complex manifold. Instead of requiring that the transition functions between charts are smooth (as in the definition of a smooth manifold), in a complex manifold they must be holomorphic, which gives a much stronger structure to Riemann surfaces.

Definition 1. *Let X be a two dimensional connected manifold. A complex chart on X is a pair (U,φ) where $U \subseteq X$ is an open set and $\varphi : U \to V$ is a homeomorphism where $V \subseteq \mathbb{C}$ is an open set. Two charts $(U_i, \varphi_i), i = 1, 2$ are said to be compatible if*

$$\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$$

is a biholomorphic complex function.

A complex atlas on X is a family of compatible complex charts $\{(U_\alpha, \varphi_\alpha)\}$ such that $\{U_\alpha\}$ are a cover of X. A complex structure on X is a maximal atlas. We say that a 2-dimensional manifold X equiped with a complex structure is a Riemann surface.

We give some examples of Riemann surfaces that will be relevant in the future.

Example 1.
(1) The complex plane $\mathbb{C}$. In this case we have a global atlas consisting simply of $(\mathbb{C}, \mathrm{id}_{\mathbb{C}})$. More generally any open set of $\mathbb{C}$ is a Riemann surface. A connected open set of $\mathbb{C}$ is called a domain. In particular the unit disk D is a domain.
(2) The Riemann sphere $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. Topologically this is the Alexandrov compactification of $\mathbb{C}$ and we can give it a complex structure with the atlas given by $(\mathbb{P}^1 \setminus \{\infty\}, \mathrm{id}_{\mathbb{C}})$ and $(\mathbb{P}^1 \setminus \{0\}, \varphi_2)$ where $\varphi_2(z) = 1/z$ and we define $1/\infty = 0$.
(3) The tori. Given a lattice $\Gamma = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ (with ω_1, ω_2 linearly independent over $\mathbb{R}$) we can give the tori $\mathbb{C}/\Gamma$ a complex structure in the following way: if $V \subseteq \mathbb{C}$ is an open set without points equivalent under Γ then the projection on the quotient $\pi : V \to \pi(V)$ is a homeomorphism and $\pi^{-1} : \pi(V) \to V$ is a complex chart. It's easy to see that such complex charts are compatible and that they cover $\mathbb{C}/\Gamma$.

Definition 2. *Let X and Y be Riemann surfaces. We say that $f : X \to Y$ is holomorphic if, for every pair of charts $\psi_1 : U_1 \to V_1$ on X and $\psi_2 : U_2 \to V_2$ on Y such that $f(U_1) \subseteq U_2$, the mapping*

$$\psi_2 \circ f \circ \psi_1^{-1} : V_1 \to V_2$$

is holomorphic (in the usual sense of a complex map). The mapping f is said to be a biholomorphism if it's injective and both f and f^{-1} are holomorphic. Two Riemann surfaces are said to be isomorphic if there is a biholomorphism between them.

In particular we have defined holomorphic functions $X \to \mathbb{C}$. The set of such functions is denoted by $\mathcal{O}(X)$ and has a natural ring structure. A meromorphic function is a function $f : X \to \mathbb{P}^1$ that is not identically ∞, and we denote by $\mathcal{M}(X)$ the set of such functions. We have a natural inclusion $\mathcal{O}(X) \hookrightarrow \mathcal{M}(X)$. We will now recover some important results of complex analysis in the context of Riemann surfaces.

Theorem 1 (Identity Theorem)**.** *Suppose that X and Y are Riemann surfaces and $f_1, f_2 : X \to Y$ are two holomorphic maps not identically equal. Then the set of points in which f_1 and f_2 coincide is discrete.*

Proof. Let A be the set of points in which they coincide and suppose by contradiction that $a \in X$ is a limit point of A. Considering a chart around a and using the identity theorem on the plane it's easy to see that there is a neighborhood W of a in which $f|W = g|W$. We define G to be the set of points in X such that there is a neighborhood W around them such that $f_1|W = f_2|W$; we already saw that $a \in G$ and it's clear that $G \subseteq A$. It's clear by definition that G is open. On the other hand, if $x \in \partial G \setminus G$ then x would be a limit point of G, and hence of A; but by the same argument we used for a it follows that $x \in G$. Thus $\partial G \subseteq G$, which shows that G in closed, open and non-empty, proving that $G = X$ and $f_1 \equiv f_2$. □

Notice that the identity theorem implies that the set of points $x \in X$ such that $f(x) = \infty$ is discrete. Such points are called poles of f. Notice also that, if X is compact a subset of X is discrete if and only if it's finite, so on a compact set f_1 and f_2 can only coincide on a finite number of points; in particular a meromorphic function can only have a finite number of zeros or poles. We proceed now to the definition of multiplicity on a Riemann surface; we use the following lemma on the local behavior of a holomorphic function.

Lemma 2. *Let $f : X \to Y$ be a non-constant holomorphic mapping, $a \in X$, $b = f(a) \in Y$. Then there is an integer $k \geq 1$ and charts (U, φ) and (V, ψ) around a and b, respectively, such that $\varphi(a) = \psi(b) = 0$, $f(U) \subseteq V$ and the map $F = \psi \circ f \circ \varphi^{-1}$ is given by $F(z) = z^k$.*

Proof (sketch). Chose local coordinates (U, z) and (V, w) around a and b, respectively, such that $z(a) = w(b) = 0$. In this charts, the function f reads $w = z^k h(z)$ for some k and some holomorphic h non-vanishing at 0. In a neighborhood around 0 we can find a holomorphic function g such that $g^k = h$. Thus $w = (zg(z))^k$ and the charts (U, z') and (V, w) where $z' = zg(z)$ are as required. □

The integer k is called the multiplicity of f at a and can be characterized in the following way: for every sufficiently small neighborhood U around a and $W \subseteq f(U)$ around $f(a)$, given $y \in W \setminus \{f(a)\}$ the equation $f(x) = y$ has exactly k solutions in U. This caracterization shows that k is well defined (i.e. is unique and doesn't depend on the parametrizations chosen); we call k the multiplicity of f at a.

Theorem 3 (Open mapping theorem)**.** *Let $f : X \to Y$ be a non-constant holomorphic map between Riemann surfaces. Then f is an open function.*

Proof. Let $a \in X$ and consider U as in lemma 2. Then it's clear that $f(U)$ is an open neighborhood of $f(a)$ since F is open and φ, ψ are homeomorphisms. □

Theorem 4. *Let $f : X \to Y$ be an injective holomorphic map between Riemann surfaces. Then f is a biholomorphic map of X onto $f(X)$.*

Proof. Since f is injective the multiplicity defined in lemma 2 is 1 for every $a \in X$. This makes clear that $f^{-1} : f(X) \to X$ is holomorphic (notice that since f is open $f(X)$ is open in Y, hence it's a Riemann surface) since in the coordinates of lemma 2 it's given by the identity. □

Theorem 5 (Maximum principle)**.** *If X is a Riemann surface and $f : X \to \mathbb{C}$ is a non-constant holomorphic function, then $|f|$ does not attain a maximum.*

Proof. By the open mapping theorem $f(X)$ is open in $\mathbb{C}$, so $\{|f(x)| : x \in X\}$ is open in $\mathbb{R}$ and therefore doesn't admit a maximum. □

Theorem 6. *Suppose X is compact and $f : X \to Y$ is a non-constant holomorphic function between Riemann surfaces. Then Y is compact and f is surjective.*

Proof. By the open mapping theorem $f(X)$ is open. But since X is compact $f(X)$ is also compact, hence closed. Thus $f(X) = Y$ and the result follows. □

Theorem 7. *Every holomorphic function $f : X \to \mathbb{C}$ on a compact X is constant.*

Proof. This follows directly from theorem 6 and the fact that $\mathbb{C}$ isn't compact. Alternatively we could use the maximum principle and Weierstrass theorem to obtain a contradiction. □

This is is a generalization of the classic Liouville theorem in complex analysis. Indeed, if $f : \mathbb{C} \to \mathbb{C}$ is bounded it can be extended to a holomorphic function $f : \mathbb{P}^1 \to \mathbb{C}$, and since $\mathbb{P}^1$ is compact if follows from theorem 7 that f is constant. With this theorem we can now give a very simples caracterization of meromorphic function on $\mathbb{P}^1$.

Corollary 8. *Every meromorphic function on $\mathbb{P}^1$ is a rational function.*

Proof. We can suppose without loss of generality that f doesn't have a pole at ∞, otherwise just consider $1/f$. Since $\mathbb{P}^1$ is compact is admits a finite set of poles $a_1, \ldots, a_n \in \mathbb{C}$. Let $h_j(z) = \sum_{i=-k_j}^{-1} c_{ji}(z - a_j)^i$ be the principal part of f around a_j. Then the function $g = f - (h_1 + \ldots + h_n)$ is a holomorphic function in $\mathbb{P}^1$, hence it's constant and the desired follows. □

2.2. **Differential forms and integration.** By identifying $\mathbb{C}$ with $\mathbb{R}^2$, a Riemann surface X is also a manifold of dimension 2 over $\mathbb{R}$. In particular we can talk about differentiable functions (that is, C^∞ functions in the real sense) $f : X \to \mathbb{C}$. We denote by $\mathcal{E}(U)$ the set of differentiable functions $f : U \to \mathbb{C}$; we have the trivial inclusion $\mathcal{O}(U) \subseteq \mathcal{E}(U)$ where $\mathcal{O}(U)$ denotes the holomorphic functions. We can also talk about the complexified tangent and cotangent spaces of X. If $z : U \to \mathbb{C}$ is a chart around $a \in U \subseteq X$, we can write $z = x + iy$ and the cotangent space at a, which we denote by T^1_a, is the complex vector space with basis $\{dx, dy\}$; that is, T^1_a is the tensor product of the real cotangent space with $\mathbb{C}$. Notice that $\{dz, d\bar{z}\}$ is also a basis for T^1_a where $dz = dx + idy$, $d\bar{z} = dx - idy$. This admits a dual basis for the (complexified) tangent space given by

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Notice that by the Cauchy-Riemann equations a function $f \in \mathcal{E}(U)$ is holomorphic if and only if $\frac{\partial f}{\partial \bar{z}} = 0$.

It can be easily shown using the Cauchy-Riemann equations that if z, z' are different charts then the spaces generated by dz and $d\bar{z}$ do not depend on our choice of chart z, so we have a decomposition $T^1_a = T^{1,0}_a \oplus T^{0,1}_a$ where $T^{1,0}_a = \mathbb{C}dz$ and $T^{0,1}_a = \mathbb{C}d\bar{z}$. We denote by $\mathcal{E}^{(1)}(U)$ the vector space of 1-forms on $U \subseteq X$; the decomposition of the cotangent space induces a decomposition $\mathcal{E}^{(1)}(U) = \mathcal{E}^{1,0}(U) \oplus \mathcal{E}^{0,1}(U)$. If a 1-form ω in U can be written locally as $\omega = f\,dz$ with f holomorphic, we say that ω is a holomorphic 1-form and we denote the vector space of such form by $\Omega(U)$. We have the obvious inclusions $\Omega(U) \subseteq \mathcal{E}^{1,0}(U) \subseteq \mathcal{E}^{(1)}(U)$. The exterior derivative of a differentiable function f is given, again in local coordinates, by

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z}.$$

Composing the operator d with the projections on the subspaces $\mathcal{E}^{1,0}$ and $\mathcal{E}^{0,1}$, respectively, we get the operators $d' : \mathcal{E} \to \mathcal{E}^{1,0}$ and $d'' : \mathcal{E} \to \mathcal{E}^{0,1}$, respectively. These are given locally by $d'f = \frac{\partial f}{\partial z}dz, d''f = \frac{\partial f}{\partial \bar{z}}d\bar{z}$ and satisfy $d = d' + d''$.

A meromorphic 1-form on U with a set of isolated poles A is a holomorphic 1-form on $U \setminus A$ that, near each pole $a \in A$, in local coordinates it can be written as $f\,dz$ where f is a meromorphic function having a pole at a; the vector space of such 1-forms is denoted by $\mathcal{M}^{(1)}U$. We shall now define the residue of a meromorphic 1-form and recover a version of the classic residue theorem in the setting of compact Riemann surfaces.

Definition 3. *Let ω be a meromorphic 1-form on Y, open set in X, and $a \in Y$. Let (U, z) be a coordinate neighborhood of a such that $z(a) = 0$ and, on $U \setminus \{a\}$,* *we can write* $\omega = f\,dz$ *with* $f \in \mathcal{O}(U \setminus \{a\})$. *Let*

$$f(z) = \sum_{j=-\infty}^{\infty} c_j z^j$$

be the Laurent series for f *around* a. *Then we define the residue of* ω *at* a *by* $\mathrm{Res}_a(\omega) = c_{-1}$.

One has to check that this definition doesn't depend on the coordinate neighborhood chosen (see [6], 9.9). It's clear that if ω is holomorphic at a then $\mathrm{Res}_a(\omega) = 0$.

We denote by $\mathcal{E}^{(2)}(U)$ the complex valued 2-forms on the open set $U \subseteq X$. Any 2-form ω can be written locally in the form

$$\omega = f dx \wedge dy = \frac{i}{2} f dz \wedge d\bar{z}, \;\; f \in \mathcal{E}(U).$$

We now give a proof of the residue theorem using the Stokes theorem.

Theorem 9 (Residue Theorem)**.** *Let* X *be a compact Riemann surface and* $\omega \in \mathcal{M}^{(1)}(X)$ *with poles at* $\{a_1, \dots, a_n\}$. *Then*

$$\sum_{k=1}^{n} \mathrm{Res}_{a_k}(\omega) = 0.$$

Proof. Consider a coordinate chart $z_k : U_k \to \mathbb{C}$ around a_k. By restricting them, we can suppose that U_k don't overlap and that in the coordinate system they are a disk $B_{r_k}(a_k)$. Let $X' = X \setminus \bigcup_{k=1}^{n} U_k$. Then X' is a compact manifold with boundary and we can apply Stokes' theorem:

$$\int_{X'} d\omega = \int_{\partial X'} \omega = \sum_{k=1}^{n} \int_{\partial U_k} \omega.$$

Since ω is a holomorphic form in X', locally $\omega = f\,dz$ with holomorphic f, thus $d\omega = \frac{\partial f}{\partial z} dz \wedge dz = 0$, so the left hand side is 0. On the other hand,

$$\int_{\partial U_k} \omega = \int_{|z|=r_k} \omega = 2\pi i \mathrm{Res}_{a_k}(\omega)$$

by the residue theorem on the plane, so we have the desired result. □

As a result, we have the following:

Theorem 10. *Let* $f : X \to \mathbb{P}^1$ *be a non-constant meromorphic function on a compact Riemann surface* X. *Then* f *has as many poles as zeros (counting with multiplicity). More generaly, there is an* $n \in \mathbb{Z}^+$ *(called the order of* f*) such that* f *takes every value of* $\mathbb{P}^1$ *exactly* n *times.*

Proof. Let $\omega = \frac{df}{f}$. It's easy to see that ω has poles precisely at the zeros and poles of f and $\mathrm{Res}_a(\omega) = k$ if a is a zero of order k or $\mathrm{Res}_a(\omega) = -k$ if a is a pole of order k, so the first claim follows from the Residue theorem. For the second, just notice that the poles of f and $f - c$ are the same and the number of zeros of $f - c$ is the number of times f takes the value c. $\square$

2.3. **Sheaf cohomology.** Given a topological space X possibly with some more structure (for example a Riemann surface or a manifold), given an open set U of X we can assign to U the space of continuous (or holomorphic or smooth, for example) functions defined in U, possibly with some extra structure (for example the vector space or ring structure of the holomorphic functions). If $V \subseteq U$ we can also consider the restriction $f|V$ of a function defined on U. The definition of sheaf intends to put these ideas in an abstract setting and it will be an important tool in discussing local to global problems. Given a topological space X denote by X^{op} the category whose objects are the open sets of X and the morphisms are the inclusions $\iota_{V,U} : V \hookrightarrow U$ when $V \subseteq U$.

