VIRASORO CONSTRAINTS ON MODULI OF SHEAVES AND VERTEX ALGEBRAS

ARKADIJ BOJKO, WOONAM LIM, AND MIGUEL MOREIRA

Abstract. In enumerative geometry, Virasoro constraints were first conjectured in Gromov-Witten theory with many new recent developments in the sheaf theoretic context. In this paper, we rephrase the sheaf-theoretic Virasoro constraints in terms of primary states coming from a natural conformal vector in Joyce's vertex algebra. This shows that Virasoro constraints are preserved under wall-crossing. As an application, we prove the conjectural Virasoro constraints for moduli spaces of torsion-free sheaves on any curve and on surfaces with only (p, p) cohomology classes.

Contents

1. Introduction

This paper concerns the Virasoro constraints on sheaf counting theories. Given a moduli space of sheaves M with a virtual fundamental class $[M]^{\mathrm{vir}}$ we may produce numerical invariants by integrating natural cohomology classes – called descendents – against the virtual fundamental class. The Virasoro conjecture predicts that these numerical invariants are constrained by some explicit and universal relations. The main result of this paper is a proof of those constraints in the following cases:

Theorem A. The Virasoro constraints (Conjecture 1.4) hold for the following moduli spaces:

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(1) The moduli spaces of stable bundles $M_C(r,d)$ on any smooth projective curve C.
(2) The moduli spaces of stable torsion-free sheaves $M_S^H(r,c_1,c_2)$ on any smooth projective surface S with $h^{1,0}(S)=h^{2,0}(S)=0$.
(3) Assuming the technical condition 5.7, the moduli spaces of stable one dimensional sheaves $M_S^H(\beta,n)$ on any smooth projective surface S with $h^{1,0}(S)=h^{2,0}(S)=0$.

The case (2) of the theorem solves the conjecture of D. van Bree in [vB]. The formulation of the other two cases is new; indeed, we provide a very general conjecture that includes many other interesting cases. Our work relies in a fundamental way on the vertex algebra that D. Joyce recently introduced [Joy1, GJT, Joy6] to study wall-crossing of sheaf moduli spaces. Indeed, we explain how the constraints can be naturally formulated in this language.

Theorem B (=Theorem 4.12 and Corollary 1.8)**.** Let X be a curve or a surface with $p_g=0$. There is a natural conformal element ω in the vertex algebra $V_\bullet^{\mathrm{pa}}$ for which the corresponding Virasoro operators L_n are dual to the Virasoro operators $\mathsf{L}_n^{\mathrm{pa}}\colon \mathbb{D}^{X,\mathrm{pa}}\to\mathbb{D}^{X,\mathrm{pa}}$ for $n\geqslant -1$ on the (pair) descendent algebra. A moduli space of sheaves (or pairs) satisfies the Virasoro constraints if and only if its class in $\check{V}_\bullet^{\mathrm{pa}}$ (or $V_\bullet^{\mathrm{pa}}$) is a primary state.

As a consequence of this description, the Joyce's wall-crossing machinery can be used to prove a compatibility between wall-crossing and the Virasoro constraints.

Theorem C (=Propositions 3.11 and 3.13)**.** The Virasoro constraints are compatible with wall-crossing.

This is the fundamental tool that we use in the proof of Theorem A.

1.1. **History.**

Virasoro constraints The study of Virasoro constraints on curve counts traces back to the origins of Gromov-Witten (GW) theory and intersection theory on the moduli space of stable curves, in Witten's foundational paper [Wit]. Witten conjectured that integrals of products of descendents – certain natural classes in $H^\bullet(\overline{\mathcal{M}}_{g,n})$ – obeyed some explicit relations. The relations he proposed were equivalent to certain differential operators, which satisfy the Virasoro bracket relation, annihilating the partition function which encodes all the integrals of descendents on $\overline{\mathcal{M}}_{g,n}$. Witten's conjecture was proven by Kontsevich [Kon] and new proofs were obtained by Okounkov-Pandharipance [OP2] and [Mir].

In [EHX], the authors extended the Virasoro conjecture to the GW invariants of a variety X. Since then, a lot of effort has been put into proving the result for some target varieties X; most notably, the conjecture is now known when X is a toric variety (or, more generally, when X has semisimple quantum cohomology) by work of Givental and Teleman [Giv, Tel] and when X is a curve by work of Okounkov-Pandharipande [OP1]. The general case, however, is still out of reach.

In [MNOP1, MNOP2], Maulik-Nekrasov-Okounkov-Pandharipande propose a deep connection between Gromov-Witten invariants and Donaldson-Thomas (DT) invariants of 3-folds. Such correspondence suggested that the DT descendent invariants should as well be constrained by some sort of Virasoro operators. Almost 15 years ago, not long after the proposal of the MNOP conjecture, Oblomkov, Okounkov and Pandharipande were able to predict the precise form for the DT Virasoro operators (at least for $X = \mathbb{P}^3$, see [Pan, Conjecture 8]) from experimental data with X toric. The understanding of the MNOP correspondence at the time, however, was not sufficiently explicit to be used effectively to relate the conjectures on the GW and on the DT sides. Recently, the GW/PT descendent correspondence has been made more effective and this allowed a proof of the DT Virasoro conjecture when X is a toric 3-fold in the stationary regime [MOOP].[1]

Taking a surface S and $X = S \times \mathbb{P}^1$ it is possible to deduce some Virasoro constraints for the Hilbert scheme of points on S from the PT Virasoro constraints on X. The third author used a universality argument in [Mor] to prove such constraints for every surface with $H^1(S) = 0$ by starting with the toric results in [MOOP]. Subsequently, van Bree proposed a generalization of the Hilbert scheme constraints to the moduli spaces of torsion-free stable sheaves on a surface S and made several non-trivial checks for toric S using localization [vB].

While the Virasoro constraints on sheaf-counting theories come historically from Gromov-Witten theory, they form a rich theory themselves as indicated by the examples where they can be studied – DT, PT, Hilbert scheme, stable torsion-free sheaves on surfaces. We show that they have an independent meaning and origin by connecting them to the geometric construction of the vertex algebras of Joyce [Joy1] which were developed to study wall-crossing.

Wall-crossing and vertex algebras When Donaldson [Don] introduced his invariants counting anti-self-dual instantons, they were intrinsically dependent on the choice of a metric g of the underlying four-manifold. Varying g leads to discontinuous jumps of the invariants along codimension one walls, a phenomenon called "wall-crossing". The precise description of the wall-crossing contributions has been given in [KM] and many further studies have been conducted.

With the goal of treating wall-crossing phenomena uniformly, Joyce [Joy2, Joy3, Joy4, Joy5] developed a theory which could be applied in large generality to abelian categories. Here the metric was replaced by stability conditions and instantons by semistable objects. Using a Lie algebra structure, he was able to define motivic

[1]The results in [MOOP] are formulated entirely in the theory of stable pairs, also known as Pandharipande-Thomas (PT) invariants, and not DT invariants. In the stationary regime for toric 3-folds, however, the two formulations of Virasoro are known to be equivalent.

invariants counting semistable objects and described how they change when varying the stability conditions. Further refinements to include DT theory of 3-folds were considered by Joyce–Song [JS] and Kontsevich-Soibelman [KS].

The (virtual) fundamental classes of sheaves are however not local invariants outside of the realm of Calabi–Yau threefolds, so their theories were not sufficient for studying other geometries. In a more recent development, Joyce [Joy1] introduced a sheaf-theoretic construction of vertex algebras (see [Bor, Kac, LL] for a gentle introduction to this topic). Vertex algebras are representation theoretic objects introduced by Borcherds [Bor] and they give an axiomatization of conformal field theories in two dimensions. The Lie bracket operation induced from the sheaf-theoretic vertex algebras was used to describe wall-crossing of virtual fundamental classes counting semistable objects, as conjectured in [GJT] and proven in many cases by Joyce [Joy6]. For surfaces, these wall-crossing formulas are related to the work of Mochizuki [Moc] where the formulas are presented without vertex algebras.

Wall-crossing has been used in [Boj1, Boj2] to give explicit formulae for all descendent invariants of punctual Quot schemes on surfaces and Calabi–Yau fourfolds, and by Bu [Bu] to study moduli spaces of vector bundles on curves. However, further structures coming from the vertex algebra remained a mystery. We fill this gap by giving a geometric interpretation of a natural conformal element in terms of Virasoro constraints. The conformal element induces a representation of the Virasoro algebra on Joyce's vertex algebra [Joy1]. The Virasoro operators $\{L_n\}_{n\in\mathbb{Z}}$ act on the homology of the stack where wall-crossing takes place, defining a smaller Lie algebra of *primary/physical states*. We show that the Virasoro constraints are precisely the statement that (virtual) fundamental classes of moduli of semistable sheaves are physical states, and thus are preserved by wall-crossing. We use this new technique, together with a rank reduction argument, to prove existing and new conjectures about Virasoro constraints.

1.2. **Moduli of sheaves and pairs.** Let X be a projective smooth variety over the complex numbers. Typically, we will restrict ourselves to small dimension X (up to dimension 3) so that the moduli spaces of sheaves that we consider have a virtual fundamental class in the sense of Behrend-Fantechi [BF].[2]

The main objects in this paper are moduli spaces M which parameterize (semistable) sheaves on X and their cohomology. Throughout the introduction, we will make some simplifying assumptions about M:

(1) M is a projective scheme of finite type.
(2) There are no $\mathbb{C}$-points of M corresponding to strictly semistable sheaves.

[2]Virasoro constraints are also expected for moduli spaces of sheaves on Calabi-Yau 4-folds, admitting a virtual fundamental class in the sense of Oh-Thomas [OT]. However, we will not consider this case here and it will be the subject of future work.

(3) There exists a universal sheaf $\mathbb{G}$ in $M \times X$; the sheaf $\mathbb{G}$ is, in principle, not unique.[3]
(4) Deformation theory at $[G] \in M$ is given by

$$\begin{aligned} \mathrm{Tan} &= \mathrm{Ext}^1(G, G) \\ \mathrm{Obs} &= \mathrm{Ext}^2(G, G) \\ 0 &= \mathrm{Ext}^{>2}(G, G)\,. \end{aligned}$$

In such conditions, M admits a virtual fundamental class $[M]^{\mathrm{vir}} \in H_\bullet(M)$ by [BF]. Assumption (2) will be removed by replacing the virtual fundamental class, which does not exist when M has strictly semistable sheaves, by Joyce's invariant class

$$[M]^{\mathrm{inv}} \in H_\bullet(\mathcal{M}_X^{\mathrm{rig}})\,,$$

which is defined by a wall-crossing formula as we will overview in Section 5.3. Assumption (3) can also be removed, as we explain in Section 2.4, but for the sake of concreteness we assume it for now. There are many examples of moduli of sheaves on curves, surfaces and Fano 3-folds where all the assumptions are satisfied.