Definition 4 (Sheaf)**.** *Let $\mathcal{C}$ be a concrete category (for example of abelian groups, rings or vector spaces). A presheaf on X with values in $\mathbb{C}$ is a contravariant functor $\mathcal{F} : X^{op} \to \mathcal{C}$. If $V \subseteq U$ we denote by $\rho^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$ (called the restriction homomorphisms) the image by $\mathcal{F}$ of the inclusion $V \hookrightarrow U$ and, given $f \in \mathcal{F}(U)$, we denote $\rho^U_V(f)$ by $f|V$.*

A presheaf is said to be a sheaf if, for every open set $U \subseteq X$ and family of open sets $(U_i)_{i\in I}$ such that $U = \bigcup_{i\in I} U_i$, the following two sheaf axioms are satisfied:

(1) If $f, g \in \mathcal{F}(U)$ are such that $f|U_i = g|U_i$ for every $i \in I$, then $f = g$.
(2) Given $f_i \in \mathcal{F}(U_i)$ such that

$$f_i|(U_i \cap U_j) = f_j|(U_i \cap U_j) \text{ for every } i, j \in I$$

there is $f \in \mathcal{F}(U)$ such that $f|U_i = f_i$.

Notice that by (1) the element $f \in \mathcal{F}(U)$ whose existence is assured by (2) is unique. We can think of axioms (1) and (2) as follows: if two elements are locally identical then they are globally the same and if a family of elements defined on a cover is compatible (in the sense that they agree on the intersections of the cover) then they can be glued together. A lot of important examples of sheaves in the context of Riemann surfaces have already appeared.

Example 2. (1) Given an open set $U \subseteq X$ let $\mathcal{O}(X)$ be the additive abelian group (respectively vector space, ring) of holomorphic functions $X \to \mathbb{C}$. With the usual restriction mappings $\mathcal{O}(U) \to \mathcal{O}(V)$ when $V \subseteq U$ this defines a sheaf of abelian groups (respectively vector spaces, rings). In the same way we can define the sheaves $\mathcal{M}$, Ω, $\mathcal{E}$, $\mathcal{E}^{(1)}$, $\mathcal{E}^{(2)}$, $\mathcal{E}^{1,0}$ and $\mathcal{E}^{0,1}$.

(2) Letting $\mathcal{O}^*(U)$ be the multiplicative group of holomorphic functions $f : U \to \mathbb{C}^*$, we also define a sheaf $\mathcal{O}^*$. In a similar way we define the sheaf $\mathcal{M}^*$ of meromorphic functions which don't vanish identically on any open set.
(3) Suppose that G is an abelian group (for instance $G = \mathbb{Z}$ or $G = \mathbb{C}$). We can consider the sheaf of functions with values in G which are locally constant (that is, which are constant in each connected component of $U \subseteq X$) and we will denote this sheaf simply by G.

Definition 5 (Stalk of a sheaf)**.** *Suppose $\mathcal{F}$ is a sheaf on X and $a \in X$. On the disjoint union*

$$\bigcup_{U \ni a} \mathcal{F}(U)$$

we can define an equivalence relation in the following way: if $f \in \mathcal{F}(U), g \in \mathcal{F}(V)$ we say that $f \overset{a}{\sim} g$ if there is an open set $W \subseteq U \cap V$ such that $f|W = g|W$. Then the stalk of $\mathcal{F}$ at a is the direct limit

$$\mathcal{F}_a = \left(\bigcup_{U \ni a} \mathcal{F}(U) \right) / \overset{a}{\sim}$$

For $f \in \mathcal{F}(U)$ we define its germ at a $\rho_a(f) \in \mathcal{F}_a$ as the equivalence class of f.

Consider for example the sheaf of holomorphic functions $\mathbb{C} \to \mathbb{C}$. Then we can identify the germ of f with its Taylor series expansion at a and the stalk at a with the ring $\mathbb{C}\{z-a\}$ of Taylor series $\sum_{j=0}^{\infty} a_j (z-a)^j$ with a non-zero radius of convergence.

From now on, we will only consider sheaves of abelian groups, unless stated otherwise. We will now define the cohomology group $H^1(X, \mathcal{F})$ of a sheaf of abelian groups $\mathcal{F}$ on X. For that we need first to introduce cochains, cocycles and coboundaries. Given a cover $\mathfrak{U} = (U_i)_{i \in I}$ of X by open sets, for $q = 0, 1, 2, \ldots$ the q-th cochain group of $\mathcal{F}$ with respect to $\mathfrak{U}$ is

$$C^q(\mathfrak{U}, \mathcal{F}) = \prod_{(i_0, \ldots, i_q) \in I^{q+1}} \mathcal{F}(U_{i_0} \cap \ldots \cap U_{i_q})$$

For instance $C^0(\mathfrak{U}, \mathcal{F}) = \{(f_i)_{i \in I} : f_i \in \mathcal{F}(U_i)\}$ and $C^1(\mathfrak{U}, \mathcal{F}) = \{(f_{ij})_{i,j \in I} : f_{ij} \in \mathcal{F}(U_i \cap U_j)\}$. We can now define the coboundary operators

$$\delta : C^0(\mathfrak{U}, \mathcal{F}) \to \mathcal{C}^1(\mathfrak{U}, \mathcal{F})$$
$$\delta : C^1(\mathfrak{U}, \mathcal{F}) \to \mathcal{C}^2(\mathfrak{U}, \mathcal{F})$$

by:

(1) If $(f_i)_{i \in I} \in C^0(\mathfrak{U}, \mathcal{F})$, define $\delta((f_i)_{i \in I}) = (g_{ij})_{i,j \in I}$ where $g_{ij} = f_i - f_j \in \mathcal{F}(U_i \cap U_j)$; here $f_i - f_j$ should be interpreted as $f_i|(U_i \cap U_j) - f_j|(U_i \cap U_j)$ so that the difference makes sense.

(2) If $(f_{ij})_{i,j\in I} \in C^1(\mathfrak{U},\mathcal{F})$, define $\delta((f_{ij})_{i,j\in I}) = (g_{ijk})_{i,j,k\in I}$ where $g_{ijk} = f_{jk} - f_{ik} + f_{ij} \in \mathcal{F}(U_i \cap U_j \cap U_k)$.

These coboundary operators δ are group homomorphisms, so we can define

$$Z^1(\mathfrak{U},\mathcal{F}) = \ker\left(C^1(\mathfrak{U},\mathcal{F}) \xrightarrow{\delta} C^2(\mathfrak{U},\mathcal{F})\right) < C^1(\mathfrak{U},\mathcal{F})$$
$$B^1(\mathfrak{U},\mathcal{F}) = \operatorname{im}\left(C^0(\mathfrak{U},\mathcal{F}) \xrightarrow{\delta} C^1(\mathfrak{U},\mathcal{F})\right) < C^1(\mathfrak{U},\mathcal{F})$$

The elements in Z^1 are called cocyles and elements in B^1 are coboundaries. An element $(f_{ij}) \in C^1(\mathfrak{U},\mathcal{F})$ is a cocyle if it satisfies $f_{ij} + f_{jk} = f_{ik}$ for every $i,j,k \in I$ (in particular $f_{ii} = 0$ and $f_{ij} = -f_{ji}$ and is a coboundary if there is $(g_i) \in C^0(\mathfrak{U},\mathcal{F})$ such that $f_{ij} = g_i - g_j$. It's clear that a coboundary is a cocyle. We say that a cocycle splits if it's a coboundary. We can now define the first cohomology group of $\mathcal{F}$ with respect to a cover as $H^1(\mathfrak{U},\mathcal{F}) = Z^1(\mathfrak{U},\mathcal{F})/B^1(\mathfrak{U},\mathcal{F})$.

We now have to define the cohomology group of the surface X. To do so we will take a direct limit of the groups $H^1(\mathfrak{U},\mathcal{F})$ by defining a direct system on the set of covers of X. We say that a cover $\mathfrak{B} = (V_k)_{k\in K}$ is finer than $\mathfrak{U} = (U_i)_{i\in I}$ (and write $\mathfrak{B} < \mathfrak{U}$) if every V_k is contained in at least a U_i; that is, if there's a function $\tau : K \to I$ such that $V_k \subseteq U_{\tau k}$ for every $k \in K$. Given τ we can define a map $t^{\mathfrak{U}}_{\mathfrak{B}} : Z^1(\mathfrak{U},\mathcal{F}) \to Z^1(\mathfrak{B},\mathcal{F})$ in the following way: given $(f_{ij}) \in Z^1(\mathfrak{U},\mathcal{F})$, define $t^{\mathfrak{U}}_{\mathfrak{B}}((f_{ij})) = (g_{kl})$ where

$$g_{kl} = f_{\tau k,\tau l}|V_k \cap V_l, \qquad \text{for } k,l \in K$$

Since $t^{\mathfrak{U}}_{\mathfrak{B}}$ send coboundaries into coboundaries, it induces a map $t^{\mathfrak{U}}_{\mathfrak{B}} : H^1(\mathfrak{U},\mathcal{F}) \to H^1(\mathfrak{B},\mathcal{F})$. It can be shown (see lemmas 12.3 and 12.4 in **??**) that $t^{\mathfrak{U}}_{\mathfrak{B}}$ doesn't depend on τ and is injective.

Define the equivalence relation $\sim$ in $\bigcup H^1(\mathfrak{U},\mathcal{F})$ by saying that $\xi \sim \eta$, with $\xi \in H^1(\mathfrak{U},\mathcal{F})$ and $\eta \in H^1(\mathfrak{U}',\mathcal{F})$, if there is $\mathfrak{B} < \mathfrak{U},\mathfrak{U}'$ such that $t^{\mathfrak{U}}_{\mathfrak{B}}(\xi) = t^{\mathfrak{U}'}_{\mathfrak{B}}(\eta)$. The first cohomology group of X with coefficients in the sheaf $\mathcal{F}$ is finally defined as the direct limit

$$H^1(X,\mathcal{F}) = \bigcup H^1(\mathfrak{U},\mathcal{F})/\sim.$$

The 0-th cohomology is simpler and will also be useful. The 0-cocycles are elements $(f_i) \in C^0(\mathfrak{U},\mathcal{F})$ such that $f_i|U_i \cap U_j = f_j|U_i \cap U_j$; by the sheaf axioms, there is a 1-to-1 correspondence between $Z^0(\mathfrak{U},\mathcal{F})$ and globally defined $f \in \mathcal{F}(X)$. Thus

$$H^0(\mathfrak{U},\mathcal{F}) = Z^0(\mathfrak{U},\mathcal{F})/B^0(\mathfrak{U},\mathcal{F}) = Z^0(\mathfrak{U},\mathcal{F}) = \mathcal{F}(X)$$

since we set $B^0(\mathfrak{U},\mathcal{F}) = 0$. This does not depend on the cover $\mathfrak{U}$, so there is no need to use a direct limit and we can simply define $H^0(X,\mathcal{F}) = \mathcal{F}(X)$.

Notice that from the fact that $t^{\mathfrak{U}}_{\mathfrak{B}}$ is injective follows easily that the obvious mapping $H^1(\mathfrak{U},\mathcal{F}) \to H^1(X,\mathcal{F})$ is also injective. In particular, $H^1(X,\mathcal{F})$ is trivial if and only if $H^1(\mathfrak{U},\mathcal{F})$ is trivial for every cover $\mathfrak{U}$. We can use this and the existence of partitions of unity to prove that the sheaf of differentiable functions has a trivial 1st cohomology.

Proposition 11. *Let X be a Riemann surface and $\mathcal{E}$ be the sheaf of differentiable functions on X. Then $H^1(X, \mathcal{E}) = 0$*

Proof. Let $\mathfrak{U} = (U_i)_{i\in I}$ be an open cover of X and $(\psi_i)_{i\in I}$ a differentiable partition of unity subordinate to $\mathfrak{U}$. Let $(f_{ij}) \in Z^1(\mathfrak{U}, \mathcal{E})$; we want to show that it splits. The function $\psi_j f_{ij}$ is defined on $U_i \cap U_j$ and can be extended differentiably to a function on U_i by setting it to be 0 outside of its support. Thus we can set

$$g_i = \sum_{j\in I} \psi_j f_{ij} \in \mathcal{E}(U_i).$$

We have

$$\begin{aligned} g_i - g_k &= \sum_{j\in I} \psi_j f_{ij} - \sum_{j\in I} \psi_j f_{kj} = \sum_{j\in I} \psi_j (f_{ij} - f_{kj}) \\ &= \sum_{j\in I} \psi_j f_{ik} = \left(\sum_{j\in I} \psi_j\right) f_{ik} = f_{ik}. \end{aligned}$$

Thus (f_{ij}) splits, as desired, proving that $H^1(\mathfrak{U}, \mathcal{E})$ for every cover $\mathfrak{U}$, as desired. $\square$

We cannot adapt the proof to show that $H^1(X, \mathcal{O})$ is trivial because a holomorphic partition of unit does not exist in general; indeed, we will later see that $H^1(X, \mathcal{O})$ is trivial if and only if X is simply connected. We state now a result that assures that the embedding $H^1(\mathfrak{U}, \mathcal{F}) \to H^1(X, \mathcal{F})$ is actually an isomorphism if the cover is "fine enough", and enables us to compute cohomology in practical cases without requiring the use of the direct limit definition.

Theorem 12 (Leray)**.** *Let $\mathcal{F}$ be a sheaf of abelian groups on topological space X. A cover $\mathfrak{U} = (U_i)_{i\in I}$ is said to be a Leray cover if $H^1(U_i, \mathcal{F}) = 0$ for every $i \in I$. If $\mathfrak{U}$ is a Leray cover, then $H^1(X, \mathcal{F}) = H^1(\mathfrak{U}, \mathcal{F})$.*

2.4. **The exact cohomology sequence and Dolbeault's lemma.** Here we will develop a very important tool in the study of Riemann surfaces, the exact cohomology sequence. With this tool, the Riemann-Roch theorem in its formulation with the cohomology groups, one of the central theorems in Riemann surfaces, will be almost straightforward. We begin by defining homomorphisms of sheaves.

Definition 6. *If $\mathcal{F}$ and $\mathcal{G}$ are sheaves of abelian groups, a sheaf homomorphism $\alpha : \mathcal{F} \to \mathcal{G}$ is a natural transformation between the functors $\mathcal{F}$ and $\mathcal{G}$. We denote by $\alpha_U : \mathcal{F}(U) \to \mathcal{G}(U)$ the group homomorphism with respect to the open set U; if the open set U is clear we may omit it and write $\alpha : \mathcal{F}(U) \to \mathcal{G}(U)$.*

Example 3.
(1) The exterior derivatives $d : \mathcal{E} \to \mathcal{E}^{(1)}$ and $d : \mathcal{E}^{(1)} \to \mathcal{E}^{(2)}$ are sheaf homomorphisms. Similarly d' and d'' are also sheaf homomorphisms.
(2) The natural inclusions $\mathbb{Z} \hookrightarrow \mathbb{C} \hookrightarrow \mathcal{O} \hookrightarrow \mathcal{E}$ and $\Omega \hookrightarrow \mathcal{E}^{1,0} \hookrightarrow \mathcal{E}^{(1)}$ are all sheaf homomorphisms.