Apart from the moduli of sheaves, we also study those of pairs. We fix a sheaf V on X and we let P be a moduli space parametrizing a sheaf F together with a map $V \to F$ (with some stability condition). We make the following assumptions:

(1) P is a projective scheme of finite type.
(2) There are no $\mathbb{C}$-points of P corresponding to strictly semistable pairs.
(3) There is a unique universal pair $q^*V \to \mathbb{F}$ in $P \times X$ where $q\colon P \times X \to X$ is the projection.
(4) Deformation theory at $[V \to F] \in P$ is given by

$$\begin{aligned} \mathrm{Tan} &= \mathrm{Ext}^0([V \to F], F) \\ \mathrm{Obs} &= \mathrm{Ext}^1([V \to F], F) \\ 0 &= \mathrm{Ext}^{>1}([V \to F], F)\,. \end{aligned}$$

In such conditions, P admits a virtual fundamental class $[P]^{\mathrm{vir}} \in H_\bullet(P)$. Various moduli of pairs on curves and surfaces satisfy these assumptions; Quot schemes with at most one dimensional quotients and moduli of μ^t-semistable pairs.

There are two important differences between moduli of sheaves and pairs.

(1) The first is the difference in obstruction theory. It is apparent from comparing Example 2.6 and the definition of the Virasoro operators in 2.3 that obstruction theory dictates their form.
(2) The second is the uniqueness or non-uniqueness of the universal object. This difference will play a crucial role in our treatment of the Virasoro constraints for moduli of sheaves and for moduli of pairs.

[3] Universal sheaf is non-unique for the following reason: given a universal sheaf $\mathbb{G}$ and a line bundle L on M, the sheaf $\mathbb{G}\otimes p^*L$ is also a universal sheaf (where $p\colon M\times X \to M$ is the projection) parametrizing the same objects.

Remark 1.1. When we refer to moduli of sheaves we are mostly thinking about moduli of sheaves without fixed determinant. This is implicit in the obstruction theory above since when the determinant is fixed the deformation theory should instead use traceless Ext groups:

$$\mathrm{Ext}^i(G,G)_0 = \ker\left(\mathrm{Ext}^i(G,G) \to H^i(\mathcal{O}_X)\right).$$

We explain how to obtain a fixed determinant version of the Virasoro constraints in Section 2.8 when $h^{1,0} \neq 0$ but $h^{p,0} = 0$ for $p > 1$. Although a conjecture for Hilbert schemes of points on surfaces with possibly $p_g = h^{2,0} > 0$ (which have traceless deformation theory) appears in [Mor], our approach in this paper is currently not suitable to understand it. We hope to pursue this direction in the future.

Remark 1.2. Virasoro constraints that we study for moduli of sheaves naturally generalize to moduli of objects in a derived category $D^b(X)$. Indeed, moduli spaces of stable pairs on a 3-fold X (with $H^i(\mathcal{O}_X) = 0$ for $i > 0$) in the sense of Pandharipande-Thomas [PT] are instances of such. We emphasize here that stable pairs on X are subject to Virasoro constraints of sheaf type rather than pair type, despite their name. This is because virtual classes are constructed using the obstruction theory governed by $\mathrm{Ext}^i(I^\bullet, I^\bullet)$ where $I^\bullet = [\mathcal{O}_X \to F] \in D^b(X)$.

1.3. **Universal sheaves and descendents.** Descendents on M are defined using a slant product construction with a universal sheaf $\mathbb{G}$ and the maps

$$\begin{tikzcd} & M \times X \arrow[dl, "p"'] \arrow[dr, "q"] & \\ M & & X\,. \end{tikzcd}$$

Definition 1.3. We let $\mathbb{D}^X$ be the supercommutative algebra generated by symbols $\mathrm{ch}_i^{\mathrm{H}}(\gamma)$ for $i \geqslant 0$, $\gamma \in H^\bullet(X)$ (see Definition 2.3). The geometric realization with respect to a universal sheaf $\mathbb{G}$ in $M \times X$ is the algebra homomorphism

$$\xi_{\mathbb{G}} \colon \mathbb{D}^X \to H^\bullet(M)$$

defined on generators $\mathrm{ch}_i^{\mathrm{H}}(\gamma)$ with $\gamma \in H^{r,s}(X)$ by

$$\xi_{\mathbb{G}}\left(\mathrm{ch}_i^{\mathrm{H}}(\gamma)\right) = p_*\left(\mathrm{ch}_{i+\dim(X)-r}(\mathbb{G})\, q^*\gamma\right).$$

The shift in the index of the Chern character using the Hodge degree of γ is non-standard, but useful for a cleaner formulation of the Virasoro operators. With this convention, we may think of $\xi_{\mathbb{G}}(\mathrm{ch}_i^{\mathrm{H}}(\gamma))$ as being in $H^{i,i-r+s}(M)$ (of course M might be singular, so a Hodge decomposition may not exist). See also Remark 2.4.

The main objects of study in this paper are descendent integrals, i.e., the enumerative invariants obtained by integrating descendents against the virtual fundamental classes

$$\int_{[M]^{\mathrm{vir}}} \xi_{\mathbb{G}}(D)\,, \quad \int_{[P]^{\mathrm{vir}}} \xi_{\mathbb{F}}(D)\,, \quad \text{for } D \in \mathbb{D}^X.$$

Note that the descendent invariants of M depend in principle on the choice of universal sheaf $\mathbb{G}$. For some D in the descendent algebra, however, they do not depend on this choice; these D form what we call the *weight 0 descendent* algebra $\mathbb{D}^X_{\mathrm{wt}_0}$ (cf. Section 2.4). For $D \in \mathbb{D}^X_{\mathrm{wt}_0}$ we will omit the geometric realization morphism and write

$$\int_{[M]^{\mathrm{vir}}} D = \int_{[M]^{\mathrm{vir}}} \xi_{\mathbb{G}}(D)$$

for any universal sheaf $\mathbb{G}$ since it does not depend on such choice.

The Virasoro constraints say that these numbers satisfy some explicit universal relations. These relations are stated using certain operators

$$\mathsf{L}_{\mathrm{wt}_0} : \mathbb{D}^X \to \mathbb{D}^X_{\mathrm{wt}_0}\,, \quad \mathsf{L}^V_k : \mathbb{D}^X \to \mathbb{D}^X\,, \quad k \geqslant -1$$

that we will introduce in Section 2.

Conjecture 1.4 (Virasoro for sheaves)**.** Let M be a moduli of sheaves as before. Then

$$\int_{[M]^{\mathrm{vir}}} \mathsf{L}_{\mathrm{wt}_0}(D) = 0 \quad \text{ for any } D \in \mathbb{D}^X.$$

Conjecture 1.5 (Virasoro for pairs)**.** Let P be a moduli of pairs as before. Then

$$\int_{[P]^{\mathrm{vir}}} \xi_{\mathbb{F}}\left(\mathsf{L}^V_k(D)\right) = 0 \quad \text{ for any } k \geqslant 0,\, D \in \mathbb{D}^X.$$

Remark 1.6. The previous Virasoro conjectures for sheaves in [MOOP, Mor, vB] require a specific choice of a universal sheaf and S_k operators.[4] Conjecture 1.4 improves the formulation by avoiding both of these, even though we prove that two formulations are equivalent (see Proposition 2.16). Conjecture 1.5 for pairs is new and we provide convincing evidences by proving it for various geometries in this paper.

1.4. **Joyce's vertex algebra.** D. Joyce recently introduced a vertex algebra and a closely related Lie algebra associated to the derived category $D^b(X)$ [Joy1, GJT, Joy6]. Joyce proposes to use his Lie algebra to study wall-crossing formulas for moduli of sheaves (or, more generally, moduli of semistable objects in a $\mathbb{C}$-linear abelian or triangulated category).

The vertex algebra is constructed using the homology of the (higher) moduli stack $\mathcal{M}_X$ parametrizing objects in the triangulated category $D^b(X)$. He defines a vertex algebra structure on

$$V_\bullet = \widehat{H}_\bullet(\mathcal{M}_X)\,,$$

where $\widehat{H}_\bullet$ is meant to denote an appropriate shift in the grading of the homology. The two most important ingredients for a vertex algebra are a translation operator T and a state-field correspondence $Y(-, z)$; we will recall the definition of a vertex

[4]The S_k operators in [MOOP, Mor, vB] do not satisfy Virasoro bracket relations, obscuring the meaning of Virasoro constraints.

algebras in Section 3.1. In our setting, the translation operator is obtained from the $B\mathbb{G}_m$ action on $\mathcal{M}_X$; the state-field correspondence $Y(-,z)$ is defined in terms of the map $\Sigma : \mathcal{M}_X \times \mathcal{M}_X \to \mathcal{M}_X$ induced by taking directs sums and a perfect complex Θ on $\mathcal{M}_X \times \mathcal{M}_X$ whose restriction to the diagonal is related to the obstruction theory of $\mathcal{M}_X$. These arise as a consequence of the master space localization technique that is commonly used for the proof of wall-crossing formulas (for instance in Mochizuki's work [Moc]); the complex Θ is closely related to the virtual normal bundle appearing in the localization formula and thus obstruction theory of $\mathcal{M}_X$. Remark 4.13 uses this observation to explain the relation of Virasoro constraints to the obstruction theory which we eluded to earlier on.