(3) We can define a sheaf homomorphism $ex : \mathcal{O} \to \mathcal{O}^*$ by $ex_U(f) = \exp(2\pi i f)$ for $f \in \mathcal{O}(U)$.

We can define the kernel of a sheaf homomorphism in a quite natural way.

Definition 7. *Given a sheaf homomorphism $\alpha : \mathcal{F} \to \mathcal{G}$ we can define its kernel. For U open, let $\mathcal{K}(U) = \ker \alpha_U$; notice that $\alpha_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is a group homomorphism, so $\mathcal{K}(U)$ is a subgroup of $\mathcal{F}(U)$. Now it can be seen that $\mathcal{K}$ defines a sheaf with the restriction homomorphisms induced by the restriction homomorphisms of $\mathcal{F}$.*

One could try to define the image of a sheaf homomorphism is a similar way, as the sheaf defined by $\mathcal{I}(U) = \text{im}\alpha_U$, but this wouldn't satisfy the second sheaf axiom. For instance, consider $ex : \mathcal{O} \to \mathcal{O}^*$ and $U_1 = \mathbb{C} \setminus \mathbb{R}_0^+$, $U_2 = \mathbb{C} \setminus \mathbb{R}_0^-$. Since we can define a logarithm in U_1 and U_2, there are functions $f_1 \in \mathcal{I}(U_1)$, $f_2 \in \mathcal{I}(U_2)$ such that $f_1(z) = z$ and $f_2(z) = z$. We have $f_1|U_1 \cap U_2 = f_2|U_1 \cap U_2$ but there is no element $f \in \mathcal{I}(U_1 \cup U_2)$ such that $f|U_i = f_i$, that is, such that $f(z) = z$ in $U_1 \cup U_2 = \mathbb{C}^*$. We can define the concept of exact sequence.

Definition 8. *Suppose $\alpha : \mathcal{F} \to \mathcal{G}$ is a sheaf homomorphism. Given $x \in X$, α induces a homomorphism of the stalks $\alpha_x : \mathcal{F}_x \to \mathcal{G}_x$. A sequence of sheaves*

$$\mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H}$$

is said to be exact if, for each $x \in X$,

$$\mathcal{F}_x \xrightarrow{\alpha_x} \mathcal{G}_x \xrightarrow{\beta_x} \mathcal{H}_x$$

is an exact sequence of groups, that is, if $\ker \beta_x = \operatorname{im} \alpha_x$. A long sequence

$$\mathcal{F}_1 \xrightarrow{\alpha_1} \mathcal{F}_2 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{n-1}} \mathcal{F}_{n-1}$$

is exact if

$$\mathcal{F}_{i-1} \xrightarrow{\alpha_{i-1}} \mathcal{F}_i \xrightarrow{\alpha_i} \mathcal{F}_{i+1}$$

is exact for $i = 2, \dots, n-1$.

We call $\alpha : \mathcal{F} \to \mathcal{G}$ a monomorphism if $0 \longrightarrow \mathcal{F} \xrightarrow{\alpha} \mathcal{G}$ is exact and an epimorphism if $\mathcal{F} \xrightarrow{\alpha} \mathcal{G} \longrightarrow 0$ is exact. Notice that α being a monomorphism is equivalent to $\ker \alpha_x = 0$ for every $x \in X$; thus, if $f \in \mathcal{F}(U)$ is such that $\alpha_U(f) = 0$ there is, for each $x \in X$, a neighborhood $U_x \subseteq U$ of f such that $f|U_x = 0$. By the first sheaf axiom it follows that $f = 0$. This shows that if α is a monomorphism then α_U is injective for every open set U; in other words, if α is a monomorphism then $\ker \alpha$ is the trivial sheaf.

On the other hand, α being an epimorphism does not imply that α_U is surjective for every U. The reason for this is that the definition of exact sequence (and therefore of epimorphism) is local. Indeed, α is an epimorphism if and only if for every $x \in X$, U neighborhood of x and $f \in \mathcal{G}(U)$ there is a smaller neighborhood $x \in V \subseteq U$ such that $f|V \in \operatorname{im} \alpha_V$. An instance of this is again the example of

$ex : \mathcal{O}(\mathbb{C}^*) \to \mathcal{O}^*(\mathbb{C}^*)$. It's clear that $ex_{\mathbb{C}^*}$ is not surjective (for example it doesn't have the identity in its image); however, ex is an epimorphism since, because $f(x) \neq 0$, we can always find a neighborhood V of x for which which we can define a holomorphic logarithm on $f(V)$; for such a V the homomorphism α_V is surjective. Let's consider now a few examples of short exact sequences of sheaves occurring naturally in Riemann surfaces.

Example 4. (1) Let $\mathcal{K} = \ker\left(\mathcal{E}^{(1)} \xrightarrow{d} \mathcal{E}^{(2)}\right)$ be the sheaf of closed 1-forms. Then the sequence

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{E} \xrightarrow{d} \mathcal{K} \longrightarrow 0$$

is exact. Here $\mathbb{C} \longrightarrow \mathcal{E}$ is the obvious inclusion, so it's clearly a monomorphism. It's also clear that the kernel of d is $\mathbb{C}$, that is, the differentiable functions f such that $df = 0$ are the locally constant ones. The fact that $d : \mathcal{E} \to \mathcal{K}$ is an epimorphism follows from the Poincaré lemma that says that, locally (in a neighborhood homeomorphic to the disk), every closed form is exact.

(2) A holomorphic version of the above is the following sequence:

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{O} \xrightarrow{d} \Omega \longrightarrow 0$$

The fact that d is an epimorphism can be proven by noticing that locally any holomorphic function has a holomorphic primitive, so if $g\,dz \in \Omega$ there is f such that $df = \frac{\partial f}{\partial z} dz = g\,dz$.

(3) The sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \xrightarrow{ex} \mathcal{O}^* \longrightarrow 0$$

is exact. ex is an epimorphism since we can always define locally a logarithm. We also have the constant version of the above with $\mathcal{O}$ substituted by $\mathbb{C}$:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{C} \xrightarrow{ex} \mathbb{C}^* \longrightarrow 0$$

(4) The following two sequences are exact:

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{E} \xrightarrow{d''} \mathcal{E}^{0,1} \longrightarrow 0$$

$$0 \longrightarrow \Omega \longrightarrow \mathcal{E}^{1,0} \xrightarrow{d} \mathcal{E}^{(2)} \longrightarrow 0.$$

That they are exact in the middle isn't hard to show: locally we can write $d''f = \frac{\partial f}{\partial \bar{z}} d\bar{z}$, which vanishes if and only if and only if f is holomorphic; a similar reasoning for the second one. However, we are not yet able to justify that d'' and d are epimorphisms; that shall follow from Dolbeault's lemma, see 17.

Lemma 13. *Suppose*

$$0 \longrightarrow \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H}$$

is an exact sequence of sheaves on X and let U be an open set on X. Then

$$0 \longrightarrow \mathcal{F}(U) \xrightarrow{\alpha_U} \mathcal{G}(U) \xrightarrow{\beta_U} \mathcal{H}(U)$$

is an exact sequence of groups.

A homomorphism of sheaves $\alpha : \mathcal{F} \to \mathcal{G}$ induces homomorphisms

$$\begin{aligned} \alpha^0 :& H^0(X, \mathcal{F}) \to H^0(X, \mathcal{G}) \\ \alpha^1 :& H^1(X, \mathcal{F}) \to H^1(X, \mathcal{G}). \end{aligned}$$

The homomorphism α^0 is simply $\alpha_X : \mathcal{F}(X) \to \mathcal{G}(X)$. The homomorphism α^1 is constructed in the following way: given a cover $\mathfrak{U} = (U_i)$ of X, let $\alpha_{\mathfrak{U}} : C^1(\mathfrak{U}, \mathcal{F}) \to C^1(\mathfrak{U}, \mathcal{G})$ be defined by $\alpha_{\mathfrak{U}}((f_{ij})) = \alpha(f_{ij})$. It's easy to see that $\alpha_{\mathfrak{U}}$ sends cocycles to cocyles and coboundaries to coboundaries, so it induces a homomorphism $\alpha_{\mathfrak{U}} : H^1(\mathfrak{U}, \mathcal{F}) \to H^1(\mathfrak{U}, \mathcal{G})$. The collection of homomorphisms $\alpha_{\mathfrak{U}}$ now induces a homomorphism $\alpha^1 : H^1(X, \mathcal{F}) \to H^1(X, \mathcal{G})$ in the obvious way.

If we fix X then H^0 and H^1 are functors from the category of sheaves on X to the category of abelian groups acting on the morphisms by sending α to α^0 and α^1, respectively. Lemma 13 with $U = X$ tells us that H^0 is a left exact functor. However, it's not right exact because $\alpha : \mathcal{F} \to \mathcal{G}$ being an epimorphism doesn't imply that $\alpha^0 : \mathcal{F}(X) \to \mathcal{G}(X)$ is surjective, as we saw in the example of *ex*; again, this is a problem of passing from local to global. With the long exact sequence we are going to prove, the 1st cohomology groups can be used to measure how H^0 fails to be right-exact.

To construct this exact sequence, we need the connecting homomorphism δ^*. If $0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \to 0$ is a short exact sequence of sheaves on X, we are going to define a homomorphism $\delta^* : H^0(X, \mathcal{H}) \to H^1(X, \mathcal{F})$ in the following way. Suppose $h \in H^0(X, \mathcal{H}) = \mathcal{H}(X)$. Since β is an epimorphism, there is a cover $\mathfrak{U} = (U_i)$ such that $h|U_i \in \operatorname{im} \beta_{U_i}$, so there is a cochain $(g_i) \in C^0(\mathfrak{U}, \mathcal{G})$ such that $\beta(g_i) = h|U_i$. But then $\beta(g_j - g_i) = 0$ in $U_i \cap U_j$, so by lemma 13 there is $f_{ij} \in \mathcal{F}(U_i \cap U_j)$ such that $\alpha(f_{ij}) = g_j - g_i$. We have $\alpha(f_{ij} + f_{jk} - f_{ik}) = 0$ in $U_i \cap U_j \cap U_k$, and since α is a monomorphism this implies that $f_{ij} + f_{jk} - f_{ik} = 0$; therefore $(f_{ij}) \in Z^1(\mathfrak{U}, \mathcal{F})$ is a cochain. We now defined δ^* by setting $\delta^*(h)$ to be the cohomology class corresponding to (f_{ij}) in $H^1(X, \mathcal{F})$. It should be checked that this is well defined, i.e. it doesn't depend on the choice of the cover $\mathfrak{U}$ and on the choice of $(g_i) \in C^0(\mathfrak{U}, \mathcal{G})$ such that $\beta(g_i) = h|U_i$.

Theorem 14 (Long exact sequence)**.** *Suppose*

$$0 \longrightarrow \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \longrightarrow 0$$

is a short exact sequence of sheaves on X. Then

$$0 \longrightarrow H^0(X,\mathcal{F}) \xrightarrow{\alpha^0} H^0(X,\mathcal{G}) \xrightarrow{\beta^0} H^0(X,\mathcal{H}) \xrightarrow{\delta^*}$$
$$\xrightarrow{\delta^*} H^1(X,\mathcal{F}) \xrightarrow{\alpha^1} H^1(X,\mathcal{G}) \xrightarrow{\beta^1} H^1(X,\mathcal{H})$$

is an exact sequence of abelian groups.

Corollary 15. *Suppose*

$$0 \longrightarrow \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \longrightarrow 0$$

is a short exact sequence of sheaves on X such that $H^1(X,\mathcal{G}) = 0$. Then

$$H^1(X,\mathcal{F}) \cong \mathcal{H}(X)/\beta\mathcal{G}(X)$$

Proof. By the long exact sequence the sequence of groups

$$\mathcal{G}(X) \xrightarrow{\beta^0} \mathcal{H}(X) \xrightarrow{\delta^*} H^1(X,\mathcal{F}) \xrightarrow{\alpha^1} 0 = H^1(X,\mathcal{G})$$

is exact. The desired follows. □

Theorem 16 (de Rham)**.** *Let $Rh^1(X)$ be the first deRahm group of X, that is, the closed 1-forms modulo the exact 1-forms, defined by*

$$Rh^1(X) = \frac{\ker\left(\mathcal{E}^{(1)} \xrightarrow{d} \mathcal{E}^{(2)}\right)}{\operatorname{im}\left(\mathcal{E} \xrightarrow{d} \mathcal{E}^{(1)}\right)}.$$

Then $Rh^1(X) \cong H^1(X,\mathbb{C})$

Proof. This follows directly from the exact sequence

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{E} \xrightarrow{d} \ker\left(\mathcal{E}^{(1)} \xrightarrow{d} \mathcal{E}^{(2)}\right) \longrightarrow 0$$

we saw in example 4, corollary 15 and the fact that $H^1(X,\mathcal{E}) = 0$. □

Although here we gave a proof only in the case of Riemann manifolds and 1-forms, this result admits a more general form with differentiable manifolds of arbitrary dimension and higher order forms. By Poincaré lemma it follows that if X is simply connected then $H^1(X,\mathbb{C})$ is trivial; using the exact sequence of example 4 (3) we conclude that $H^1(X,\mathbb{Z})$ is also trivial if X is simply connected.

We will state and sketh the proof of an analytic result that will have interesting consequences in our study of cohomology in Riemann surfaces.

Lemma 17 (Dolbeault)**.** *Let $X = \{z \in \mathbb{C} : |z| < R\}$ with $0 < R \le \infty$ and $g \in \mathcal{E}(X)$. Then there is $f \in \mathcal{E}(X)$ such that $g = \frac{\partial f}{\partial \bar{z}}$.*

Proof (sketch). If $R < \infty$, we can define the function

$$f(\zeta) = \frac{1}{2\pi i} \int_X \frac{g(z)}{z - \zeta} dz \wedge d\bar{z}.$$

This integral exists, depends differentiably on ζ and f has the desired property, but we won't show this here. A complete proof can be found in [6], Lemma 13.1.

In the case $R = \infty$ the above integral may not converge, so we construct a sequence of functions (g_n) with support in $B_{n+1}(0)$ such that $g_n | B_n(0) = g$. By the above we can find (f_n) such that $\bar{\partial} f_n = g_n$. We can then modify the sequence (f_n) to a sequence $(\tilde{f}_n)$ that still obeys $\bar{\partial} \tilde{f}_n = g_n$ and converges uniformly to a function f – such f has the desired property. Again, the details may be found in [6], Theorem 13.2. □

Thanks to this, locally every form in $\mathcal{E}^{0,1}$ can be written as $g\, d\bar{z} = \frac{\partial f}{\partial \bar{z}} d\bar{z} = d'' f$, which shows that d'' in example 4 (4) is an epimorphism. Similarly, $g\, d\bar{z} \wedge dz = \frac{\partial f}{\partial \bar{z}} d\bar{z} \wedge dz = d\,(f\, dz)$, proving that d in the second sequence is also an epimorphism. With the sequences in example 4 (4) and corollary 15 we get the following two isomorphisms

$$H^1(X, \mathcal{O}) \cong \mathcal{E}^{0,1} / d'' \mathcal{E}$$
$$H^1(X, \Omega) \cong \mathcal{E}^{(2)} / d\mathcal{E}^{1,0}.$$

Finally, we use Dolbeault lemma to show that the 1st cohomology of $H^1(X, \mathcal{O})$ is trivial when $X = D, \mathbb{C}, \mathbb{P}^1$.

Theorem 18. *Let $X = D, \mathbb{C}$ or $\mathbb{P}^1$ be either the unit disk, the complex plane or the Riemann sphere. Then $H^1(X, \mathcal{O}) = 0$.*

Proof. We consider first the case $X = D$ or $X = \mathbb{C}^2$. By Dolbeault's lemma the following sequence is exact:

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{E} \xrightarrow{\bar{\partial}} \mathcal{E} \longrightarrow 0.$$

By corollary 15, $H^1(X, \mathcal{O}) \cong \mathcal{E} / \bar{\partial} \mathcal{E}$. But by Dolbeault's lemma $\bar{\partial} \mathcal{E} = \mathcal{E}$, proving the desired in this case.

Now suppose that $X = \mathbb{P}^1$ and consider the cover $\mathfrak{U} = \{U_1, U_2\}$ where $U_1 = \mathbb{P}^1 \setminus \{\infty\} = \mathbb{C}, U_2 = \mathbb{P}^1 \setminus \{0\}$. By what we just saw this is a Leray cover, so it's enough to show that $H^1(\mathfrak{U}, \mathcal{O})$ is trivial. Indeed, if $(f_{ij}) \in Z^1(\mathfrak{U}, \mathcal{O})$ we can write f_{12} as

$$f_{12}(z) = \sum_{j=-\infty}^{\infty} a_j z^j = g_1(z) - g_2(z)$$

where $g_1(z) = \sum_{j=0}^{\infty} a_j z^j \in \mathcal{O}(U_1)$ and $g_2(z) = \sum_{j=-\infty}^{-1} a_j z^j \in \mathcal{O}(U_2)$. Thus (f_{ij}) splits, as desired. □

2.5. **The Riemann-Roch theorem.** We are going to prove the Riemann-Roch theorem, a central theorem in compact Riemann surfaces. Essentially, it says how many linearly independent meromorphic functions we can find with a "bounded" amount of poles. A proof of the following theorem can be found in [6], corollary 14.10.

Theorem 19. *Suppose X is a compact Riemann surface. Then $H^1(X, \mathcal{O})$ is finite dimensional.*

Thanks to the above theorem, we can define the genus of Riemann surface.

Definition 9. *The genus of a compact surface X is $g = \dim H^1(X, \mathcal{O})$.*

In fact, the genus of X is a topological invariant. Indeed, the genus can also be given by $2g = \dim H^1(X, \mathbb{C})$ and, since the sheaf $\mathbb{C}$ only depends on the topological structure of X, so does the genus. This identity follows from the decomposition (see chapter 19 of [6])

$$\ker\left(\mathcal{E}^{(1)} \xrightarrow{d} \mathcal{E}^{(2)}\right) = d\mathcal{E} \oplus \Omega(X) \oplus \bar{\Omega}(X)$$

together with the deRham theorem and the fact that $\dim \Omega(X) = \dim \bar{\Omega}(X) = g$ which we will prove later; here $\bar{\Omega}(X)$ denotes the 1-forms which are locally of the form $f(\bar{z})\, d\bar{z}$ for holomorphic f. Topologically, X is a two-dimensional compact and orientable manifold, thus by the classification theorem X is a connected sum of g tori (if $g = 0$, X is S^2) for some g; this g is precisely the genus of X and can be seen as the "number of holes" in X.

We want to introduce some terminology to state and prove the Riemann-Roch theorem.

Definition 10. *A divisor on X is a map $D : X \to \mathbb{Z}$ such that $D(x) = 0$ except for finitely many points $x \in X$ We denote by $\operatorname{Div}(X)$ the abelian group of divisors. We say that $D \leq D'$ if $D(x) \leq D'(x)$ for every $x \in X$.*

Notice that a divisor can also be seen as an element in $\mathbb{Z}[X]$, the abelian group generated by X, by identifying D with $\sum_{x \in X} D(x)x \in \mathbb{Z}[X]$. Given a meromorphic function $f \neq 0$, its order at a, denoted by $\operatorname{ord}_a(f)$ is 0 if f is holomorphic and non-zero at a, k if f has a zero of order k at a and $-k$ if f has a pole of order k at a; for $f = 0$ set $\operatorname{ord}_x(f) = -\infty$. If X is compact, the number of zeros and poles of f is finite. Therefore the mapping $x \to \operatorname{ord}_x(f)$ is a divisor; we denote it by (f). We also define the order of a meromorphic 1-form $\omega \in \mathcal{M}^{(1)}(X)$ in a similar way; in this case, the order is given by $\operatorname{ord}_x(\omega) = \operatorname{ord}_x(f)$ where, in a neighborhood around x, $\omega = f\, dz$. We call a divisor of the form (f) for some $f \in \mathcal{M}(X) \setminus \{0\}$ principal and of the form (ω) for some $\omega \in \mathcal{M}^{(1)}(X) \setminus \{0\}$ canonical. We say that two divisors D and D' are equivalent if $D - D'$ is principal.

The degree is the homomorphism $\deg : \operatorname{Div}(X) \to \mathbb{Z}$ given by

$$\deg\, D = \sum_{x \in X} D(x).$$

By the definition of divisor, only a finite number of terms in the sum are non-zero. By theorem **??** the degree of a principal divisor is 0.

Definition 11. *Given a divisor D, we define its sheaf of meromorphic functions $\mathcal{O}_D$ by*

$$\mathcal{O}_D(U) = \{f \in \mathcal{M}(U) : (f) \geq -D|U\}.$$

Similarly, define the sheaf Ω_D of meromorphic 1-forms by

$$\Omega_D(U) = \{\omega \in \mathcal{M}^{(1)}(U) : (\omega) \geq -D|U\}.$$

Notice that if D and D' are equivalent then $D - D' = (\psi)$ and we can check that the mapping $\mathcal{O}_D \to \mathcal{O}_{D'}$ that sends f to ψf is an isomorphism.

The sheaf $\mathcal{O}_D$ corresponds to meromorphic functions with restrictions on their poles and zeros. For instance, if $D = 0$ the sheaf $\mathcal{O}_D$ is the sheaf of functions with no poles, that is, $\mathcal{O}_D = \mathcal{O}$. Thus, the 0-th cohomology of such sheaves $H^0(X, \mathcal{O}_D)$ consists of the globally defined functions the given restrictions on zeros and poles. If $\deg D < 0$ there are no such functions.

Proposition 20. *Let D be a divisor with* $\deg D < 0$. *Then $H^0(X, \mathcal{O}_D) = 0$.*

Proof. Suppose $f \in H^0(X, \mathcal{O}_D) = \mathcal{O}_D(X)$. Then $(f) \geq -D$, thus $0 = \deg(f) \geq \deg -D > 0$, contradiction. $\square$

The Riemann-Roch theorem gives information about the dimension of $H^0(X, \mathcal{O}_D)$.

Theorem 21 (Riemann-Roch)**.** *Let X be a compact Riemann surface with genus g and D a divisor on X. Then $H^0(X, \mathcal{O}_D)$ and $H^1(X, \mathcal{O}_D)$ are finite dimensional vector spaces and*

$$\dim H^0(X, \mathcal{O}_D) - \dim H^1(X, \mathcal{O}_D) = 1 - g + \deg\, D.$$

The proof will be a quite simple induction after we construct an exact sequence relating $\mathcal{O}_D$ and $\mathcal{O}_{D+P}$ where P is the divisor that is 1 in P and 0 otherwise.

Given a point $P \in X$, let $\mathbb{C}_P$ be the skyscraper sheaf defined by $\mathbb{C}_P(U) = \mathbb{C}$ if $P \in U$ and 0 otherwise with the obvious restriction morphisms. Suppose that D is a divisor on X. We have a natural inclusion $\mathcal{O}_D \to \mathcal{O}_{D+P}$ (here P denotes a divisor that is 1 in P and 0 elsewhere). Let $k = D(P)$. We define a sheaf homomorphism $\beta : \mathcal{O}_{D+P} \to \mathbb{C}_P$ as follows: if $P \notin U$, $\beta_U = 0$. If $P \in U$, f has a Laurent series of the form $f = \sum_{n=-k-1}^{\infty} a_n z^n$ around P (with $z(P) = 0$) and we set $\beta_U(f) = c_{-k-1} \in \mathbb{C} = \mathbb{C}_P(U)$. It's clear that β is an epimorphism. Also, $f \in \ker \beta$ if and only if $c_{-k-1} = 0$, which is equivalent to the pole of f at P having order at most k; thus $\ker \beta = \mathcal{O}_D$ and we have the short exact sequence

$$0 \longrightarrow \mathcal{O}_D \longrightarrow \mathcal{O}_{D+P} \xrightarrow{\beta} \mathbb{C}_P \longrightarrow 0.$$

Lemma 22. *The cohomology groups of $\mathbb{C}_P$ are $H^0(X, \mathbb{C}_P) = \mathbb{C}$ and $H^1(X, \mathbb{C}_P) = 0$.*

Proof. The first statement is trivial since $H^0(X, \mathbb{C}_P) = \mathbb{C}_P(X) = \mathbb{C}$ as $P \in X$. For the second one, we prove that $H^1(\mathfrak{U}, \mathbb{C}_P) = 0$. Consider a refinement $\mathfrak{B}$ of $\mathfrak{U}$ such that exactly one open set of $\mathfrak{B}$ contains P; for example take $\mathfrak{B} = \{U\} \cup \{V \setminus \{P\} : V \in \mathfrak{U}\}$ where $U \in \mathfrak{U}$ contains P. Since $t^{\mathfrak{U}}_{\mathfrak{B}} : H^1(\mathfrak{U}, \mathbb{C}_P) \to H^1(\mathfrak{B}, \mathbb{C}_P)$ is injective it's enough to show that $H^1(X, \mathfrak{B}) = 0$. But any cocycle $(f_{ij}) \in Z^1(\mathfrak{B}, \mathbb{C}_P)$ is zero since $f_{ii} = 0$ and for $i \neq j$ we have $f_{ij} \in \mathbb{C}_P(B_i \cap B_j) = 0$ because $P \notin B_i \cap B_j$. □

Proof (Riemann-Roch). We check the base case $D = 0$. Here $\dim H^1(X, \mathcal{O}_D) = \dim H^1(X, \mathcal{O}) = g$, by definition, and $\dim H^0(X, \mathcal{O}_D) = \dim \mathcal{O}(X) = 1$ because by theorem 7 $\mathcal{O}(X)$ consists of the constant function, so for $D = 0$ the result is correct.

Now we suppose that the result holds for D (respectively $D + P$) and we prove it for $D + P$ (respectively D). Since every divisor can be written in the form $D = P_1 + \ldots + P_n - Q_1 - \ldots - Q_m$ where P_i, Q_j are points, this is enough to prove the theorem.

Applying the long exact sequence theorem to the short exact sequence

$$0 \longrightarrow \mathcal{O}_D \longrightarrow \mathcal{O}_{D+P} \xrightarrow{\beta} \mathbb{C}_P \longrightarrow 0$$

and using lemma 22 we have the following exact sequence:

$$\begin{aligned} 0 \longrightarrow H^0(X, \mathcal{O}_D) \longrightarrow H^0(X, \mathcal{O}_{D+P}) \longrightarrow \mathbb{C} \longrightarrow \\ \longrightarrow H^1(X, \mathcal{O}_D) \longrightarrow H^1(X, \mathcal{O}_{D+P}) \longrightarrow 0. \end{aligned}$$

Therefore

$$\dim H^0(X, \mathcal{O}_D) + \dim \mathbb{C} + \dim H^1(X, \mathcal{O}_{D+P}) = \dim H^0(X, \mathcal{O}_{D+P}) + \dim H^1(X, \mathcal{O}_D),$$

which is the same as

$$\dim H^0(X, \mathcal{O}_{D+P}) - \dim H^1(X, \mathcal{O}_{D+P}) = \dim H^0(X, \mathcal{O}_D) - \dim H^1(X, \mathcal{O}_D) + 1.$$

This clearly shows the desired. □

The utility of Riemann-Roch is mainly to compute (or estimate) the term $H^0(X, \mathcal{O}_D)$ and we regard $H^1(X, \mathcal{O}_D)$ as an "error" term. In particular, Riemann-Roch implies the inequality

$$\dim H^0(X, \mathcal{O}_D) \geq 1 - g + \deg D.$$

For instance, if we chose $D = (g + 1)P$ we have $\dim H^0(X, \mathcal{O}_D) \geq 2$ and, as a result, there is a non-constant meromorphic function on X with a pole only at P with order at most $g + 1$. In particular this ensures the existence of a non-trivial meromorphic function on every compact Riemann surface, and a non-trivial meromorphic 1-form by applying the exterior derivative. The above observation for $g = 0$ gives the compact case of the uniformization theorem.

Theorem 23. *If X is a compact Riemann surface of genus $g = 0$ then X is biholomorphic to the Riemann sphere $\mathbb{P}^1$.*

Proof. By the observation above, there is a meromorphic mapping $f : X \to \mathbb{C}$ which has only a single pole at at a point P. Regarding f as a holomorphic function $X \to \mathbb{P}^1$, f takes the value ∞ exactly once, so by theorem 10 f is injective. By theorem 4 f is a biholomorphism. □

This theorem is a special case of the more general uniformization theorem, whose proof is out of the scope of this paper.

Theorem 24 (Uniformization Theorem)**.** *A simply connected Riemann surface is biholomorphic to either the Riemann sphere $\mathbb{P}^1$, the complex plane $\mathbb{C}$ or the unit disk D.*

The uniformization theorem gives a complete description of Riemann surfaces. Given a Riemann surface X, we can give a complex structure to its universal covering $\tilde{X}$. Then X is isomorphic to $\tilde{X}/G$ where $\tilde{X} \in \{\mathbb{P}^1, \mathbb{C}, D\}$ and G is a discrete group acting freely on $\tilde{X}$. If $\tilde{X} = \mathbb{P}^1$ there is no such non-trivial group G. If $\tilde{X} = \mathbb{C}$ then G is trivial or $G = \omega_1\mathbb{Z}$ or $G = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ acts by translations; in the first case we get the punctured plane $\mathbb{C} \setminus \{0\}$ and in the second the torus. If $\tilde{X} = D \cong H$ where $H = \{z \in \mathbb{C} : \text{Im z} > 0\}$ is the hyperbolic plane, G must be a discrete subgroup of $\text{PSL}(2, \mathbb{R})$, called a Fuchsian group; in the second part of this paper we will study some examples of those. A Riemann surface is called elliptic, parabolic or hyperbolic, respectively, if its universal cover is $\mathbb{P}^1, \mathbb{C}$ or $D \cong H$, respectively.

A nicer interpretation for the "error" term $\dim H^1(X, \mathcal{O}_D)$ is given by Serre duality. In particular, it will enable us to show that it's 0 in a lot of cases.

Theorem 25 (Serre Duality)**.** *Let X be a compact Riemann surface, D a divisor on X and K be a canonical divisor. Then*

$$\dim H^1(X, \mathcal{O}_D) = \dim H^0(X, \Omega_{-D}) = \dim H^0(X, \mathcal{O}_{K-D}).$$

Proof (sketch). We begin by the second equality, which is easy. If ω is a meromorphic 1-form with $(\omega) = K$, it can be easily verified that the mapping $H^0(X, \mathcal{O}_{K-D}) \to H^0(X, \Omega_{-D})$ sending $f \to f\omega$ is a group isomorphism, as required.