Associated to the vertex algebra $V_\bullet$ is the Lie algebra obtained as the quotient by the translation operator:

$$\check{V}_\bullet = V_{\bullet+2}/TV_\bullet\,.$$

The Lie bracket on $\check{V}_\bullet$ is a shadow of the vertex algebra structure on $V_\bullet$ and is obtained by a well-known construction due to Borcherds [Bor]. Alternatively, it can be constructed as the homology $H_\bullet(\mathcal{M}^{\mathrm{rig}})$ of the rigidification $\mathcal{M}_X^{\mathrm{rig}} = \mathcal{M}_X /\!\!/ B\mathbb{G}_m$; the two definitions agree when restricted to complexes with non-trivial numerical class, see Lemma 4.10. The Lie algebra $\check{V}_\bullet$ is a natural place where we can compare virtual fundamental classes of moduli spaces of sheaves; given a moduli space M of stable sheaves (or more generally of objects in $D^b(X)$) containing no strictly semistable sheaves, there is an open embedding $M \hookrightarrow \mathcal{M}_X^{\mathrm{rig}}$. If M admits a virtual fundamental class, we may push it forward along this embedding to obtain a class

$$[M]^{\mathrm{vir}} \in \check{H}_\bullet(\mathcal{M}_X^{\mathrm{rig}}) = \check{V}_\bullet.$$

If we fix a choice of a universal sheaf $\mathbb{G}$ in $M \times X$, by the universal property of $\mathcal{M}_X$ we get a map $f_{\mathbb{G}} \colon M \to \mathcal{M}_X$ lifting $M \hookrightarrow \mathcal{M}_X^{\mathrm{rig}}$, and thus a natural lift of the virtual fundamental class to the vertex algebra

$$[M]^{\mathrm{vir}}_{\mathbb{G}} := (f_{\mathbb{G}})_*[M]^{\mathrm{vir}} \in V_\bullet.$$

Crucially, Joyce defines more general classes

$$[M]^{\mathrm{inv}} \in \check{V}_\bullet$$

even when strictly semistable sheaves exist; when $[M]^{\mathrm{vir}}$ is defined, both classes agree.

The classes $[M]^{\mathrm{vir}} \in \check{V}_\bullet$ or $[M]^{\mathrm{vir}}_{\mathbb{G}} \in V_\bullet$ contain essentially the information of the (invariant) descendent integrals on M. This is made precise by J. Gross' [Gro] explicit description of $V_\bullet$, which we recall in Section 4.2. The cohomologies $H^\bullet(\mathcal{M}_X)$ and $H^\bullet(\mathcal{M}_X^{\mathrm{rig}})$ are closely related to the algebras of descendents $\mathbb{D}^X$ and $\mathbb{D}^X_{\mathrm{wt}_0}$ (see Section 2.4 for the definition of $\mathbb{D}^X_{\mathrm{wt}_0}$), respectively; see Lemmas 4.8 and 4.10 for the precise statements. The pairing between cohomology and homology then recovers the descendent integrals

$$\begin{aligned} H^\bullet(\mathcal{M}_X)\otimes H_\bullet(\mathcal{M}_X) &\to \mathbb{C}\\ (D,[M]^{\mathrm{vir}}_{\mathbb{G}}) &\mapsto \int_{[M]^{\mathrm{vir}}} \xi_{\mathbb{G}}(D)\,, \end{aligned}$$

and

$$\begin{aligned} H^\bullet(\mathcal{M}^{\mathrm{rig}}_X)\otimes H_\bullet(\mathcal{M}^{\mathrm{rig}}_X) &\to \mathbb{C}\\ (D,[M]^{\mathrm{vir}}) &\mapsto \int_{[M]^{\mathrm{vir}}} D\,, \end{aligned}$$

where the second integral is independent of the choice of $\mathbb{G}$.

1.5. **Conformal element and Virasoro constraints.** A vertex operator algebra is a vertex algebra $V_\bullet$ equipped with a conformal element $\omega \in V_4$. The main property of a conformal element (see Section 3.1 for a precise definition) is that the operators $\{L_n\}_{n\in\mathbb{Z}}$ on $V_\bullet$ induced from ω via the state-field correspondence satisfy the Virasoro bracket

$$\big[L_n, L_m\big] = (n-m)L_{m+n} + \frac{n^3-n}{12}\delta_{n+m,0}\cdot C$$

for some constant $C\in\mathbb{C}$ called the central charge of $(V_\bullet,\omega)$. It is thus natural to expect that the Virasoro operators in the descendent algebra previously studied might be explained by the existence of a conformal element ω on Joyce's vertex algebra (or some slight variation, namely the pair vertex algebra). Due to the mysterious role that the Hodge degrees play in the Virasoro operators in [Mor], we do not know how to do so in complete generality, but only under the following assumption:

Assumption 1.7. We assume that the Hodge cohomology groups $H^{p,q}(X)$ vanish whenever $|p-q|>1$.

This assumption is satisfied for curves, surfaces with $p_g=0$ and Fano 3-folds, hence covering the majority of the target varieties in Donaldson-Thomas theory.

The result of J. Gross [Gro] shows that, under certain assumption (satisfied for curves, surfaces and rational 3-folds), Joyce's vertex algebra $V_\bullet$ is naturally isomorphic to a lattice vertex algebra from $(K^\bullet(X), K^0_{\mathrm{sst}}(X), \chi_{\mathrm{sym}})$; here

$$K^\bullet(X) = K^0(X)\oplus K^1(X)\cong H^\bullet(X)$$

is the topological K-theory of X with $\mathbb{C}$-coefficient, $K^0_{\mathrm{sst}}(X)$ is the semi-topological K-theory[5] with $\mathbb{Z}$-coefficient and χ_{sym} is the symmetric pairing

$$\chi_{\mathrm{sym}}(v,w) = \int_X \mathrm{ch}(v^\vee)\mathrm{ch}(w)\mathrm{td}(X) + \int_X \mathrm{ch}(w^\vee)\mathrm{ch}(v)\mathrm{td}(X).$$

[5]The zeroth semi-topological K-theory $K^0_{\mathrm{sst}}(X)$ is defined as a Grothendieck's group of vector bundles modulo algebraic equivalence. By definition, we have $K^0_{\mathrm{sst}}(X)\simeq \pi_0(\mathcal{M}_X)$.

The construction of a vertex algebra from such data is recalled and summarized in Theorem 3.5 and follows Kac [Kac]; it uses Kac's bosonic vertex algebra construction in the even part $K^0(X)$ and the anti-fermionic vertex algebra construction in the odd part $K^1(X)$.

Kac's construction produces a conformal element when the pairing χ_{sym} is non-degenerate; unfortunately, due to the symmetrization this is not often the case. It turns out that this issue can be overcome by using the larger vertex algebra $V_\bullet^{\mathrm{pa}}$. The vector space underlying $V_\bullet^{\mathrm{pa}}$ is the homology of the stack of pairs $\mathcal{P}_X \simeq \mathcal{M}_X \times \mathcal{M}_X$:

$$V_\bullet^{\mathrm{pa}} = \widehat{H}_\bullet(\mathcal{P}_X)\,.$$

The construction of the conformal element requires a choice of an isotropic decomposition of the fermionic part; this construction is reminiscent of the bosonization procedure in physics. This decomposition is where the Hodge degrees and Assumption 1.7 come into play, because $K^1(X)$ splits into the isotropic subspaces

$$K^1(X) := K^{\bullet,\bullet+1} \oplus K^{\bullet+1,\bullet}$$

which via the chern character isomorphism correspond to

$$K^{\bullet,\bullet+1} \cong \bigoplus_{p\geqslant 0} H^{p,p+1}(X) \quad \text{and} \quad K^{\bullet+1,\bullet} \cong \bigoplus_{p\geqslant 0} H^{p+1,p}(X)\,.$$

It is for the construction of such conformal element that we use the vertex algebra over the complex numbers while the result of J. Gross [Gro] works over any field containing rational numbers, as it relies on Künneth decomposition. We prove that the Virasoro operators induced by this conformal element ω are dual to the pair Virasoro operators defined in the algebra of descendents, see Theorem B and Section 4.3.

One remarkable aspect of Theorem B is that, while the operators $\mathsf{L}_n^{\mathrm{pa}}$ on the descendent algebra were previously only defined for $n \geqslant -1$, a conformal element provides fields L_n^{pa} for every $n \in \mathbb{Z}$ and thus a complete representation of the Virasoro algebra. In particular, this representation now has a non-trivial central charge $2\chi(X)$ which the positive branch $\{L_n\}_{n\geqslant -1}$ does not detect. We note that the factor of 2 appears due to working with the pair vertex algebra $V_\bullet^{\mathrm{pa}}$; if the pairing χ_{sym} were non-degenerate we would get a conformal element in $V_\bullet$ with central charge $\chi(X)$. Remarkably, the Virasoro operators on the Gromov-Witten theory of X (at least if $\dim(X)$ is even) are also known to admit an extension to a full representation of the Virasoro algebra; the central charge in the Gromov-Witten case is $\chi(X)$ [Get, Section 2.10].

This description of the Virasoro operators provides a beautiful formulation of the Virasoro constraints for sheaves and for pairs in terms of well-known notions in the theory of vertex operator algebras, namely primary states (also known as physical states):

$$\check{P}_0 \subset \check{V}_\bullet, \quad P_0^{\mathrm{pa}} \subset V_\bullet^{\mathrm{pa}}.$$

Corollary 1.8. Assume X is in class D (see Remark 4.2) and satisfies Assumption 1.7. Then

(1) Conjecture 1.4 is equivalent to the class

$$[M]^{\mathrm{vir}} \in \check{V}_\bullet \subseteq \check{V}_\bullet^{\mathrm{pa}}$$

being a primary state in $\check{P}_0$ (cf. Definition 3.9).

(2) Conjecture 1.5 is equivalent to the class

$$[P]^{\mathrm{vir}}_{(q^*V,\mathbb{F})} \in V_\bullet^{\mathrm{pa}}$$

being a primary state in P_0^{pa} (cf. Definition 3.9). Here, the class $[P]^{\mathrm{vir}}_{(q^*V,\mathbb{F})}$ denotes the lift of $[P]^{\mathrm{vir}} \in \check{V}_\bullet^{\mathrm{pa}}$ to $V_\bullet^{\mathrm{pa}}$ induced by the universal pair $q^*V \to \mathbb{F}$.

We prove in Proposition 3.11, 3.13 that the space of physical states interact nicely with Lie bracket operations: $\check{P}_0 \subset \check{V}_\bullet$ is a Lie subalgebra and $P_0^{\mathrm{pa}} \subset V_\bullet^{\mathrm{pa}}$ is a Lie submodule over $\check{P}_0$. Since wall-crossing formulas in [Joy6] are always written using the Lie bracket, these Lie algebraic statements prove a compatibility between wall-crossing and the Virasoro constraints.