Recall that we have an isomorphism $H^1(X, \Omega) \cong \mathcal{E}^{(2)}(X)/d\mathcal{E}^{1,0}(X)$. Given $\xi \in H^1(X, \Omega)$ let $\omega \in \mathcal{E}^{(2)}(X)$ be a representative of ξ under the above isomorphism and define

$$\text{Res}(\xi) = \frac{1}{2\pi i}\int_X \omega.$$

We can then define a bilinear mapping $\langle , \rangle : H^0(X, \Omega_{-D}) \times H^1(X, \mathcal{O}_D) \to \mathbb{R}$ by $\langle \omega, \xi \rangle = \text{Res}(\xi\omega)$. This bilinear mapping induces a linear mapping $H^0(X, \Omega_{-D}) \to H^1(X, \mathcal{O}_D)^*$ into the dual space. This mapping is actually an isomorphism (see [6], 17.9), which shows that

$$\dim H^0(X, \Omega_{-D}) = \dim H^1(X, \mathcal{O}_D)^* = \dim H^1(X, \mathcal{O}_D). \quad \square$$

In particular, for $D = 0$ one gets

$$g = \dim H^1(X, \mathcal{O}) = \dim H^0(X, \Omega) = \dim H^0(X, \mathcal{O}_K).$$

Corollary 26. *Let ω be a (non-vanishing) meromorphic 1-form on a compact Riemann surface of genus g. Then $\deg(\omega) = 2g - 2$.*

Proof. By the Riemann-Roch for $K = (\omega)$ we have

$$\dim H^0(X, \mathcal{O}_K) - \dim H^1(X, \mathcal{O}_K) = 1 - g + \deg K.$$

By the above observation $\dim H^0(X, \mathcal{O}_K) = g$ and by Serre duality $\dim H^1(X, \mathcal{O}_K) = \dim H^0(X, \mathcal{O}) = 1$, proving the desired result. □

We use this result to show that $H^1(X, \mathcal{O}_D)$ is zero in some cases.

Corollary 27. *If D is a divisor with $\deg D > 2g - 2$ then $H^1(X, \mathcal{O}_D) = 0$.*

Proof. By Serre duality $H^1(X, \mathcal{O}_D) = H^0(X, \mathcal{O}_{K-D})$ and $\deg(K - D) = 2g - 2 - \deg D < 0$ by corollary 26, so the desired follows from proposition 20. □

We end this section with the Hurwitz formula. Consider a holomorphic map $f : X \to Y$ between compact Riemann surfaces such that there is an integer n such that each value $y \in Y$ is taken exactly n times (counting with multiplicity); see theorem 10. We say that f is an n-sheeted covering map. We denote by e_x the multiplicity with which f takes the value $f(x)$ at $x \in X$; this is the value k in lemma 2. If $e_x > 1$ we say that f is branched at x; this happens if and only if $f'(x) = 0$, and since X is compact that may only happen finitely often.

Theorem 28 (Hurwitz formula)**.** *Let $f : X \to Y$ be an n-sheeted holomorphic covering map between compact Riemann surfaces. Let g and g' be the genera of X and Y respectively. Then*

$$2g - 2 = n(2g' - 2) + \sum_{P \in X} (e_P - 1).$$

Proof. Let ω be a non-vanishing meromorphic 1-form on Y. Then $f^*\omega$ is a non-vanishing 1-form on X, so by corollary 26 we have $\deg(\omega) = 2g' - 2$ and $\deg(f^*\omega) = 2g - 2$.

Suppose that $x \in X$ and let $y = f(x)$. Let $k = e_x$ and choose charts (U, z) and (V, w) around x and y, respectively, such that in terms of the charts f is given by $w = z^k$, which is possible by lemma 2. Locally, we can write $\omega = \psi(w)dw$. Thus,

$$f^*\omega = \psi(z^k)dz^k = \psi(z^k)kz^{k-1}dz.$$

Therefore

$$\operatorname{ord}_x(f^*\omega) = k \operatorname{ord}_y(\psi) + k - 1 = e_x \operatorname{ord}_y(\omega) + (e_x - 1).$$

Summing over $x \in f^{-1}(y)$,

$$\sum_{x \in f^{-1}(y)} \operatorname{ord}_x(f^*\omega) = \operatorname{ord}_y(\omega) \left(\sum_{x \in f^{-1}(y)} e_x \right) + \sum_{x \in f^{-1}(y)} (e_x - 1).$$

But $\sum_{x \in f^{-1}(y)} e_x = n$ because it's the number of times (counting with multiplicity) that f takes y. Summing now over y we get

$$\begin{aligned} \deg(f^*\omega) &= \sum_{x \in X} \operatorname{ord}_x(f^*\omega) = n \sum_{y \in Y} \operatorname{ord}_y(\omega) + \sum_{x \in X} (e_x - 1) \\ &= n \deg(\omega) + \sum_{x \in X} (e_x - 1), \end{aligned}$$

which gives the desired result. $\square$

3. Elliptic functions and modular forms

3.1. **Elliptic functions.** We turn our attentions now to the study of the complex structures on the torus. By the classification of Riemann surfaces given by the uniformization theorem, the complex structures of the torus can be given by $\mathbb{C}/\Lambda$ where $\Lambda = \Lambda(\omega_1, \omega_2) = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ (with ω_1, ω_2 linearly independent over $\mathbb{R}$) or by H/G where G is a Fuchsian group; but $G \cong \pi_1(T^2) \cong \mathbb{Z} \times \mathbb{Z}$ and there is no Fuchsian group isomorphic to $\mathbb{Z} \times \mathbb{Z}$ (an abelian Fuchsian group must be cyclic). The torus has genus 1, as one would expect; indeed, the 1-form dz on $\mathbb{C}$ induces a holomorphic 1-form ω on $\mathbb{C}/\Lambda$ with no zeros. By corollary 26, $2g - 2 = \deg \omega = 0$.

Definition 12. *A meromorphic function $f : \mathbb{C} \to \mathbb{C}$ is double periodic if it admits a pair of periods (ω_1, ω_2) with ω_1, ω_2 linearly independent over $\mathbb{R}$.*

Such a function obeys $f(z+\omega) = f(z)$ for every $\omega \in \Lambda(\omega_1, \omega_2)$. Thus, it induces a meromorphic function on the torus $\mathbb{C}/\Lambda$ and we have the entire first part of theory about such functions! By theorem 7 a non-constant double periodic function can't be holomorphic. By the residue theorem a double periodic function cannot have a single pole having order 1, since in that case its residue would be nonzero. We call the positive integer n given by theorem 10 of a double periodic function its order; thus the order of a double periodic function is at least 2.

Proposition 29. *Let $P_1, \dots, P_n, Q_1, \dots, Q_n \in \mathbb{C}/\Lambda$ and $n \geq 2$ be two sets of points with possible repetitions but with $P_i \neq Q_j$. Then there is a double-periodic function f such that $P_1, \dots, P_n$ are exactly its poles and $Q_1, \dots, Q_n$ are exactly its zeros (with the correct multiplicities) if and only if*

$$\sum_{i=1}^{n} P_i \equiv \sum_{i=1}^{n} Q_i \bmod \Lambda.$$

Proof. We start with the "only if" part. Let $C = C_1 \cup C_2 \cup C_3 \cup C_4$ be the boundary of a fundamental paralelogram of Λ where C_1, C_2, C_3 and C_4 are the edges from 0 to ω_1 to $\omega_1 + \omega_2$ to ω_2 to 0. The function $g(z) = \frac{zf'(z)}{f(z)}$ has poles at the zeros and poles of f and $\mathrm{Res}_a(g) = ka$ if a is a zero of multiplicity k and $-ka$ if a is a pole of multiplicity k. By the residue theorem,

$$\begin{aligned}
\sum_{i=1}^{n} Q_i - \sum_{i=1}^{n} P_i &= \frac{1}{2\pi i}\int_C g(z)dz \\
&= \frac{1}{2\pi i}\int_{C_1} (g(z) - g(z+\omega_2))dz + \frac{1}{2\pi i}\int_{C_2} (g(z) - g(z-\omega_1))dz \\
&= -\frac{\omega_2}{2\pi i}\int_{C_1} \frac{f'(z)}{f(z)}dz + \frac{\omega_1}{2\pi i}\int_{C_2} \frac{f'(z)}{f(z)}dz \\
&= -\frac{\omega_2}{2\pi i}\int_{f(C_1)} \frac{1}{w}dw + \frac{\omega_1}{2\pi i}\int_{f(C_1)} \frac{1}{w}dw.
\end{aligned}$$

But $\frac{1}{2\pi i}\int_{f(C_k)} \frac{1}{w}dw$ is an integer since it is the winding number of the closed curve $f(C_k)$ around 0, and the desired follows.

For the if part we consider the divisor $D = \sum_{i=1}^{n} P_i - \sum_{i=1}^{n-1} Q_i$. By the Riemann-Roch theorem $\dim H^0(\mathbb{C}/\Lambda, \mathcal{O}_D) \geq 1 - g + \deg D = 1$, so there is a double periodic function f such that $(f) \geq -D$. Therefore f has zeros at $Q_1, \ldots, Q_{n-1}$ and poles at most at $P_1, \ldots, P_n$, so f has order $n-1$ or n. If f has order $n-1$, its poles are P_i for $i = 1, \ldots, t-1, t+1, \ldots, n$; but by the "only if" part and the hypothesis it would follow that $P_t = Q_n$, a contradiction. If f has order n its poles are precisely $P_1, \ldots, P_n$ and its zeros are $Q_1, \ldots, Q_{n-1}, \tilde{Q}_n$ for some $\tilde{Q}_n$. By the "only if" part and the hypothesis it follows that $Q_n = \tilde{Q}_n$, so f is the desired function. □

Now we are going to construct a double periodic function with order 2 explicitly, the Weierstrass $\wp$ function. The Weierstrass function has a double pole at every point in Λ and is even. The idea is to use a sum over Λ; the obvious approach would be to define the function as $\sum_{\omega\in\Lambda} \frac{1}{(z-\omega)^2}$, but this does not converge absolutely.

Lemma 30. *The sum*

$$\sum_{\omega\in\Lambda\setminus\{0\}} \frac{1}{\omega^\alpha}$$

converges absolutely if and only if $\alpha > 2$.

Proof. We can easily see that $\left|\frac{1}{(m\omega_1+n\omega_2)^\alpha}\right| \asymp (m^2+n^2)^{-\alpha/2}$, so the sum above converges absolutely if and only if $\sum_{(m,n)\neq(0,0)}(m^2+n^2)^{-\alpha/2}$ converges. By the integral test, this happens if and only if

$$\int\int_{x^2+y^2>1} (x^2+y^2)^{-\alpha/2}\,dx\,dy = \int_0^{2\pi}\int_1^\infty r^{-2\alpha+1}\,dr\,d\theta$$

converges, which happens if and only if $-2\alpha+1 < -1$, as desired. □

Definition 13. *The Weierstrass $\wp$ function is defined by*

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega\in\Lambda\setminus\{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right).$$

This series converges absolutely since we can bound

$$\left|\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right| = \left|\frac{z(2\omega - z)}{(z-\omega)^2\omega^2}\right| = O\left(\frac{1}{\omega^3}\right)$$

and use lemma 30. The function is holomorphic except at the points in Λ since it is the uniform limit of holomorphic functions; at the points in Λ the function has a pole of order 2. Since $\omega \in \Lambda$ if and only if $-\omega \in \Lambda$ it's also clear that $\wp$ is even. We compute the Taylor expansion of $\wp$ around 0 and use it to derive an important diferential equation that $\wp$ obeys.

Proposition 31. *Let $r = \min\{|\omega| : \omega \in \Lambda \setminus \{0\}\}$. Then, for $0 < |z| < r$,*

$$\wp(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty}(n+1)G_{2n+1}z^{2n}$$

where

$$G_m = \sum_{\omega \in \Lambda\setminus\{0\}} \frac{1}{\omega^m} \text{ for } m \geq 3.$$

Proof. This is simply a matter of computing the Taylor expansion of $\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}$, summing over $\omega \neq 0$ and interchanging the summations (which we can, since they converge absolutely). The odd coefficients vanish since $\wp$ is even. □

Proposition 32. *The function $\wp$ satisfies the differential equation*

$$[\wp'(z)]^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$$

where $g_2 = 60G_4$ and $g_3 = 140G_6$.

Proof. Consider the difference between the left and the right side. Using proposition 31 we can see that this difference vanishes at 0, so it's a double periodic holomorphic function with a zero at 0. Hence, by theorem 7, it's constant equal to 0, proving the result. □

By the differential equation we expect that g_2 and g_3 determine the function $\wp$. Indeed, differentiating the equation we get that $\wp''(z) = 6\wp(z)^2 - g_2/2$ and this enables us to write a recursive definition for G_n by comparing the Laurent expansion of both sides; with this we can write G_n as a polynomial function in g_2 and g_3. Given prescribed zeros and poles (like in proposition 29), we can construct explicitly a meromorphic function with those zeros and poles as a rational function in $\wp$ and $\wp'$. Since two meromorphic functions with the same zeros and poles differ by a constant, every meromorphic function can be written in that form. Therefore the field of meromorphic functions on the torus $\mathbb{C}/\Lambda$ is

$$\mathbb{C}(\wp, \wp') \cong \mathbb{C}(x, y)/\langle y^2 - 4x^3 + g_2x + g_3\rangle.$$

Let

$$e_1 = \wp(\omega_1/2),\ e_2 = \wp(\omega_2/2),\ e_3 = \wp((\omega_1 + \omega_2)/2).$$

Proposition 33. *We have*

$$4\wp(z)^3 - 60G_4\wp(z) - 140G_6 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3).$$

Moreover the roots e_1, e_2, e_3 are distinct and $g_2^3 - 27g_3^2 \neq 0$.

Proof. Since $\wp$ is even and double periodic, we have $\wp(z) = \wp(\omega - z)$. Differentiating and plugging $z = \omega/2$ we get $\wp'(\omega/2) = 0$, so $\omega_1/2, \omega_2/2, (\omega_1 + \omega_2)/2$ are zeros of $\wp'$. By the differential equation e_1, e_2 and e_3 are roots of the polynomial $4x^3 - g_2x - g_3$. They are distinc because, if for instance $e_1 = e_2$ then $\wp(z) - e_1$

would have a double zero both at ω_1 and ω_2, contradiction with $\wp$ having order 2; this proves the factorization since e_1, e_2, e_3 are 3 distinct roots of the polynomial. The last observation follows from the fact that $g_2^3 - 27g_3^2$ is the discriminant of the polynomial $4x^3 - g_2x - g_3$ which doesn't have repeated roots. $\square$

We now regard g_2, g_3 as functions of ω_1, ω_2, that is, $g_k = g_k(\omega_1, \omega_2)$. It's easy to see that we have $g_2(\lambda\omega_1, \lambda\omega_2) = \lambda^{-4}g_2(\omega_1, \omega_2)$ and $g_3(\lambda\omega_1, \lambda\omega_2) = \lambda^{-6}g_2(\omega_1, \omega_2)$ (that is, they are homogeneous of degrees -4 and -6, respectively). Thus, g_2 and g_3 can be reduced to the 1-variable functions $g_2(\tau) = g_2(\tau, 1)$ and $g_3(\tau) = g_3(\tau, 1)$ defined on $\mathbb{C} \setminus \mathbb{R}$, where $\tau = \omega_1/\omega_2$. By possibly switching ω_1 with ω_2 we may suppose that $\tau \in H = \{z \in \mathbb{C} : \text{Im } z > 0\}$, so g_2, g_3 are functions $H \to \mathbb{C}$. Recall that we have

$$g_2(\tau) = 60G_4(\tau, 1) = 60 \sum_{(m,n)\neq(0,0)} \frac{1}{(m+n\tau)^4} \text{ and}$$
$$g_3(\tau) = 140G_4(\tau, 1) = 140 \sum_{(m,n)\neq(0,0)} \frac{1}{(m+n\tau)^6}.$$

Definition 14. *We define the functions $\Delta = g_2^3 - 27g_3^2$ and $J = \frac{g_2^3}{\Delta}$ which, like g_2 and g_3, can be regarded both as 2-variable functions or as 1-variable functions $H \to \mathbb{C}$.*

By proposition 33 the function Δ doesn't vanish, so J is well defined. As g_2 and g_3 are homegeneous of degrees -4 and -6, respectively, Δ is homogeneous of degree -12 and J is homogeneous of degree 0. Thus $J(\omega_1, \omega_2) = J(\omega_1/\omega_2)$. The functions g_2, g_3, Δ and J are all of them holomorphic in H; this is proven in theorem 1.15 in [2] by showing that the series defining g_2 and g_3 converge uniformly at the strips $\{x + iy : |x| < A, y > \delta > 0\}$. These functions will be of main interest in the rest of the paper.