1.6. **Proof of Theorem A and other results.** The main result of the paper is Theorem A, i.e. a proof of Conjecture 1.4 for stable sheaves on curves and on surfaces with $h^{0,1} = h^{0,2} = 0$. Part (2) of the Theorem solves the conjecture proposed by van Bree [vB, Conjecture 1.4] (see Remark 2.17 for a comparison with van Bree's formulation). The main ingredient in the proof is an inductive rank reduction argument via wall-crossing. This is the content of Section 5. In each of the 3 cases, we consider the moduli spaces of Bradlow pairs P_α^t which depend on a stability parameter $t > 0$. Assuming that M_α contains no strictly semi-stable sheaves, when $0 < t \ll 1$ is small there is a map $P_\alpha^{0+} := P_\alpha^{t\ll 1} \to M_\alpha$ which is a (virtual) projective bundle; that is equivalent to wall-crossing at the Joyce–Song wall

$$[P_\alpha^{0+}]^{\mathrm{vir}} = \big[[M_\alpha]^{\mathrm{vir}}, e^{(1,0)}\big]\,.$$

On the other hand, for large $t \gg 1$ the moduli spaces $P_\alpha^\infty := P_\alpha^{t\gg 1}$ are easier to understand (and sometimes empty). We prove not only that M_α satisfy the sheaf Virasoro constraints (i.e. Theorem A) but also that the moduli spaces of Bradlow pairs P_α^t satisfy the pair analogue of the constraints:

Theorem 1.9. The moduli spaces of Bradlow pairs P_α^t (see Definition 5.2) satisfy the pair Virasoro constraints (Conjecture 2.18) for every $t > 0$ in the 3 settings of Theorem A, i.e. $(m,d) = (2,2),(1,1)$ and $(m,d) = (2,1)$ provided Assumption 5.7 holds.

To prove Theorems A and 1.9 we need the following steps:

(1) Prove that P_α^∞ satisfies Conjecture 1.5. In case (1), we only need to prove the statement for symmetric powers of curves which we do by a direct computation in Proposition 6.1. Cases (2) and (3) can be reduced to the Hilbert scheme of points, which was obtained in [Mor].
(2) Use the wall-crossing formula between P_α^∞ and P_α^{0+} to show that P_α^{0+} satisfies Virasoro constraints for pairs as well. We use induction on α to have the Virasoro constraints on the wall-crossing terms together with the compatibility between wall-crossing and Virasoro (Propositions 3.11 and 3.13).
(3) Finally, we use a projective bundle compatibility for $P_\alpha^{0+} \to M_\alpha$ to show that the pair Virasoro constraints on P_α^{0+} imply the sheaf Virasoro constraints on M_α.

A crucial point in the argument is that we must include moduli spaces M_α admitting strictly semistable sheaves in the induction since they unavoidably appear as wall-crossing terms. That is, we must prove that $[M]^{\mathrm{inv}}$ is in the Lie algebra of primary states. Because of that, in step (3) we don't exactly have a projective bundle. However, by the very definition of the invariant classes $[M]^{\mathrm{inv}}$, what we have to prove is essentially the same as in the projective bundle case. We do this in Theorem 5.11.

We also use the wall-crossing compatibility together with the results of the first author in [Boj1] to prove that the pair Virasoro constraints hold for punctual Quot schemes.

Theorem 1.10. Let X be a curve or a surface and let V be a torsion-free sheaf on X. Then the punctual Quot scheme $\mathrm{Quot}_X(V, n)$ satisfies the pair Virasoro constraints (Conjecture 2.18).

1.7. **Notation and conventions.** Except the semi-topological K-group $K^0_{\mathrm{sst}}(X)$ which we consider over $\mathbb{Z}$-coefficient, all cohomology and K-theory groups are assumed to have coefficients in $\mathbb{C}$ unless stated otherwise. We write

$$K^\bullet(X) = K^0(X) \oplus K^1(X)$$

for the topological K-theory and we denote by

$$\mathrm{ch}\colon K^\bullet(X) \to H^\bullet(X)$$

the isomorphism between topological K-theory and cohomology. The total cohomology of a topological space is always understood to be the direct product of the cohomology groups in each degree, while homology is the direct sum. We will use the cap product with the cohomology acting on the left. This is the convention followed in [Dol]; it differs from the more usual convention with cohomology on the right by a sign, i.e., $\gamma \cap u = (-1)^{|\gamma||u|} u \cap \gamma$ for $\gamma \in H^\bullet(X)$, $u \in H_\bullet(X)$.

α, β	Semi-topological K-theory classes in $K^0_{\mathrm{sst}}(X)$.
γ, δ	Cohomology class on X.
v, w	Elements of $K^\bullet(X)$.
$\deg(-)$	Degree for any graded vector space.
$\lvert - \rvert$	Super-grading taking value in $\{0, 1\}$.
$\mathrm{ch}_i^{\mathrm{H}}(\gamma)$	The holomorphic descendent in degree $2i-p+q$ depending on the Hodge degree of $\gamma \in H^{p,q}(X)$.
$\mathrm{ch}_i(\gamma)$	The topological descendent in degree $2i - \lvert\gamma\rvert$.
$L_n, T_n, R_n,$	Virasoro operators on homology and vertex algebra.
$\mathsf{L}_n, \mathsf{T}_n, \mathsf{R}_n,$	Dual operator notation on cohomology and descendent algebra.
$\mathsf{L}_{\mathrm{wt}_0}$	weight 0 Virasoro operator on descendent algebra.

1.8. **Future directions.** There are several open directions regarding the Virasoro constraints for sheaves. The first obvious direction is to try to improve Theorem A by removing the assumptions $h^{0,1} = h^{0,2} = 0$. The arguments in this paper show that we can get the constraints for $h^{0,1} > 0$ as long as we can prove them for the moduli of rank 1 sheaves (isomorphic to the Hilbert scheme of points times the Jacobian). Finding an argument that works in general for the Hilbert scheme of points and does not go through Gromov-Witten theory would be highly desirable. Removing the assumption $h^{0,2} = 0$ requires a better understanding of the constraints in the setting of reduced virtual fundamental classes for fixed determinant theory (see Remark 2.23).

The authors are working on understanding and proving the constraints for moduli spaces of quiver representations (possibly with relations) and Quot-schemes with 1-dimensional quotients. Many other moduli spaces might be approachable in the near future, such as more general nested Hilbert schemes and Fano 3-folds.

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2. Virasoro constraints

2.1. **Supercommutative algebras.** Before we move onto geometry, we note down some useful observations about freely generated *supercommutative* algebras and derivations on them. Let $D_\bullet$ be a supercommutative $\mathbb{Z}$-*graded unital algebra* over $\mathbb{C}$

with degree $\deg(v) = i$ for any $v \in D_i$. Supercommutativity means that multiplication satisfies

$$v \cdot w = (-1)^{|v||w|} w \cdot v\,,$$

where

$$|v| \in \{0,1\}\,, \quad \text{such that} \quad |v| \equiv \deg(v) \mod 2\,.$$

The unit of $D_\bullet$ is always going to be denoted by 1 and in general, we will omit specifying it in the notation.

A *superderivation* of degree r on $D_\bullet$ is a $\mathbb{Z}$-graded linear map

$$R : D_\bullet \longrightarrow D_{\bullet+r}$$

satisfying the *graded Leibniz rule*

$$R(v \cdot w) = R(v) \cdot w + (-1)^{r|v|}\, v \cdot R(w)\,.$$

Definition 2.1. Let $C_\bullet$ be a $\mathbb{Z}$-graded $\mathbb{C}$-vector space. We denote by

$$D_\bullet = \mathrm{SSym}[C_\bullet]$$

the freely generated unital supercommutative algebra generated by $C_\bullet$. Denote by $C^\bullet$ the graded dual of $C_\bullet$. We define the dual of $D_\bullet$ as a completion of $\mathrm{SSym}[C^\bullet]$ with respect to the degree. More precisely, the dual is

$$D^\bullet := \mathrm{SSym}[\![C^\bullet]\!] = \prod_{i\geqslant 0} \mathrm{SSym}[C^\bullet]^i$$

where $\mathrm{SSym}[C^\bullet]^i$ denotes the degree i part of $\mathrm{SSym}[C^\bullet]$ with the degree induced by the one on $C^\bullet$.[6]

Given a linear map $f\colon C_\bullet \to B_{\bullet+r}$ of degree r, there is a unique way to extend f to an algebra homomorphism and to a derivation of degree r.

The pairing between $C_\bullet$ and $C^\bullet$ can be promoted to a cap product between $D_\bullet$ and $D^\bullet$.

Definition 2.2. Fix $C_\bullet$ and $C^\bullet$ dual vector spaces and let $\langle -,-\rangle\colon C^\bullet \times C_\bullet \to \mathbb{C}$ be the pairing. Let $D_\bullet$ and $D^\bullet$ be as in Definition 2.1. We define a cap product

$$\cap\colon C^\bullet \times D_\bullet \longrightarrow D_\bullet$$

by letting $\nu \cap (-)$ for $\nu \in C^\bullet$ act as a superderivation of degree $-\deg(\nu)$ on $D_\bullet$ restricting to $\langle \nu, -\rangle : C_\bullet \to \mathbb{C}$. The cap product extends uniquely to

$$\cap\colon D^\bullet \times D_\bullet \longrightarrow D_\bullet\,,$$

by requiring that $(\mu\nu) \cap u = \mu \cap (\nu \cap u)$. Notice that this makes $D_\bullet$ into a left $D^\bullet$-module.

By composing with the projection $D_\bullet \to \mathbb{C}$, we recover a non-degenerate pairings

$$\langle -,-\rangle : D^\bullet \times D_\bullet \to \mathbb{C}\,.$$

[6]We follow here the convention that the total homology is the direct sum of the homology groups in each degree, while cohomology is the product.

An explicit description of the cap product can be obtained after fixing a basis $B \subset C_\bullet$ and observing that

$$\nu \cap (-) = \sum_{v \in B} \langle \nu, v \rangle \frac{\partial}{\partial v}$$

where $\frac{\partial}{\partial v}$ is the super-derivation of degree $-\deg(v)$ acting on the elements of the basis B by $\frac{\partial}{\partial v}(v) = 1$ and $\frac{\partial}{\partial v}(w) = 0$ for $w \in B \backslash \{v\}$.

2.2. **Descendent algebra.** Let X be a smooth projective variety over $\mathbb{C}$.