Remark 1. An elliptic curve is an algebraic curve of the form $y^2 = x^3 - ax - b$ with no repeated roots. Knowing that J is surjective (which we will prove in the next section) it's easy to show that we can always find ω_1, ω_2 such that $g_2(\omega_1, \omega_2) = a$ and $g_3(\omega_1, \omega_2) = b$. Thus there is a correspondence between elliptic curves and complex tori. Moreover, the mapping $z \to [\wp(z) : \wp'(z) : 1]$ defines a bihololorphism between the torus $\mathbb{C}/\Lambda$ and the projective curve defined by $Y^2Z = X^3 - g_2XZ^2 - g_3Z^3$. Since there is a natural group structure on the torus, this isomorphism induces a group structure on the elliptic curve, which is the well known elliptic curve group law (see for example [7]).

3.2. **The modular group and modular functions.**

We have constructed functions g_2, g_3, Δ and J. Notice that these functions only depend on the lattice generated, that is, if (ω_1, ω_2) and (ω_1', ω_2') generate the same lattice $\Lambda = \Lambda(\omega_1, \omega_2) =$

$\Lambda(\omega_1', \omega_2')$ then $f(\omega_1, \omega_2) = f(\omega_1', \omega_2')$ for $f = g_2, g_3, \Delta, J$. The following proposition tells us when this happens.

Proposition 34. *The pairs* (ω_1, ω_2) *and* (ω_1', ω_2') *generate the same lattices if and only if there is a matrix*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z})$$

such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix}.$$

This shows that if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z})$ then $f(a\omega_1 + b\omega_2, c\omega_1 + d\omega_2) = f(\omega_1, \omega_2)$ if f only depends on the lattice. Now the next theorem tells when do two lattices give the same torus.

Theorem 35. *Let* Λ, Λ' *be two lattices. The Riemann surfaces* $\mathbb{C}/\Lambda$ *and* $\mathbb{C}/\Lambda'$ *are isomorphic if and only if there is* $a \in \mathbb{C}^*$ *such that* $\Lambda' = a\Lambda$.

Proof. The "if" is obvious as the transformation $z \to az$ induces a biholomorphism $\mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$. Suppose that $f : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$ is a biholomorphism; up to translation we can suppose that $f(0) = 0$. In that case f lifts to a map $F : \mathbb{C} \to \mathbb{C}$ such that $F(0) = 0$, $F(\Lambda) = \Lambda'$ and $F(z + \omega) - F(z) \in \Lambda'$ for every $z \in \mathbb{C}, \omega \in \Lambda$. As Λ' is discrete, $F(z + \omega) - F(z)$ is constant for fixed ω, thus $F'(z + \omega) = F'(z)$. Hence F' is a double periodic holomorphic function, therefore it's constant and $F(z) = az$. $\square$

Geometrically this corresponds to rotate and scale the lattice. By the theorem, every torus can be written in the form $\mathbb{C}/\Lambda$ where $\Lambda = \Lambda(\tau, 1) = \tau\mathbb{Z} + \mathbb{Z}$ for some $\tau \in H$. When do two parameters $\tau, \tau' \in H$ give the same torus? By proposition 34 and the theorem above, this happens if and only if there is $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z})$ such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \tau \\ 1 \end{pmatrix}$ and $\begin{pmatrix} \tau' \\ 1 \end{pmatrix}$ are a multiple of each other, that is, if $\tau' = \frac{a\tau+b}{c\tau+d}$. Notice that

$$\mathrm{Im}\, \frac{a\tau + b}{c\tau + d} = \frac{(ad - bc)\mathrm{Im}\, \tau}{|c\tau + d|^2}.$$

Thus, if $\tau \in H$ then $\tau' = \frac{a\tau+b}{c\tau+d} \in H$ if and only if $ad - bc = 1$, that is, if the matrix is in $SL_2(\mathbb{Z})$. Hence $SL_2(\mathbb{Z})$ acts on H and, since the action of A and $-A$ is the same, we actually have a faithful action of $PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z})/\langle -I \rangle$ on H. We denote $\Gamma = PSL_2(\mathbb{Z})$ and call this the modular group. By our observations, there is a one to one correspondence between H/Γ and the complex structures on the torus up to biholomorphism. This correspondence is defined by $[\tau]_\Gamma \to \mathbb{C}/(\tau\mathbb{Z}+\mathbb{Z})$.

We return now to our functions $f = g_2, g_3, \Delta, J$ and see how they behave under Γ. We have

$$f(\tau) = f(\tau, 1) = f(a\tau + b, c\tau + d) = (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right).$$

Here $-k$ is he degree of homogeneity of f, which is $4, 6, 12$ and 0 for g_2, g_3, Δ, J, respectively. In particular, J is invariant under the action of the modular group Γ.

Definition 15. *A meromorphic function $f : H \to \mathbb{C}$ is said to be a modular form of weight k if it satisfies*

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau)$$

for every $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and has a Fourier expansion of the form

$$f(\tau) = \sum_{n=-N}^{\infty} c(n) e^{2\pi i n\tau}.$$

A modular form of weight 0 is called a modular function.

The last condition may seem strange, but it will be soon clarified; for now, let's ignore it. We consider for now just the case of modular functions. The action of Γ in H is not free, but we will be able to define a complex structure on H/Γ. This will be done essentially in the same way as described in [11]; notice that in our case the group is not discrete, but we'll see soon that Γ has "small isotropy", in the sense that the points with nontrivial isotropy are discrete and, for those, the isotropy group is finite, so proposition 3.3 in [11] still applies. Then a modular function (forgetting for now the last condition) is a meromorphic function in H/Γ. To describe H/Γ and its complex structure we need a description of Γ.

Proposition 36. *The group Γ is generated by the matrices*

$$S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Also, we have the relations $T^2 = 1$, $(ST)^3 = 1$ and $\Gamma \cong \langle S, T | T^2 = (ST)^3 = 1\rangle \cong \mathbb{Z}_2 * \mathbb{Z}_3$.

The proof that S and T generate Γ is found in [2], 2.1. In [1] there is a proof that $\Gamma \cong \mathbb{Z}_2 * \mathbb{Z}_3$.

We say that $R_\Gamma \subseteq H$ is a fundamental region of Γ if R_Γ is an open and connected space such that no two points in R_Γ are equivalent under Γ and every point in H is equivalent to some point in the closure $\overline{R_\Gamma}$ of R_Γ.

Proposition 37. *The set*

$$R_\Gamma = \{z \in H : |z| > 1, |z + \bar{z}| < 1\}$$

is a fundamental region of Γ. *Moreover, the only points* $\tau \in \overline{R}_\Gamma$ *with non-trivial isotropy are* i *and* $\rho, \rho + 1$*; the isotropy of* i *is* $\{I, S\}$ *and the isotropy of* ρ *is* $\{I, TS, (TS)^2\}$.

Proof. The proof is done in [2], theorem 2.3, except for the part of the isotropy groups on ∂R_Γ. A simple extension of the the same reasoning shows that we can't have isotropy except at $i, \rho, \rho + 1$. To compute the isotropy of ρ, for example, notice that $A\rho = \rho$ is equivalent to

$$\frac{a\rho + b}{c\rho + d} = \rho \Leftrightarrow c\rho^2 + (d - a)\rho - b = 0.$$

If $A \neq I$ this implies that $c = d - a = -b$, because $x^2 + x + 1$ is the minimal polynomial of ρ. But then $1 = ad - bc = a(a + c) + c^2$ implies that $(a, c) = (0, 1)$ or $(a, c) = (1, 0)$, which give the solutions TS and $(TS)^2$, respectively. $\square$

Denote by $\pi : H \to H/\Gamma$ the projection in the quotient. We shall now give a complex structure to H/Γ. By the proposition, if $x \in H$ is not equivalent to i or ρ it has no isotropy, so we can chose a neighborhood U of x such that $\pi|U$ is a homeomorphism and define a coordinate chart around $\pi(x)$ by $(\pi(U), \pi^{-1})$. It remains to construct coordinate charts around $\pi(i)$ and $\pi(\rho)$. To do so, notice that the map $z \to \frac{z-i}{z+i}$ sends a small disk U (small enough so that the only points in U equivalent under Γ are equivalent under the isotropy group of i) around i to a small disk around 0. By this map, the action of T in U is transformed in the action $z \to -z$. Thus, the mapping $z \to \left(\frac{z-i}{z+i}\right)^2$ is invariant under S, and we take it to be the coordinate function around $\pi(i)$. Similarly, the mapping $z \to \left(\frac{z-\rho^2}{z-\rho}\right)^3$ gives coordinates around $\pi(\rho)$.

With this complex structure, it's easy to see that π has multiplicity 2 at i, multiplicity 3 at ρ and 1 at points not equivalent to either i or ρ. The surface we obtain is not compact. However, we can compactify it by adding a point $i\infty$ – a cusp – which we think of as a point in the end of the imaginary axis. To define a complex structure in $\overline{H}/\Gamma = H/\Gamma \cup \{i\infty\}$ let $U = \{z \in H : \operatorname{Im} z > 1\} \cup \{i\infty\}$ and $q : U \to \mathbb{C}$ be defined by $q(z) = e^{2\pi i z}$ and $q(i\infty) = 0$. By proposition 37 the only points in U equivalent by Γ are equivalent by $\langle S \rangle$ and q is invariant by S, so q induces a function $q : \pi(U) \to \mathbb{C}$; we take $(\pi(U), q)$ to be the coordinate neighborhood around $i\infty$. We can see that $\overline{R}_\Gamma \cup \{i\infty\}$ is the Alexandroff compactification of $\overline{R}_\Gamma$, so $\overline{H}/\Gamma = \pi\left(\overline{R}_\Gamma \cup \{i\infty\}\right)$ is compact, as desired.

Now the condition on the Fourier expansion in the definition of modular function should be clear: the Fourier expansion is the Laurent expansion of f with respect to the coordinate $q = e^{2\pi i z}$, so the condition that its principal part is a finite sum says that we can extend f to a meromorphic function $f : \overline{H}/\Gamma \to \mathbb{C}$; if the Fourier series was infinite to the left, $i\infty$ would be an essential singularity of f. Thus, there is a 1-to-1 correspondence between modular functions and meromorphic functions on the compact Riemann surface $\overline{H}/\Gamma$.

Thanks to this observation, we can now apply the results about compact Riemann surfaces in the first section of this paper to modular functions. By theorem 10 a modular function (when seen as a meromorphic function in $\overline{H}/\Gamma$) takes every value exactly the same amount of times. However, one has to be careful because if we regard a modular function as a meromorphic function in H we have to divide the multiplicity of a zero/pole at i, ρ by 2 and 3, respectively, because π has multiplicity 2 and 3 at those points; also, we have to consider the multiplicity of the point $i\infty$, which is seen by looking at the Fourier series of the function (that is, the Laurent series with respect to q).

We can also give a similar geometric interpretation for modular forms of weight $2k > 0$. Consider a meromorphic k-form on H given by $\omega = f\,dz^k$. Notice that

$$f\left(\frac{az+b}{cz+d}\right) d\left(\frac{az+b}{cz+d}\right)^k = f\left(\frac{az+b}{cz+d}\right)(cz+d)^{-2k}dz.$$

Hence, ω is invariat by the action of Γ if and only if f is modular of weight $2k$, so there is a correspondence between modular forms of degree $2k$ and meromorphic k-forms.

We compute the Fourier expansion of the functions g_2, g_3, Δ and J to analyze their behavior near $i\infty$.

Proposition 38. *We have the following Fourier expansions:*

(1)

$$g_2(\tau) = \frac{4\pi^4}{3}\left(1 + 240\sum_{k=1}^{\infty}\sigma_3(k)e^{2\pi i k\tau}\right)$$

(2)

$$g_3(\tau) = \frac{8\pi^6}{27}\left(1 - 504\sum_{k=1}^{\infty}\sigma_3(k)e^{2\pi i k\tau}\right)$$

(3)

$$\Delta(\tau) = (2\pi)^{12}\sum_{k=1}^{\infty}\tau(k)e^{2\pi i k\tau}$$

(4)

$$12^3 J(\tau) = e^{-2\pi i\tau} + 744 + \sum_{k=1}^{\infty}c(n)e^{2\pi i n\tau}$$

where $\sigma_\alpha(k) = \sum_{d|k} d^\alpha$ and $\tau(k), c(k)$ are sequences of integers.

A proof of these can be found in 1.18, 1.19 and 1.20 in [2] and it relies essentially on the identity

$$\pi \cot \pi n\tau = \frac{1}{n\tau} + \sum_{k\in\mathbb{Z}\setminus\{0\}} \left(\frac{1}{n\tau + m} - \frac{1}{m}\right).$$

Comparing this identity to the Fourier expansion of cot, differentiating repeatedly and summing over n gives the expressions for g_2 and g_3.

This shows that J has a simple pole at $i\infty$. We can see, by looking at the identifications on the edges of $\overline{R}_\Gamma$ given by the action of Γ, that topologically $\overline{H}/\Gamma$ is a sphere, and therefore simply connected. By the uniformization theorem, $\overline{H}/\Gamma$ is biholomorphic to the Riemann sphere. Indeed, J defines such a biholomorphism.

Theorem 39. *The function J defines a biholomorphism from $\overline{H}/\Gamma$ to the Riemann sphere. Also, $J(\rho) = 0$ and $J(i) = 1$ with multiplicities 3 and 2, respectively.*

Proof. The only pole of J is a simple pole at $i\infty$. Thus J has order 1. Hence it's injective, and by theorem 4 it's a biholomorphism. The values at ρ and i can be verified by checking that $g_2(\rho) = g_3(i) = 0$ and their multplicities follow from the fact that J has order 1. □

This shows that $\mathbb{C}/\Lambda(\omega_1, \omega_2) \to J(\omega_1/\omega_2)$ defines a correspondence between the possible complex structures on the torus and $\mathbb{C}$. The space of complex structures on the torus is called the moduli space of the torus and is identified with $\mathbb{C}$. For $g > 1$ the moduli space of the topological surface with genus g has dimension $3g - 3$.

Corollary 40. *Every modular function is a rational function of J.*

Proof. This is straightforward from theorem 39 and corollary 8. □

3.3. **Hecke operators.**

Hecke operators are an important tool in the theory of modular forms. They were introduced to study modular forms with coefficients obeying certain multiplicative relations. Hecke operators can be defined either directly from their formula (this is done in [2]) or as abstract operators on lattices (as in [12]); we follow the later approach. Here we will use Hecke operators to show that τ defined in proposition 38 is a multiplicative function (and satisfies a more general multiplicative property), but won't explore much more of their applications; such applications can be found in chapter 6 of [2].

Denote by $\mathcal{L}$ the space of lattices of $\mathbb{C}$ and by $\mathbb{Z}[\mathcal{L}]$ the abelian free group generated by $\mathcal{L}$.

Definition 16. *Let $n \geq 1$ be an integer. We define the Hecke operator $T_n : \mathbb{Z}[\mathcal{L}] \to \mathbb{Z}[\mathcal{L}]$ by*

$$T_n\Lambda = \sum_{(\Lambda:\Lambda')=n} \Lambda' \quad \text{for } \Lambda \in \mathcal{L}.$$

The (formal) sum runs every lattice $\Lambda' \subseteq \Lambda$ with index n in Λ. Given $\lambda \in \mathbb{C}^*$ we also define the scalar operators $R_\lambda : \mathbb{Z}[\mathcal{L}] \to \mathbb{Z}[\mathcal{L}]$ by

$$R_\lambda \Lambda = \lambda \Lambda.$$

The sum defining the Hecke operators is finite since $(\Lambda : \Lambda') = n$ implies that $n\Lambda \subseteq \Lambda' \subseteq \Lambda$. Hence there is a 1-to-1 correspondence between lattices with $(\Lambda : \Lambda') = n$ and subgroups of index n of $\Lambda/n\Lambda \cong \mathbb{Z}_n^2$. We have the following identities about the composition of Hecke and scalar operators.

Proposition 41. *Let $n, m, k \in \mathbb{Z}^+$, let p be a prime and $\lambda \in \mathbb{C}^*$. Then we have*

(1) $R_\lambda R_\mu = R_{\lambda\mu}$;
(2) $R_\lambda T_n = T_n R_\lambda$;
(3) If $(m, n) = 1$ then $T_m T_n = T_{mn}$;
(4) $T_{p^k} T_p = T_{p^{k+1}} + pT_{p^{k-1}} R_p$;
(5) $T_m T_n = \sum_{d|(m,n)} dT_{\frac{mn}{d^2}} R_d$.

Proof (sketch). The first four are proved in proposition 10 of chapter VII in [12]. To prove the last one we can show, using (4), by induction on l that the formula holds when $n = p^k, m = p^l$ are powers of the same prime. After that, using (3) we get the general case easily. $\square$

The next proposition describes what are the lattices Λ' such that $(\Lambda : \Lambda') = n$.

Proposition 42. *Let $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ be a lattice. There is a bijection between the set of matrices of the form $A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ with $a, d \geq 0$, $ad = n$ and $0 \leq b < d$ and the set of lattices Λ' such that $(\Lambda : \Lambda') = n$ given by*

$$A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \to \Lambda(a\omega_1 + b\omega_2, d\omega_2) \equiv \Lambda(A).$$

Proof. Since $\det A = n$ it's clear that $(\Lambda : \Lambda(A)) = n$. If Λ' is such that $(\Lambda : \Lambda') = n$, define

$$Y_1 = \Lambda/(\Lambda' + \mathbb{Z}\omega_2) \text{ and } Y_2 = \mathbb{Z}\omega_2/(\mathbb{Z}\omega_2 \cap \Lambda').$$

These are cyclic groups generated by the images of ω_1 and ω_2, respectively; let a and d be their orders. By the second isomorphism theorem

$$(\Lambda' + \mathbb{Z}\omega_2)/\Lambda' \cong \mathbb{Z}\omega_2/(\mathbb{Z}\omega_2 + \Lambda') = Y_2.$$

By the third isomorphism theorem $(\Lambda/\Lambda')/Y_2 \cong Y_1$, and therefore $ad = |\Lambda/\Lambda'| = n$. Since Y_2 has order d we know that $d\omega_2 \in \Lambda'$ and since Y_1 has order a we get that $a\omega_1 \in \Lambda' + \mathbb{Z}\omega_2$, so there is $b \in \mathbb{Z}$ such that $0 \leq b < d$ and $a\omega_1 + b\omega_2 \in \Lambda'$. This defines a map from the set of lattices with $(\Lambda : \Lambda') = n$ to the set of described matrices which can be easily seen to be an inverse of the map $A \to \Lambda(A)$. $\square$

We can also look at Hecke operators as operators on the space of holomorphic modular forms. Recall that a modular form f of weight k can be regarded as a function $F : \mathcal{L} \to \mathbb{C}$ homogeneous of degree $-k$ by defining

$$F(\Lambda(\omega_1, \omega_2)) = \omega_2^{-k} f(\omega_1/\omega_2).$$

We can then extend linearly F to $F : \mathbb{Z}[\Lambda] \to \mathbb{C}$. By (2) in proposition 41 the function $T_n^* F = F \circ T_n : \mathbb{Z}[\Lambda] \to \mathbb{C}$ is also homogeneous of degree $-k$, so it corresponds to a modular function; we define the Hecke operator as this function multiplied by n^{k-1} (this factor serves only to simplify formulae). Concretely, we are defining

$$(T_n f)(\tau) = n^{k-1}(F \circ T_n)(\Lambda(\tau, 1)).$$

Using proposition 42 we can give an explicit formula for $T_n f$:

$$\begin{aligned}(T_n f)(\tau) &= n^{k-1} \sum_{\substack{ad=n \\ 0 \le b < d}} F(\Lambda(a\tau + b, d)) = n^{k-1} \sum_{\substack{ad=n \\ 0 \le b < d}} d^{-k} f\left(\frac{a\tau + b}{d}\right) \\ &= \frac{1}{n} \sum_{\substack{ad=n \\ 0 \le b < d}} a^k f\left(\frac{a\tau + b}{d}\right)\end{aligned}$$

From this formula it's definitely not evident that if f is a modular form then $T_n f$ is also a modular form, but considering its construction from a homogeneous function on $\mathcal{L}$ that should be clear. It's also obvious from the formula that if f is meromorphic in H then so is $T_n f$. At last, the following lemma computes the Fourier expansion of $T_n f$ showing that it's also meromorphic at the cusp $i\infty$.

Lemma 43. *Let $f : H \to \mathbb{C}$ be a modular form of weight k with Fourier series*

$$f(\tau) = \sum_{m=-N}^{\infty} c(m) e^{2\pi i m \tau}.$$

Then $T_n f$ is also a modular form of weight k with Fourier series

$$(T_n f)(\tau) = \sum_{m=-N}^{\infty} \gamma_n(m) e^{2\pi i m \tau}.$$

where γ is given by

$$\gamma_n(m) = \sum_{d|(n,m)} d^{k-1} c\left(\frac{mn}{d^2}\right).$$

Also, if f is holomorphic, then $T_n f$ is also holomorphic.

The proof is a simple computation after plugging the Fourier series of f in the formula for $T_n f$ and can be found in proposition 11 of chaper VII of [12]. With these results, we can now prove easily an interesting identity.

Theorem 44. *The coefficients $\tau(n)$ in the Fourier series of Δ in proposition 38 satisfy the following multiplicative property:*

$$\tau(n)\tau(m) = \sum_{d|(m,n)} d^{11}\tau\left(\frac{mn}{d^2}\right).$$

In particular τ is a multiplicative function.

Proof. Recall that Δ is a holomorphic modular form of weight 12, so $T_n\Delta$ is also a holomorphic modular form of weight 12. By proposition 38 Δ has a zero of order 1 at $i\infty$. By lemma 43 $T_n\Delta$ also has a zero of order 1 at $i\infty$. Therefore $\frac{T_n\Delta}{\Delta}$ is a holomorphic modular function, hence it's constant and $T_n\Delta = \lambda(n)\Delta$ for some $\lambda(n) \in \mathbb{C}$. Comparing the Fourier expansions using lemma 43 it follows that

$$\lambda(n)\tau(m) = \sum_{d|(m,n)} d^{11}\tau\left(\frac{mn}{d^2}\right).$$

Plugging $m = 1$ we get that $\lambda(n) = \tau(n)$ and the stated identity follows. □

3.4. **Congruence subgroups.** In this last section we will study functions which are not invariant under the action of Γ but are invariant under the action of certain subgroups G of Γ, namely the congruence subgroups. Such functions are called automorphic functions under G; in particular a automorphic function under Γ is a modular function.

Definition 17. *The n-congruence subgroup $\Gamma_0(n)$ is the subgroup*

$$\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : n|c\right\} < \Gamma.$$

The next proposition describes the cosets of $\Gamma_0(p)$ in Γ

Proposition 45. *Let p be a prime. Then the set $\{I, S, ST, \ldots, ST^{p-1}\}$ is a set of representatives of the right cosets of $\Gamma_0(p)$ in Γ. That is, for every $A \in \Gamma$ either $A \in \Gamma_0(p)$ or there is a unique $0 \leq k < p$ such that $V(ST^k)^{-1} \in \Gamma_0(p)$.*

Proof. Notice that if $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, W = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \Gamma$ then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d' & -b' \\ -c' & a' \end{pmatrix} = \begin{pmatrix} * & * \\ cd' - dc' & * \end{pmatrix}.$$

Thus $VW^{-1} \in \Gamma_0(p)$ if and only if $cd' - dc' \equiv_p 0$. Suppose that $V \notin \Gamma_0(p)$ (that is, $p \nmid c$). Noticing that $ST^k = \begin{pmatrix} 0 & -1 \\ 1 & k \end{pmatrix}$, $V(ST^k)^{-1} \in \Gamma_0(p)$ if and only if $ck \equiv_p d$, which has a unique solution $0 \leq k < p$. □

From this, a description of a fundamental region of $\Gamma_0(p)$ follows straightforwardly.

Proposition 46. *For a prime p the set*

$$R_{\Gamma_0(p)} = R_\Gamma \cup \bigcup_{k=0}^{p-1} ST^k(R_\Gamma)$$

is a fundamental region of $\Gamma_0(p)$.

We can give $H/\Gamma_0(p)$ a complex structure the same way we gave a complex structure to H/Γ. We can think of this complex structure as the unique one such that the natural projection $f : H/\Gamma_0(p) \to H/\Gamma$ is holomorphic. However, if $p > 1$ in order to compactify H/Γ it's not enough to add the point $i\infty$; this is because $0 (= S(i\infty))$ is in the closure of $R_{\Gamma_0(p)}$ as a subset of $\mathbb{C}$. Hence we define $\overline{H}/\Gamma_0(p) = H/\Gamma_0(p) \cup \{0, i\infty\}$ for $p > 1$. We can define a chart around 0 by $z \to e^{-2\pi i/z}$; notice that this is just the composition of S with the chart $z \to e^{2\pi i z}$ around $i\infty$. The Riemann surface obtained is compact since $R_{\Gamma_0(p)} \cup \{0, i\infty\}$ is compact.

Remark 2. In general, if G is a subgroup of Γ, to compactify H/G we must add cusps. Cusps are the points in the extended hyperbolic plane (which is $H \cup \mathbb{R} \cup \{i\infty\}$) which are fixed points of parabolic elements of Γ; here we are considering Γ acting on the extended hyperbolic plane in the natural way, in particular with $S(i\infty) = 0$ and $T(i\infty) = i\infty$. The cusps for Γ are easily seen to be $\mathbb{Q} \cup \{0\}$. Therefore, in general we have a compactification $\overline{H}/G$ of H/G where $\overline{H} = H \cup \mathbb{Q} \cup \{0\}$. Under Γ every cusp is equivalent to $i\infty$, so $\overline{H}/\Gamma = H/\Gamma \cup \{i\infty\}$; however, in $\Gamma_0(p)$ for $p > 1$ the cusps 0 and $i\infty$ are not equivalent because $S \notin \Gamma_0(p)$. Given a fundamental region of G, the cusps which are in the boundary of G give a set of representatives for the equivalence classes of cusps under the action of G.

Recall that compact Riemann surfaces are topologically determined by their genus. We compute the genus of the surface $\overline{H}/\Gamma_0(p)$.

Theorem 47. *Let p be a prime and denote by g_p the genus of the Riemann surface $\overline{H}/\Gamma_0(p)$. Then $g_2 = g_3 = 0$ and, for $p > 3$,*

$$g_p = \begin{cases} (p-13)/12 & \text{if } p \equiv_{12} 1 \\ (p-5)/12 & \text{if } p \equiv_{12} 5 \\ (p-7)/12 & \text{if } p \equiv_{12} 7 \\ (p+1)/12 & \text{if } p \equiv_{12} 11. \end{cases}$$

Proof. We will skip the cases $p = 2, 3$ as the same method applies. The inclusion $\Gamma_0(p) \subseteq \Gamma$ gives a natural projection $f : \overline{H}/\Gamma_0(p) \to \overline{H}/\Gamma$; f sends the cusp 0 to $f(0) = i\infty \in \overline{H}/\Gamma$. We denote by π_1 and π_2 the projections of H in $\overline{H}/\Gamma$ and in $\overline{H}/\Gamma_0(p)$, respectively; we have $f \circ \pi_2 = \pi_1$. Recall that by theorem 39 the surface $\overline{H}/\Gamma$ is isomorphic to $\mathbb{P}^1$. Thus, by theorem 10 we know that f is an n-sheeted holomorphic function for some n. But proposition 46 gives $n = p + 1$ because, if $\pi_1(y) \in H/\Gamma$ and $\pi_1(y) \neq \pi_1(i), \pi_1(\rho)$ then f takes the value $\pi_1(y)$ with multiplicity

1 (because at those points f is locally an isomorphism, since those points have trivial isotropy) at the $p+1$ distinct points $\pi_2(y), \pi_2(Sy), \ldots, \pi_2\left(ST^{p-1}y\right)$. Hence we can apply the Hurwitz formula to get

$$2g_p - 2 = -2(p+1) + \sum_{P \in \overline{H}/\Gamma_0(p)} (e_P - 1).$$

Since $\sum_{P \in f^{-1}(y)} e_P = p+1$, we can also write the sum as

$$\sum_{P \in \overline{H}/\Gamma_0(p)} (e_P - 1) = \sum_{y \in \overline{H}/\Gamma} \sum_{P \in f^{-1}(y)} (e_P - 1) = \sum_{y \in \overline{H}/\Gamma} (p + 1 - |f^{-1}(y)|).$$

If $y \neq i, \rho, i\infty$ (here we are writing i, ρ for $\pi_1(i), \pi_1(\rho)$, respectively) we already saw that the corresponding term in the sum is 0. It's clear that $f^{-1}(i\infty) = \{i\infty, 0\}$.

We now compute $|f^{-1}(i)|$. By proposition 45 every point $x \in f^{-1}(i)$ has the form $\pi_2(i)$ or $\pi_2(ST^k i)$ for some $0 \leq k < p$. However, some of those points are the same. Clearly $\pi_2(i) = \pi_2(Si)$. We shall see for what pairs $0 \leq k, l < p$ we have $\pi_2(ST^k i) = \pi_2(ST^l i)$. This is equivalent to the existence of some $A \in \Gamma_0(p)$ such that $ST^k i = AST^l i$, that is, $(T^{-k}SAST^l)i = i$. But since the isotropy of i is just $\{I, S\}$ we either have $T^{-k}SAST^l = I$ or $T^{-k}SAST^l = S$. For the first case,

$$A = ST^{k-l}S = \begin{pmatrix} -1 & 0 \\ k-l & -1 \end{pmatrix}.$$

which is in $\Gamma_0(p)$ if and only if $l = k$, since $0 < k, l < p$. For the second case,

$$A = ST^kST^{-l}S = \begin{pmatrix} l & 1 \\ -1-kl & -k \end{pmatrix}.$$

Hence, $\pi_2(ST^k i) = \pi_2(ST^l i)$ if and only if $k = l$ or $p | kl + 1$, that is, $l \equiv_p k$ or $k \neq 0$ and $l \equiv_p -1/k$. If $p \equiv_3 4$ there is no $0 < \alpha < p$ such that $p | \alpha^2 + 1$, so we can group the $p - 1$ points $ST^k i$ in pairs with the same image by π_2. Thus, in this case $|f^{-1}(i)| = 1 + \frac{p-1}{2} = \frac{p+1}{2}$. If $p \equiv_4 1$ there is an α such that $p | \alpha^2 + 1$. For $k \equiv_p \alpha, -\alpha$ the only $0 < l < p$ such that $\pi_2(ST^k i) = \pi_2(ST^l i)$ is $l = k$; the remaining $p - 3$ points $ST^k i$ are again grouped in pairs, so in this case we get $|f^{-1}(i)| = 1 + 2 + \frac{p-3}{2} = \frac{p+3}{2}$.