Definition 2.3. Let CH^X denote the infinite dimensional vector space over $\mathbb{C}$ generated by symbols called *holomorphic descendents* of the form

$$\mathrm{ch}_i^{\mathrm{H}}(\gamma) \quad \text{for} \quad i \geqslant 0,\, \gamma \in H^\bullet(X)$$

subject to the linearity relations

$$\mathrm{ch}_i^{\mathrm{H}}(\lambda_1\gamma_1 + \lambda_2\gamma_2) = \lambda_1 \mathrm{ch}_i^{\mathrm{H}}(\gamma_1) + \lambda_2 \mathrm{ch}_i^{\mathrm{H}}(\gamma_2)$$

for $\lambda_1, \lambda_2 \in \mathbb{C}$. We define the *cohomological* $\mathbb{Z}$-grading on CH^X by

$$\deg \mathrm{ch}_i^{\mathrm{H}}(\gamma) = 2i - p + q \quad \text{for} \quad \gamma \in H^{p,q}(X)\,. \tag{1}$$

Finally, we let $\mathbb{D}^X$ be the $\mathbb{Z}$-graded *algebra of holomorphic descendents*

$$\mathbb{D}^X = \mathrm{SSym}[\![\mathrm{CH}^X]\!]\,,$$

which is the completion of the supercommutative algebra generated by $\mathrm{ch}_i^{\mathrm{H}}(\gamma)$. We will write $\mathrm{ch}_\bullet^{\mathrm{H}}(\gamma)$ for the element

$$\mathrm{ch}_\bullet^{\mathrm{H}}(\gamma) = \sum_{i \geqslant 0} \mathrm{ch}_i^{\mathrm{H}}(\gamma) \in \mathbb{D}^X\,.$$

Remark 2.4. This algebra of descendents is very similar to the one introduced in [MOOP] with two small differences. Firstly, we now take a completion with respect to degree; this makes little difference in practice, but it is important in the comparison between $\mathbb{D}^X$ and $H^\bullet(\mathcal{M}_X)$ (cf. Lemma 4.8) since we follow the standard convention that the total cohomology is a product of the groups in each degree. The second difference is in the grading; in our notation, given $\gamma \in H^{p,q}(X)$, the symbol $\mathrm{ch}_i^{\mathrm{H}}(\gamma)$ should be thought of as having Hodge degree $(i, i-p+q)$ (recall this from Definition 1.3) so that the geometric realization is degree preserving. The superscript H stands for holomorphic part of the Hodge degree and is used to indicate this degree convention. The original convention appearing in loc. cit. defined the descendents $\mathrm{ch}_i^{\mathrm{old}}(\gamma)$ as

$$\mathrm{ch}_i^{\mathrm{old}}(\gamma) = \mathrm{ch}_{i+p-\dim(X)}^{\mathrm{H}}(\gamma)$$

for $\gamma \in H^{p,q}(X)$. The reason for introducing holomorphic descendents is to give a natural looking expression for the operator T_k in §2.3.

It will sometimes be useful to also consider the shift

$$\mathrm{ch}_i(\gamma) := \mathrm{ch}^{\mathrm{H}}_{i+\lfloor\frac{p-q}{2}\rfloor}(\gamma) = \mathrm{ch}^{\mathrm{old}}_{i-\lceil\frac{\deg\gamma}{2}\rceil+\dim X}(\gamma)$$

so that $\deg \mathrm{ch}_i(\gamma) = 2i - |\gamma|$; we recall that $|\gamma| \in \{0,1\}$ is the parity of γ as in Section 2.1.

Definition 2.5. Let $\alpha \in K^0_{\mathrm{sst}}(X)$ be a topological type. We define CH^X_α to be the graded vector space generated by symbols

$$\mathrm{ch}_i(\gamma) \quad \text{for} \quad i \in \mathbb{Z}_{>0},\, \gamma \in H^\bullet(X).$$

We let $\mathbb{D}^X_\alpha$ be $\mathrm{SSym}[\![\mathrm{CH}^X_\alpha]\!]$. The algebra $\mathbb{D}^X_\alpha$ comes equipped with an algebra homomorphism $p_\alpha\colon \mathbb{D}^X \to \mathbb{D}^X_\alpha$ sending

$$\mathrm{ch}^{\mathrm{H}}_i(\gamma) \mapsto \begin{cases} \mathrm{ch}_{i-\lfloor\frac{p-q}{2}\rfloor}(\gamma) & \text{if } \deg \mathrm{ch}^{\mathrm{H}}_i(\gamma) > 0 \\ \int_X \gamma\cdot \mathrm{ch}(\alpha) & \text{if } \deg \mathrm{ch}^{\mathrm{H}}_i(\gamma) = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Note that abstractly the algebras $\mathbb{D}^X_\alpha$ are independent of α, but the morphisms p_α depend on α by their behavior on the descendents of degree 0. If M_α is a moduli space parametrizing sheaves of topological type α with a universal sheaf $\mathbb{G}$, then the geometric realization factors through p_α:

(2)
$$\begin{tikzcd} \mathbb{D}^X \arrow[r, "p_\alpha"] \arrow[d, "\xi_{\mathbb{G}}"'] & \mathbb{D}^X_\alpha \arrow[dl, "\xi_{\mathbb{G}}"] \\ H^*(M_\alpha) & \end{tikzcd}$$

where we still denote by $\xi_{\mathbb{G}}$ the factoring map, which is defined as

(3)
$$\xi_{\mathbb{G}}(\mathrm{ch}_i(\gamma)) = p_*\left(\mathrm{ch}_{i-\lceil\frac{\deg\gamma}{2}\rceil+\dim X}(\mathbb{G})q^*\gamma\right) \in H^{2i-|\gamma|}(M_\alpha)\,.$$

The map factors since $\xi_{\mathbb{G}}(\mathrm{ch}^{\mathrm{H}}_i(\gamma)) = \int_X \gamma\cdot\mathrm{ch}(\alpha) \in H^0(M_\alpha)$ when $\deg(\mathrm{ch}^{\mathrm{H}}_i(\gamma)) = 0$ and the degree of $\xi_{\mathbb{G}}\big(\mathrm{ch}^{\mathrm{H}}_i(\gamma)\big)$ is identical to (1). Given $D, D' \in \mathbb{D}^X$ we say that "$D = D'$ in $\mathbb{D}^X_\alpha$" if $p_\alpha(D) = p_\alpha(D')$.

Example 2.6. One may lift the Chern classes of the virtual tangent bundle of M to $\mathbb{D}^X$ or $\mathbb{D}^X_\alpha$. Recall that the virtual tangent bundle is defined as the K-theory class

$$T^{\mathrm{vir}}M = -Rp_*\mathrm{RHom}(\mathbb{G},\mathbb{G}) + \mathcal{O}_M\,.$$

Using $\sum_t \gamma^L_t \otimes \gamma^R_t$ to denote the Künneth decomposition of $\Delta_*\mathrm{td}(X)$, where $\Delta\colon X \to X\times X$ is the diagonal, and applying Grothendieck-Riemann-Roch, one computes that

$$\mathrm{ch}(TM^{\mathrm{vir}}) = -\xi_{\mathbb{G}}\left(\sum_{i,j\geqslant 0}\sum_t (-1)^{i-p^L_t+\dim(X)}\mathrm{ch}^{\mathrm{H}}_i(\gamma^L_t)\mathrm{ch}^{\mathrm{H}}_j(\gamma^R_t)\right) + 1\,,$$

where $\gamma^L_t \in H^{p^L_t, q^L_t}(X)$. The reason for the existence of this lift will become apparent from Lemma 4.8. The similarity with Virasoro constraints bellow is a general phenomenon which can be used to guess their correct formulation.

2.3. **Virasoro operators.** In this section we define the Virasoro operators

$$\mathsf{L}_k \colon \mathbb{D}^X \to \mathbb{D}^X \quad , \text{for } k \geqslant -1,$$

which produce the Virasoro constraints. These operators have two terms, a derivation term R_k and a linear term T_k which is quadratic in $\mathrm{ch}_i^{\mathrm{H}}$. The full Virasoro operators are $\mathsf{L}_k = \mathsf{R}_k + \mathsf{T}_k$, where

(1) $\mathsf{R}_k \colon \mathbb{D}^X \to \mathbb{D}^X$ is an even (of degree $2k$) derivation extended from $\mathsf{R}_k \colon \mathrm{CH}^X \to \mathrm{CH}^X$, where it is defined by

$$\mathsf{R}_k \mathrm{ch}_i^{\mathrm{H}}(\gamma) := \left(\prod_{j=0}^{k}(i+j)\right) \mathrm{ch}_{i+k}^{\mathrm{H}}(\gamma)\,.$$

We take the following conventions: the above product is 1 if $k = -1$ and $\mathrm{ch}_{i+k}^{\mathrm{H}}(\gamma) = 0$ if $i + k < 0$.

(2) $\mathsf{T}_k \colon \mathbb{D}^X \to \mathbb{D}^X$ is the operator of multiplication by the element of $\mathbb{D}^X$ given by

$$\mathsf{T}_k := \sum_{i+j=k} (-1)^{\dim X - p^L} i!j!\mathrm{ch}_i^{\mathrm{H}}\mathrm{ch}_j^{\mathrm{H}}(\mathrm{td}(X)).$$

In the formula above, $(-1)^{\dim X - p^L}\mathrm{ch}_i^{\mathrm{H}}\mathrm{ch}_j^{\mathrm{H}}(\mathrm{td}(X))$ is defined as follows: let $\Delta \colon X \to X \times X$ be the diagonal map and let

$$\sum_t \gamma_t^L \otimes \gamma_t^R = \Delta_* \mathrm{td}(X)$$

be a Kunneth decomposition of $\Delta_*\mathrm{td}(X)$ such that $\gamma_t^L \in H^{p_t^L, q_t^L}(X)$ for some p_t^L, q_t^L. Then

$$(-1)^{\dim X - r^L}\mathrm{ch}_i^{\mathrm{H}}\mathrm{ch}_j^{\mathrm{H}}(\mathrm{td}(X)) := \sum_t (-1)^{\dim X - p_t^L}\mathrm{ch}_i^{\mathrm{H}}(\gamma_t^L)\mathrm{ch}_j^{\mathrm{H}}(\gamma_t^R).$$

Remark 2.7. The operator $\mathsf{L}_{-1} = \mathsf{R}_{-1}$ plays a special role and has a particularly nice geometric interpretation in terms of $\mathbb{G}_m$-gerbes over M_α which we describe in Lemma 4.9.

The operators $\{\mathsf{L}_k\}_{k \geqslant -1}$ satisfy the Virasoro bracket relations

$$[\mathsf{L}_k, \mathsf{L}_\ell] = (\ell - k)\mathsf{L}_{k+\ell}.$$

This was noted in [MOOP]; for a detailed proof see [vB, Proposition 2.10]. The unusual constant factor $(\ell - k)$, instead of $(k - \ell)$, suggests that there might be another set of more natural Virasoro operators to which $\{\mathsf{L}_k\}_{k \geqslant -1}$ are dual. This observation is made into a precise statement in Theorem 4.12.

2.4. **Weight zero descendents.** One of the issues that arise when dealing with descendent invariants is that a priori they depend on a choice of universal sheaf $\mathbb{G}$ on $M \times X$. Given a universal sheaf $\mathbb{G}$, the possible universal sheaves are of the form $\mathbb{G}' = \mathbb{G} \otimes p^* L$ where L is a line bundle on M.