To compute $|f^{-1}(\rho)|$ we proceed in a similar way. We have $\rho = ST\rho$ so we just have to see for what pairs $0 \leq k, l < p$ we have $\pi_2(ST^k\rho) = \pi_2(ST^l\rho)$. Since the isotropy of ρ is $\{I, ST, (ST)^2\}$, by an argument similar to the above one this happens if and only if $T^{-k}SAST^l \in \{I, ST, (ST)^2\}$. Again the case of the identity only gives $k = l$. For the remaining two, we compute the matrices

$$ST^k(ST)T^{-l}S = \begin{pmatrix} -1+l & 1 \\ -1+k-kl & -k \end{pmatrix} \text{ and } ST^k(ST)^2T^{-l}S = \begin{pmatrix} -l & 1 \\ -1+(1-k)l & -k \end{pmatrix}.$$

We conclude that $\pi_2(ST^k\rho) = \pi_2(ST^l\rho)$ if and only if $k = l$, $\{k, l\} = \{0, 1\}$ or $l, k \neq 0, 1$ and

$$l \equiv_p (-1 + k)/k, 1/(1 - k).$$

If $p \equiv_3 2$ there is no β such that $-1 + \beta - \beta^2 \equiv_p 0$, hence each of the sets $\{k, (-1 + k)/k, 1/(1 - k)\}$ has 3 distinct elements mod p and induce a partition of $\{2, \dots, p-1\}$. In this case, $|f^{-1}(\rho)| = 1+\frac{p-2}{3} = \frac{p+1}{3}$. If $p \equiv_3 1$ there is a β such that $-1+\beta-\beta^2 \equiv_p 0$; for $k \equiv \beta, 1-\beta$ the only $0 < l < p$ such that $\pi_2(ST^k i) = \pi_2(ST^l i)$ is $l = k$; the remaining $p-4$ points $ST^k\rho$ (with $k \neq 0, 1, \beta, 1-\beta$) are again grouped in sets of three elements, so in this case we get $|f^{-1}(\rho)| = 1 + 2 + \frac{p-4}{3} = \frac{p+5}{2}$.

Now the result follows from analyzing each case. For instance, if $p \equiv_{12} 1$ then $p \equiv_4 1$ and $p \equiv_3 1$ and the Hurwitz formula reads

$$2g_p-2 = -2(p+1)+(p+1-2)+\left(p+1-\frac{p+3}{2}\right)+\left(p+1-\frac{p+5}{3}\right) = \frac{p-13}{6}-2$$

which is equivalent to the stated answer. The other cases are similar. □

As a consequence of the above result, the surface $\overline{H}/\Gamma_0(p)$ is simply connected (has genus 0) if and only if $p = 2, 3, 5, 7, 13$. By the uniformization theorem, this means that $\overline{H}/\Gamma_0(p)$ is biholomorphic to $\mathbb{P}^1$. Thus, for such primes there is a fuction Φ automorphic under $\Gamma_0(p)$ having order 1; this function Φ plays the same role in $\Gamma_0(p)$ that J plays in Γ.

The idea to define Φ is to consider the function $\varphi(\tau) = \frac{\Delta(p\tau)}{\Delta(\tau)}$. This function is automorphic under $\Gamma_0(p)$ because, if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(p)$, we can write $c = pc'$ and

$$\Delta\left(p\frac{a\tau+b}{c\tau+d}\right) = \Delta\left(\frac{a(p\tau)+bp}{c'(p\tau)+d}\right) = (c\tau+d)^{12}\Delta(p\tau).$$

Since we also have $\Delta\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{12}\Delta(\tau)$, the invariance of φ follows. Since Δ has no zeros and no poles in H, φ only has a zero at $i\infty$. The zero of Δ at $i\infty$ has multiplicity 1, hence the zero of φ at $i\infty$ has multiplicity $p-1$. We would like to take a $(p-1)$-root of φ and define $\Phi = \varphi^{1/(p-1)}$. We will be able to do this for the primes $p = 2, 3, 5, 7, 13$ using the Dedekind eta function.

Definition 18. *The Dedekind eta function is the function $\eta : H \to \mathbb{C}$ defined by*

$$\eta(\tau) = e^{\pi i\tau/12}\prod_{n=1}^{\infty}\left(1 - e^{2\pi i n\tau}\right).$$

When $\tau \in H$ we have $|e^{2\pi i\tau}| < 1$, so the product converges absolutely and is non-zero. The convergence is uniform on compact sets of H, hence η is holomorphic in H. The Dedekind eta function satisfies the following transformations under the action of the modular group:

Theorem 48. *The Dedekind eta function* η *satisfies the identities*

$$\eta(\tau+1) = e^{\pi i/12}\eta(\tau) \text{ and } \eta\left(-\frac{1}{\tau}\right) = (-i\tau)^{1/2}\eta(\tau).$$

More generally, for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right) = \varepsilon(a,b,c,d)(-i(c\tau+d))^{1/2}\eta(\tau)$$

where

$$\varepsilon(a,b,c,d) = \exp\left(\pi i\left(\frac{a+d}{12c} + s(-d+c)\right)\right)$$

and $s(h,k)$ *are the Dedekind sums given by*

$$s(h,k) = \sum_{r=1}^{k-1} \frac{r}{k}\left(\frac{hr}{k} - \left\lfloor \frac{hr}{k} \right\rfloor - \frac{1}{2}\right).$$

A proof of these can be found in theorems 3.1 and 3.4 of [2]. Now we have the following remarkable relation between the functions η and Δ.

Theorem 49. *We have, for* $\tau \in H$ *and* $x = e^{2\pi i\tau}$ *the equality*

$$\Delta(\tau) = (2\pi)^{12}\eta(\tau)^{24} = (2\pi)^{12}x\prod_{k=1}^{\infty}(1-x^n)^{24}.$$

Proof. Using theorem 48 it's clear that η^{24} is a modular form of weight 12. The last equality is obvious from the definition of η and tells that η has a zero at $i\infty$. Using the same argument we used in the proof of theorem 44 there is a constant λ such that $\Delta = \lambda\eta^{24}$. Comparing the first term in the Fourier expansion of both sides (recall proposition 38) we get $\lambda = (2\pi)^{12}$. □

The primes $p = 2, 3, 5, 7, 13$ are precisely the primes such that $p-1|24$. Considering the identity we just proved, we now have a very natural way of defining a $p-1$ root of φ.

Proposition 50. *Let* $p = 2, 3, 5, 7$ *or* 13 *and let* $r = 24/(p-1)$*. Then the function*

$$\Phi(\tau) = \left(\frac{\eta(p\tau)}{\eta(\tau)}\right)^r$$

is automorphic under $\Gamma_0(p)$*. Moreover it defines a biholomorphism between* $H/\Gamma_0(q)$ *and* $\mathbb{P}^1$*.*

The fact that Φ is automorphic is not immediate and requires theorem 48 and some congruence relations satisfied by the Dedekind sums. A proof can be found in [2], theorem 4.9. Since $\Phi^{q-1} = \varphi$, Φ has a single 0 at $i\infty$, so it has order 1 and is a biholomorphism.

Remark 3. For composite $n > 1$ the group $\Gamma_0(n)$ has a different structure: for instance its index in Γ is not $n+1$ and $\overline{H}/\Gamma_0(n)$ has more cusps than just 0 and $i\infty$. In [5] there is a list of the subgroups of Γ with a corresponding Riemann surface of genus at most 24 and some of their properties such as their index in Γ and number of cusps. The composite numbers $n > 1$ for which $\overline{H}/\Gamma_0(n)$ has genus 0 are $n = 4, 6, 8, 9, 10, 12, 16, 18, 25$. For those n we can still construct a biholomorphism to $\mathbb{P}^1$ using the Dedekind η function. For example the function

$$\Phi(\tau) = \frac{\eta(8\tau)^6}{\eta(4\tau)^2\eta(16\tau)^8}$$

is automorphic under $\Gamma_0(4)$ and defines a biholomorphism $\overline{H}/\Gamma_0(n) \to \mathbb{P}^1$ (see [9]).

Finally, we will use the function Φ we constructed to prove some congruence relations that are satisfied by the coefficients c of the Fourier expansion of $j = 12^3 J$ which are given in proposition 38; more precisely, we will prove that $c(pn)$ is divisible by certain powers of p for $p = 2, 3, 5, 7$. To do this, we construct automorphic functions under $\Gamma_0(p)$ from modular functions in the following way:

Definition 19. *Suppose that f is a modular function and p is prime. We define f_p by*

$$f_p(\tau) = \frac{1}{p}\sum_{b=0}^{p-1} f\left(\frac{\tau+b}{p}\right) = (T_p f)(\tau) - \frac{1}{p} f(p\tau)$$

where T_p is the Hecke operator.

Proposition 51. *If f is a modular function with Fourier expansion $f(\tau) = \sum_{n=-N}^{\infty} a(n)e^{2\pi i n\tau}$ then $f_p(\tau)$ is a function automorphic under $\Gamma_0(p)$ with Fourier expansion*

$$f_p(\tau) = \sum_{n=-\lfloor N/p \rfloor}^{\infty} a(np)e^{2\pi i n\tau}.$$

Proof. Since $T_p f$ is a modular function it's enough to show that $\tau \to f(p\tau)$ is automorphic under $\Gamma_0(p)$, which follows from the same argument we used to prove that φ is automorphic under $\Gamma_0(p)$.

The Fourier expansion is a straightforward computation with the Fourier expansion of $T_p f$ given in lemma 43. $\square$

This proposition shows that we have the Fourier expansion $j_p(\tau) = \sum_{n=0}^{\infty} c(pn)x^n$ where $x = e^{2\pi i\tau}$. In particular, j_p is holomorphic at $\tau = i\infty$; therefore j_p has a pole only at 0. By corollary 8 and the fact that Φ is a biholomorphism between $\overline{H}/\Gamma_0(p)$ for $p = 2, 3, 5, 7, 13$ we expect to be able to write j_p has a rational function of Φ; but since both j_p and Φ only have poles at 0, this rational function should actually be a polynomial. To find its form we analyze the behavior of j_p near 0.

Lemma 52. *If p is a prime and $\tau \in H$ then*

$$j_p\left(-\frac{1}{p\tau}\right) = j_p(p\tau) + \frac{1}{p}j(p^2\tau) - \frac{1}{p}j(\tau).$$

Hence, denoting $x = e^{2\pi i\tau}$ we have the Fourier expansion

$$pj_p\left(-\frac{1}{p\tau}\right) = x^{-p^2} - x^{-1} + I(x)$$

were $I(x)$ is a power series in x with integer coefficients.

Proof. Since $T_p f$ is modular,

$$j_p\left(-\frac{1}{\tau}\right) + \frac{1}{p}j\left(-\frac{p}{\tau}\right) = (T_p f)\left(-\frac{1}{\tau}\right) = (T_p f)(\tau) = j_p(\tau) + \frac{1}{p}j(p\tau).$$

Substituting $\tau \to p\tau$ and noticing that $j\left(-\frac{1}{\tau}\right) = j(\tau)$ gives the stated identity.

To get the Fourier expansion we simply plug the Fourier expansions of j and j_p obtained from propositions 38 and 51. $\square$

When $\tau \to i\infty$ we have $-1/(p\tau) \to 0$ and $x \to 0$. To get the behavior of Φ near 0 notice that, by theorem 48

$$\Phi\left(-\frac{1}{p\tau}\right) = \frac{\eta\left(-\frac{1}{\tau}\right)^r}{\eta\left(-\frac{1}{p\tau}\right)^r} = \frac{(-i\tau)^{r/2}\eta(\tau)^r}{(-ip\tau)^{r/2}\eta(p\tau)^r} = \frac{p^{-r/2}}{\Phi(\tau)}.$$

The Fourier series of $\Phi(\tau)$ is the expansion of $x\prod_{j=1}^{\infty}\left(\frac{1-x^{pk}}{1-x^k}\right)^r$, which clearly has integer coefficients. This shows that we have the Fourier expansion

$$\Psi(\tau) \equiv p^{r/2}\Phi\left(-\frac{1}{p\tau}\right) = \frac{1}{\Phi(\tau)} = x^{-1} + I(x)$$

where $I(x)$ is a power series in $x = e^{2\pi i\tau}$ with integer coefficients.

Theorem 53. *For $n > 0$ the coefficients in the Fourier expansion of $j(\tau)$ satisfy the following congruences:*

$$\begin{aligned} c(2n) &\equiv 0 \,(\mathrm{mod}\ 2^{11}) \\ c(3n) &\equiv 0 \,(\mathrm{mod}\ 3^5) \\ c(5n) &\equiv 0 \,(\mathrm{mod}\ 5^2) \\ c(7n) &\equiv 0 \,(\mathrm{mod}\ 7) \end{aligned}$$

Proof. We will now form a linear combination of powers of Ψ in order to cancel the principal part of the Fourier expansion of $pj_p\left(-\frac{1}{p\tau}\right)$, making j_p holomorphic at 0. Since $\Psi(\tau)^k = x^{-k} + \ldots$ (where the remaining coefficients are integers) and

$pj_p\left(-\frac{1}{p\tau}\right) = x^{-p^2} - x^{-1} + I(x)$ by lemma 52, we can find integers $b_1, \dots, b_{p^2}$ such that

$$f\left(-\frac{1}{p\tau}\right) \equiv pj_p\left(-\frac{1}{p\tau}\right) - b_{p^2}\Psi(\tau)^{p^2} - b_{p^2-1}\Psi(\tau)^{p^2-1} - \dots - b_1\Psi(\tau)$$

is holomorphic at $x = 0$. Replacing τ by $-1/(p\tau)$ we get that

$$f(\tau) = pj_p(\tau) - b_{p^2}\left(p^{r/2}\Phi(\tau)\right)^{p^2} - b_{p^2-1}\left(p^{r/2}\Phi(\tau)\right)^{p^2-1} - \dots - b_1p^{r/2}\Phi(\tau)$$

is holomorphic at $\tau = i\infty$; hence it's an holomorphic function in the compact Riemann surface $\overline{H}/\Gamma_0(p)$, so f is constant and, computing f at $i\infty$ we get $f(\tau) = pc(0)$ for every τ, showing that

$$pj_p(\tau) = b_{p^2}\left(p^{r/2}\Phi(\tau)\right)^{p^2} + b_{p^2-1}\left(p^{r/2}\Phi(\tau)\right)^{p^2-1} + \dots + b_1p^{r/2}\Phi(\tau) + pc(0).$$

Comparing the Fourier expansions and considering that Φ has a Fourier expansion with integer coefficients it follows that $p^{r/2-1}|c(pn)$ for $n > 0$ where $r = 24/(p-1)$. $\square$

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