Lemma 2.8. Let $\mathsf{E}\colon \mathbb{D}^X \to \mathbb{D}^X[\![\zeta]\!]$ be the algebra homomorphism defined by

$$E = e^{\zeta \mathsf{R}_{-1}}\,,$$

then given two universal sheaves $\mathbb{G}$ and $\mathbb{G}' = \mathbb{G} \otimes p^*L$, the geometric realizations concerning the two are related by

$$\xi_{\mathbb{G}'} = \xi_{\mathbb{G}} \circ \mathsf{E}|_{\zeta = c_1(L)}\,.$$

Proof. We may write

$$\mathsf{E}\big(\mathrm{ch}^{\mathrm{H}}_i(\gamma)\big) = \sum_{n=0}^{i} \frac{\zeta^n}{n!} \mathrm{ch}^{\mathrm{H}}_{i-n}(\gamma) = [e^{\zeta} \mathrm{ch}^{\mathrm{H}}_{\bullet}(\gamma)]_i$$

which after comparing to

$$\xi_{\mathbb{G}'}(\mathrm{ch}^{\mathrm{H}}_{\bullet}(\gamma)) = p_*\left(\mathrm{ch}(\mathbb{G} \otimes p^*L) q^*\gamma\right) = e^{c_1(L)} \xi_{\mathbb{G}}(\mathrm{ch}^{\mathrm{H}}_{\bullet}(\gamma)). \qquad \square$$

yields the result.

In particular, it follows that if D is such that $\mathsf{R}_{-1}(D) = 0$ then $\xi_{\mathbb{G}}(D) = \xi_{\mathbb{G}'}(D)$. This leads to the definition of the algebra of weight 0 descendents. Its geometric interpretation is summarized in Lemma 4.10 and is related to taking rigidification of moduli stacks. Roughly speaking, these classify families of sheaves on $S \times X$ up to twisting by line bundles on $L \to S$ and the above discussion formulates the precise interaction between the twisting and the descendents.

Definition 2.9. For a topological type $\alpha \in K^0_{\mathrm{sst}}(X)$, we will also denote by

$$\mathsf{R}_{-1}\colon \mathbb{D}^X_\alpha \to \mathbb{D}^X_\alpha$$

the derivation defined on generators by $\mathsf{R}_{-1}\mathrm{ch}_i(\gamma) = \mathrm{ch}_{i-1}(\gamma)$, where $\mathrm{ch}_0(\gamma)$ is interpreted as $\int_X \gamma \cdot \mathrm{ch}(\alpha)$. We then define

$$\begin{aligned} \mathbb{D}^X_{\mathrm{wt}_0} &= \{D \in \mathbb{D}^X : \mathsf{R}_{-1}(D) = 0\}\,, \\ \mathbb{D}^X_{\mathrm{wt}_0,\alpha} &= \{D \in \mathbb{D}^X_\alpha : \mathsf{R}_{-1}(D) = 0\}\,. \end{aligned}$$

For any weight 0 descendent $D \in \mathbb{D}^X_{\mathrm{wt}_0}$, by Lemma 2.8, the geometric realization $\xi_{\mathbb{G}}(D)$ does not depend on the choice of $\mathbb{G}$. Thus, when $D \in \mathbb{D}^X_{\mathrm{wt}_0}$ we will often omit specifying the realization map and write

$$\int_{[M]^{\mathrm{vir}}} D = \int_{[M]^{\mathrm{vir}}} \xi_{\mathbb{G}}(D)$$

for *any* choice of universal sheaf $\mathbb{G}$. The morphism $\mathbb{D}^X_{\mathrm{wt}_0} \to H^\bullet(M)$ is defined even without assuming the existence of any universal sheaf $\mathbb{G}$. This fact can be proven using Lemma 4.10; if $M = M_\alpha$ is a moduli space of topological type α, then it admits an open embedding $\iota\colon M \hookrightarrow \mathcal{M}^{\mathrm{rig}}_\alpha$ and thus we get a map

$$\mathbb{D}^X_{\mathrm{wt}_0} \xrightarrow{p_\alpha} \mathbb{D}^X_{\mathrm{wt}_0,\alpha} \xrightarrow{\sim} H^\bullet(\mathcal{M}^{\mathrm{rig}}_\alpha) \xrightarrow{\iota^*} H^\bullet(M).$$

Example 2.10. Given $\gamma_1, \gamma_2 \in H^\bullet(X)$ we have

$$\mathrm{ch}_1^{\mathrm{H}}(\gamma_1)\mathrm{ch}_0^{\mathrm{H}}(\gamma_2) - \mathrm{ch}_0^{\mathrm{H}}(\gamma_1)\mathrm{ch}_1^{\mathrm{H}}(\gamma_2) \in \mathbb{D}^X_{\mathrm{wt}_0}.$$

One can also check that the lift of $\mathrm{ch}(T^{\mathrm{vir}}M)$ to $\mathbb{D}^X$ is in $\mathbb{D}^X_{\mathrm{wt}_0}$ using the expression in Example 2.6. A geometric reason for this is going to be given in Example 4.11.

2.5. **Virasoro constraints for sheaves.** To formulate the Virasoro constraints for moduli of sheaves, without a canonical choice of universal sheaf, we must produce relations among weight 0 descendents. This is achieved by combining the Virasoro operators previously introduced in the way which we now describe:

Definition 2.11. The weight 0 Virasoro operator $\mathsf{L}_{\mathrm{wt}_0}\colon \mathbb{D}^X \to \mathbb{D}^X$ is defined by

$$\mathsf{L}_{\mathrm{wt}_0} = \sum_{j \geqslant -1} \frac{(-1)^j}{(j+1)!}\mathsf{L}_j\mathsf{R}_{-1}^{j+1}.$$

The operator $\mathsf{L}_{\mathrm{wt}_0}$ maps $\mathbb{D}^X$ to $\mathbb{D}^X_{\mathrm{wt}_0}$. Indeed, using that $[\mathsf{R}_{-1}, \mathsf{L}_j] = (j+1)\mathsf{L}_{j-1}$ we find

$$\mathsf{R}_{-1} \circ \mathsf{L}_{\mathrm{wt}_0} = \sum_{j \geqslant -1} \frac{(-1)^j}{(j+1)!}\mathsf{L}_j\mathsf{R}_{-1}^{j+2} + \sum_{j \geqslant -1} \frac{(-1)^j}{(j+1)!}(j+1)\mathsf{L}_{j-1}\mathsf{R}_{-1}^{j+1} = 0\,.$$

In particular, for any $D \in \mathbb{D}^X$ the integral $\int_M \xi_{\mathbb{G}}(\mathsf{L}_{\mathrm{wt}_0}(D))$ does not depend on the universal sheaf $\mathbb{G}$ so we omit the realization homomorphism $\xi_{\mathbb{G}}$ from the notation.

Conjecture 2.12. Let M be a moduli of sheaves as in Section 1.2. Then

$$\int_{[M]^{\mathrm{vir}}} \mathsf{L}_{\mathrm{wt}_0}(D) = 0 \quad \text{for any } D \in \mathbb{D}^X.$$

This formulation is different from the ones which appear in previous works, namely [MOOP, Mor, vB]. There, Virasoro constraints are formulated using a choice of universal sheaf that is natural in each of the moduli spaces considered.

Definition 2.13. Let $M = M_\alpha$ be a moduli space of sheaves of topological type α and let $\delta \in H^\bullet(X, \mathbb{Z})$ be an algebraic class such that $\int_X \delta \cdot \mathrm{ch}(\alpha) \neq 0$. We say that a universal sheaf $\mathbb{G}$ is δ-normalized if

$$\xi_{\mathbb{G}}(\mathrm{ch}_1^{\mathrm{H}}(\delta)) = 0. \tag{4}$$

Given δ as in the definition, we define the operators

$$\mathsf{L}_k^\delta = \mathsf{R}_k + \mathsf{T}_k + \mathsf{S}_k^\delta, \quad k \geqslant -1\,,$$

where

$$\mathsf{S}_k^\delta = -\frac{(k+1)!}{\int_X \delta \cdot \mathrm{ch}(\alpha)}\mathsf{R}_{-1} \circ \mathrm{ch}_{k+1}^{\mathrm{H}}(\delta).$$

Remark 2.14. If δ is such that $\int_X \delta\cdot\mathrm{ch}(\alpha)\neq 0$ then a δ-normalized universal sheaf always exists (and is unique) as an element of the rational K-theory of $M\times X$. If $\mathbb{G}$ is any universal sheaf then

$$\mathbb{G}_\delta = \mathbb{G}\otimes e^{-\xi_{\mathbb{G}}(\mathrm{ch}_1^{\mathrm{H}}(\delta))/\int_X \delta\cdot\mathrm{ch}(\alpha)}$$

is a well-defined class in the rational K-theory of $M\times X$ that can be thought of as the unique δ-normalized universal sheaf; here we use e^c to denote a rational line bundle with first Chern class equal to c for some algebraic class $c\in H^\bullet(X,\mathbb{Q})$.

The geometric realization with respect to $\mathbb{G}_\delta$ is given by $\xi_{\mathbb{G}_\delta}=\xi_{\mathbb{G}}\circ\eta$ where $\eta\colon\mathbb{D}^X\to\mathbb{D}^X_{\mathrm{wt}_0}$ is

$$\eta=\sum_{j\geqslant 0}\left(-\frac{\mathrm{ch}_1^{\mathrm{H}}(\delta)}{\int_X\delta\cdot\mathrm{ch}(\alpha)}\right)^j\mathsf{R}_{-1}^j\,.$$

Thus, we can talk about the geometric realization with respect to a δ-normalized sheaf even if such a sheaf does not exist in the usual sense. Conjecture 2.15 still makes sense in this setting and the proof of Proposition 2.16 goes through as well.

Conjecture 2.15. Let $M=M_\alpha$ be a moduli of sheaves as in Section 1.2 and let $\mathbb{G}$ be a δ-normalized universal sheaf. Then

$$\int_{[M]^{\mathrm{vir}}}\xi_{\mathbb{G}}(\mathsf{L}_k^\delta(D))=0\quad\text{for any } k\geqslant -1,\ D\in\mathbb{D}^X.$$

Proposition 2.16. Conjectures 2.12 and 2.15 are equivalent.

Proof. We begin by observing that we have the identity

$$\mathsf{L}_{\mathrm{wt}_0}=\sum_{j\geqslant -1}\frac{(-1)^j}{(j+1)!}\mathsf{L}_j^\delta\mathsf{R}_{-1}^{j+1}\,,\tag{5}$$

that follows from

$$\begin{aligned}&\sum_{j\geqslant -1}\frac{(-1)^j}{(j+1)!}\mathsf{S}_j^\delta\mathsf{R}_{-1}^{j+1}\\&\qquad=-\frac{1}{\int_X\delta\cdot\mathrm{ch}(\alpha)}\sum_{j\geqslant -1}(-1)^j\left(\mathrm{ch}_j^{\mathrm{H}}(\delta)\mathsf{R}_{-1}^{j+1}+\mathrm{ch}_{j+1}^{\mathrm{H}}(\delta)\mathsf{R}_{-1}^{j+2}\right)=0.\end{aligned}$$

By (5), Conjecture 2.15 clearly implies 2.12.

For the reverse implication we use (backward) induction on k. For every $k>\mathrm{virdim}(M)$ the statement of Conjecture 2.15 is clear by degree reasons. Assume now that the result holds for every $k'>k$, and let $F=\mathrm{ch}_1^{\mathrm{H}}(\delta)\in\mathbb{D}^X$ satisfying $\xi_{\mathbb{G}}(F)=0$ by (4). The weight 0 Virasoro operator applied to $F^{k+1}D$ gives

$$0=\int_{[M]^{\mathrm{vir}}}\xi_{\mathbb{G}}\left(\mathsf{L}_{\mathrm{wt}_0}(F^{k+1}D)\right)=\sum_{j\geqslant -1}\frac{(-1)^j}{(j+1)!}\int_{[M]^{\mathrm{vir}}}\xi_{\mathbb{G}}\left(\mathsf{L}_j^\delta\mathsf{R}_{-1}^{j+1}(F^{k+1}D)\right).\tag{6}$$

Since R_{-1} is a derivation and $\mathsf{R}_{-1}(F) = \mathrm{ch}_0^{\mathrm{H}}(\delta) = \int_X \delta \cdot \mathrm{ch}(\alpha)$ in $\mathbb{D}_\alpha^X$, we have

$$\begin{aligned} \mathsf{R}_{-1}^{j+1}(F^{k+1}D) &= \sum_{s=0}^{j+1} \binom{j+1}{s} \mathsf{R}_{-1}^{s}(F^{k+1}) \mathsf{R}_{-1}^{j+1-s}(D) \\ &= \sum_{s=0}^{\min(k+1,j+1)} \binom{j+1}{s} \frac{(k+1)!}{(k+1-s)!} r^s F^{k+1-s} R_{-1}^{j+1-s}(D) \end{aligned} \tag{7}$$

in $\mathbb{D}_\alpha^X$, where we denote $r = \int_X \delta \cdot \mathrm{ch}(\alpha)$. Note also that the operators L_j^δ satisfy the property that $\mathsf{L}_j^\delta(FD) = F\,\mathsf{L}_j^\delta(D)$ in $\mathbb{D}_\alpha^X$ for every D. Since by (4) the geometric realization of F with respect to $\mathbb{G}$ is zero, the only terms of (7) contributing to (6) are the ones with $s = k+1$, thus (6) becomes

$$0 = \sum_{j \geqslant k} \frac{(-1)^j}{(j-k)!} r^{k+1} \int_{[M]^{\mathrm{vir}}} \xi_{\mathbb{G}} \left(\mathsf{L}_j^\delta \mathsf{R}_{-1}^{j-k}(D) \right)$$

By the induction hypothesis, all the terms with $j > k$ vanish and thus the term with $j = k$ also vanishes, showing that

$$0 = \int_{[M]^{\mathrm{vir}}} \xi_{\mathbb{G}}(\mathsf{L}_k^\delta(D)),$$

which concludes the induction step. □

Remark 2.17. For surfaces or 3-folds X with $H^i(\mathcal{O}_X) = 0$ for $i > 0$, the Hilbert scheme of points and the moduli of stable pairs, respectively, are moduli of sheaves in the sense of Section 1.2. The canonical universal sheaves in each cases are precisely the pt-normalized universal sheaves and the formulations in [MOOP, Mor] coincide with Conjecture 2.15. In [vB], the author uses a geometric realization with respect to the sheaf $\mathbb{G} \otimes \det(\mathbb{G})^{-1/r}$ that is not a universal sheaf for M; however,

$$\mathbb{G} \otimes \det(\mathbb{G})^{-1/r} \otimes \Delta^{1/r}$$

recovers the pt-normalized universal sheaf of Remark 2.14. The equivalence between van Bree's formulation of Virasoro for the moduli of stable sheaves of positive rank and Conjecture 2.15 is explained by Lemma 2.19.

2.6. **Virasoro constraints for pairs.** Since for moduli spaces of pairs we have a uniquely defined universal object $q^*V \to \mathbb{F}$, there is no need to use the weight 0 Virasoro operator for pairs. We need, however, to slightly modify the operators to account for the different obstruction theory.

Let V be a fixed sheaf on X and let P as in Section 1.2 be a moduli space of pairs parametrizing sheaves F together with a map $V \to F$. The moduli P comes equipped with a (unique) universal sheaf $\mathbb{F}$ on $P \times X$ and a universal map $q^*V \to \mathbb{F}$. We conjecture that descendent invariants obtained by integration on moduli of pairs are constrained by Virasoro operators which are similar to the ones introduced in Section 2.3. Define the operators

$$\mathsf{L}_k^V : \mathbb{D}^X \to \mathbb{D}^X$$

by $\mathsf{L}_k^V = \mathsf{R}_k + \mathsf{T}_k^V$ where

$$\mathsf{T}_k^V = \mathsf{T}_k - k!\mathrm{ch}_k^{\mathrm{H}}\left(\mathrm{ch}(V)^\vee \mathrm{td}(X)\right).$$

The operator L_k^V can be described in an alternative way that should make its definition more natural and that will be closer to the vertex algebra language that we will introduce later. Let

$$\mathbb{D}^{X,\mathrm{pa}} = \mathbb{D}^X \otimes \mathbb{D}^X$$

be the algebra of pair descendents. We denote the generators of the first copy of $\mathbb{D}^X$ by $\mathrm{ch}_i^{\mathrm{H},\mathcal{V}}(\gamma)$ and the generators of the second copy by $\mathrm{ch}_i^{\mathrm{H},\mathcal{F}}(\gamma)$. Given the universal pair $q^*V \to \mathbb{F}$ on $P \times X$, we have a geometric realization map

$$\xi_{(q^*V,\mathbb{F})} \colon \mathbb{D}^{X,\mathrm{pa}} \to H^\bullet(P)$$

that is defined by

$$\xi_{(q^*V,\mathbb{F})}\left(\mathrm{ch}_i^{\mathrm{H},\mathcal{V}}(\gamma)\right) = \xi_{q^*V}\left(\mathrm{ch}_i^{\mathrm{H}}(\gamma)\right) = \begin{cases} \int_X \gamma \cdot \mathrm{ch}(V) & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\xi_{(q^*V,\mathbb{F})}\left(\mathrm{ch}_i^{\mathrm{H},\mathcal{F}}(\gamma)\right) = \xi_{\mathbb{F}}\left(\mathrm{ch}_i^{\mathrm{H}}(\gamma)\right).$$

This geometric realization map factors through

$$\begin{tikzcd} \mathbb{D}^{X,\mathrm{pa}} \arrow[d, "\xi_V"'] \arrow[dr, "\xi_{(q^*V,\mathbb{F})}"] & \\ \mathbb{D}^X \arrow[r, "\xi_{\mathbb{F}}"] & H^\bullet(P) \end{tikzcd}$$

where ξ_V is defined to send

$$\mathrm{ch}_i^{\mathrm{H},\mathcal{V}}(\gamma) \mapsto \delta_i \int_X \gamma \cdot \mathrm{ch}(V) \text{ and } \mathrm{ch}_i^{\mathrm{H},\mathcal{F}}(\gamma) \mapsto \mathrm{ch}_i^{\mathrm{H}}(\gamma).$$

We define the pair Virasoro operators $\mathsf{L}_k^{\mathrm{pa}} \colon \mathbb{D}^{X,\mathrm{pa}} \to \mathbb{D}^{X,\mathrm{pa}}$, for $k \geqslant -1$, as the sum $\mathsf{R}_k^{\mathrm{pa}} + \mathsf{T}_k^{\mathrm{pa}}$ where

(1) $\mathsf{R}_k^{\mathrm{pa}}$ is a derivation defined on generators in the same way as R_k; in other words,

$$\mathsf{R}_k^{\mathrm{pa}} = \mathsf{R}_k \otimes \mathrm{id} + \mathrm{id} \otimes \mathsf{R}_k.$$

(2) $\mathsf{T}_k^{\mathrm{pa}}$ is the operator of multiplication by the element

$$\mathsf{T}_k^{\mathrm{pa}} = \sum_{i+j=k} (-1)^{\dim X - p^L} i!j!\mathrm{ch}_i^{\mathrm{H},\mathcal{F}-\mathcal{V}}\mathrm{ch}_j^{\mathrm{H},\mathcal{F}}(\mathrm{td}(X)) \in \mathbb{D}^{X,\mathrm{pa}}$$

where

$$\mathrm{ch}_i^{\mathrm{H},\mathcal{F}-\mathcal{V}} := \mathrm{ch}_i^{\mathrm{H},\mathcal{F}} - \mathrm{ch}_i^{\mathrm{H},\mathcal{V}}.$$

The definition of $\mathsf{T}_k^{\mathrm{pa}}$ suggests an intimate relation between the linear part of the Virasoro operators and the obstruction theory of the moduli spaces we consider; recall that the deformation-obstruction theory of the pair moduli space P at $V \to F$ is governed by

$$\mathrm{RHom}\left([V \to F], F\right)$$

and use Example 2.6.

The operators L_k^V are obtained from $\mathsf{L}_k^{\mathrm{pa}}$ by the following commutative diagram:

$$\begin{tikzcd}
\mathbb{D}^{X,\mathrm{pa}} \arrow[r, "\mathsf{L}_k^{\mathrm{pa}}"] \arrow[d, "\xi_V"] & \mathbb{D}^{X,\mathrm{pa}} \arrow[d, "\xi_V"] \\
\mathbb{D}^X \arrow[r, "\mathsf{L}_k^V"] & \mathbb{D}^X .
\end{tikzcd}$$

This holds due to the identity

$$\begin{aligned}\sum_s (-1)^{\dim X - p_t^L}\left(\int_X \gamma_t^L \cdot \mathrm{ch}(V)\right)\mathrm{ch}_k^{\mathrm{H}}(\gamma_t^R) &= \sum_t \left(\int_X \gamma_t^L \cdot \mathrm{ch}(V)^\vee\right)\mathrm{ch}_k^{\mathrm{H}}(\gamma_t^R)\\ &= \mathrm{ch}_k^{\mathrm{H}}(\mathrm{ch}(V)^\vee \mathrm{td}(X)).\end{aligned}$$

It can be shown that the operators $\{\mathsf{L}_k^V\}_{k\geqslant -1}$ and $\{\mathsf{L}_k^{\mathrm{pa}}\}_{k\geqslant -1}$ satisfy the Virasoro bracket relations.

Conjecture 2.18. Let P be a moduli of pairs as in Section 1.2 with universal pair $q^*V \to \mathbb{F}$ on $P \times X$. Then

$$\int_{[P]^{\mathrm{vir}}} \xi_{\mathbb{F}}\big(\mathsf{L}_k^V(D)\big) = 0 \quad \text{for any } k \geqslant 0,\ D \in \mathbb{D}^X.$$

Equivalently, the pair Virasoro constraints can be formulated as

$$\int_{[P]^{\mathrm{vir}}} \xi_{(q^*V,\mathbb{F})}\big(\mathsf{L}_k^{\mathrm{pa}}(D)\big) = 0 \quad \text{for any } k \geqslant 0,\ D \in \mathbb{D}^{X,\mathrm{pa}}.$$

2.7. **Invariance under twist.** Suppose that $M = M_\alpha$ is the moduli space of slope semistable sheaves with respect to a polarization H with topological type α. Then, the moduli space $M' = M_{\alpha(H)}$ is isomorphic to M by sending a sheaf $[F] \in M$ to $[F'] = [F \otimes H]$. As expected, the Virasoro constraints on M and M' are equivalent as we now proceed to verify.

Suppose that $\mathbb{G}$ is a universal sheaf on $M \times X$. The universal sheaf on $M' \times X$ is identified with $\mathbb{G}' = \mathbb{G} \otimes q^*H$ via the isomorphism $M' \times X \cong M \times X$. Define an algebra isomorphism $\mathsf{F}\colon \mathbb{D}^X \to \mathbb{D}^X$ by

$$\mathsf{F}(\mathrm{ch}_i^{\mathrm{H}}(\gamma)) = \mathrm{ch}_i^{\mathrm{H}}(e^{c_1(H)}\gamma)\,.$$

Then the following diagram commutes:

$$\begin{tikzcd}
\mathbb{D}^X \arrow[r, "\xi_{\mathbb{G}'}"] \arrow[d, "\mathsf{F}"] & H^*(M') \arrow[d, "\wr"] \\
\mathbb{D}^X \arrow[r, "\xi_{\mathbb{G}}"] & H^*(M)
\end{tikzcd}$$

Lemma 2.19. The isomorphism F commutes with the Virasoro operators, i.e.

$$\mathsf{L}_k \circ \mathsf{F} = \mathsf{F} \circ \mathsf{L}_k \text{ for } k \geqslant -1 \quad \text{and} \quad \mathsf{L}_{\mathrm{wt}_0} \circ \mathsf{F} = \mathsf{F} \circ \mathsf{L}_{\mathrm{wt}_0}.$$

Thus, Conjecture 2.12 holds for M if and only if it holds for M'.

Proof. It is enough to show that L_k and F commute. Commutativity with $\mathsf{L}_{\mathrm{wt}_0}$ follows immediately from its definition; the equivalence of Virasoro constraints follows from the commutativity and the diagram before the Lemma.

The commutativity with the derivation part, i.e. $\mathsf{R}_k \circ \mathsf{F} = \mathsf{F} \circ \mathsf{R}_k$, is straightforward. To show commutativity with the operator T_k we need $\mathsf{F}(\mathsf{T}_k) = \mathsf{T}_k$ (recall that T_k denotes both an element in $\mathbb{D}^X$ and the operator which is multiplication by that element).

$$\begin{aligned}
\mathsf{F}(\mathsf{T}_k) &= \sum_{i+j=k} \sum_t (-1)^{\dim X - p_t^L} i!j! \mathrm{ch}_i^{\mathrm{H}}(e^{c_1(H)}\gamma_t^L)\mathrm{ch}_j^{\mathrm{H}}(e^{c_1(H)}\gamma_t^R) \\
&= \sum_{i+j=k} \sum_t \sum_{a,b\geqslant 0} (-1)^{\dim X - p_t^L} i!j! \frac{1}{a!b!} \mathrm{ch}_i^{\mathrm{H}}(c_1(H)^a \gamma_t^L)\mathrm{ch}_j^{\mathrm{H}}(c_1(H)^b\gamma_t^R) \\
&= \sum_{i+j=k} \sum_{a,b\geqslant 0} (-1)^{\dim X - p^L - a} i!j! \frac{1}{a!b!} \mathrm{ch}_i^{\mathrm{H}}\mathrm{ch}_j^{\mathrm{H}}(c_1(H)^{a+b}\mathrm{td}(X)) \\
&= \sum_{i+j=k} (-1)^{\dim X - p^L} i!j! \mathrm{ch}_i^{\mathrm{H}}\mathrm{ch}_j^{\mathrm{H}}(\mathrm{td}(X)) = \mathsf{T}_k.
\end{aligned}$$

The last line uses the fact that for fixed $c > 0$ the sum $\sum_{a+b=c} \frac{(-1)^a}{a!b!}$ vanishes. $\square$

2.8. **Variants for the fixed determinant theory.** As we pointed out in Remark 1.1, this paper is mostly concerned with moduli spaces of sheaves without fixed determinant; in other words, our obstruction theories use full Ext groups instead of traceless Ext groups. This contrasts with the situation for Pandharipande-Thomas stable pairs or Hilbert schemes of points studied in [Mor]. There, we see a new term appearing in the Virasoro operators which we now recall.

Definition 2.20. Given a class $\gamma \in H^\bullet(X)$ denote by $\mathsf{R}_{-1}[\gamma]$ the superderivation (of degree $\deg(\gamma) - 2$) acting on generators by

$$\mathsf{R}_{-1}[\gamma]\mathrm{ch}_i^{\mathrm{H}}(\gamma') = \mathrm{ch}_{i-1}^{\mathrm{H}}(\gamma\gamma').$$

For $k \geqslant -1$ we define the operator $\mathsf{S}_k \colon \mathbb{D}^X \to \mathbb{D}^X$ by

$$\mathsf{S}_k = (k+1)! \sum_{p_t^L = 0} \mathsf{R}_{-1}[\gamma_t^L]\mathrm{ch}_{k+1}^{\mathrm{H}}(\gamma_t^R)$$

where the sum runs over the terms $\gamma_t^L \otimes \gamma_t^R$ in the Künneth decomposition of $\Delta_* 1$ such that $p_t^L = 0$.

Remark 2.21. The part of S_k corresponding to the Künneth component $1 \otimes \mathsf{pt}$ is equal to $-r\mathsf{S}_k^{\mathsf{pt}}$ where $r = \mathrm{ch}_0(\alpha) = \mathrm{rk}(\alpha)$. When $h^{0,q}(X) = 0$ for $q > 0$ the appearance of the operator S_k is already explained in Remark 2.17 as being related to the pt-normalized universal sheaf.

We now proceed to explain the appearance of the operator S_k in the more general case in which $h^{0,1} \neq 0$ but $h^{0,q} = 0$ for $q > 1$. Let α be such that $r = \mathrm{rk}(\alpha) > 0$ and $M = M_\alpha$ be a moduli space parametrizing semistable sheaves with topological type α. Let $\Delta \in \mathrm{Pic}(X)$ be a fixed line bundle such that $c_1(\Delta) = c_1(\alpha)$. We let $M_\Delta \subseteq M$ be the moduli space of sheaves on $F \in M$ with fixed determiant $\det(F) = \Delta$; i.e., M_Δ is the pullback

$$\begin{tikzcd} M_\Delta \arrow[r, hook] \arrow[d] & M \arrow[d, "\det"] \\ \{\Delta\} \arrow[r, hook] & \mathrm{Pic}^{c_1(\alpha)}(X). \end{tikzcd} \tag{8}$$

The moduli space M_Δ has a 2-term obstruction theory given by

$$\begin{aligned} \mathrm{Tan} &= \mathrm{Ext}^1(G,G)_0 \\ \mathrm{Obs} &= \mathrm{Ext}^2(G,G)_0 = \mathrm{Ext}^2(G,G) \\ 0 &= \mathrm{Ext}^{>2}(G,G). \end{aligned}$$

Given such a moduli space, there is a unique universal sheaf (possibly rational, in the same sense of Remark 2.14) $\mathbb{G}_\Delta$ on $M_\Delta \times X$ such that $\det(\mathbb{G}_\Delta) = q^*\Delta$. We suppose also that the Jacobian $\mathrm{Pic}^0(X)$ acts on M in the natural way; that is,

$$[F] \in M \Rightarrow [F \otimes L] \in M$$

for $L \in \mathrm{Pic}^0(X)$. The two main examples to keep in mind are

(1) $M = M_C(r,d)$ a moduli of stable bundles on a curve C and Δ a line bundle of degree d;
(2) $M = M_S^H(r, c_1, c_2)$ a moduli of stable sheaves on a surface S with $p_g(S) = 0$ and Δ a line bundle with $c_1(\Delta) = c_1$. Note that if we take $r = 1, c_1 = 0$ and $\Delta = \mathcal{O}_S$ we recover the Hilbert scheme of points on S
$$M_S^H(1, \mathcal{O}_S, n) = S^{[n]}.$$

Proposition 2.22. Suppose that X is such that $h^{0,q} = 0$ for $q > 1$ and M, M_Δ are as described before. Suppose also that the Virasoro constraints (Conjecture 2.12) hold for M. Then, we have

$$\int_{[M_\Delta]^{\mathrm{vir}}} \xi_{\mathbb{G}_\Delta}\big(\mathcal{L}_k(D)\big) = 0 \quad \text{for any } k \geqslant -1,\ D \in \mathbb{D}^X$$

where

$$\mathcal{L}_k = \mathsf{L}_k - \frac{1}{r}\mathsf{S}_k.^7$$

Proof. Let

$$\pi\colon \widetilde{M} = M_\Delta \times \mathrm{Pic}^0(X) \to M$$

be the étale map sending (F, L) to $F \otimes L$. Let $\mathcal{P}$ be the Poincaré bundle on $\mathrm{Pic}^0(X) \times X$ (i.e., the pt-normalized universal bundle of $\mathrm{Pic}^0(X)$) and let $\mathbb{G}$ be the pt-normalized universal sheaf on $M \times X$. Since $\widetilde{\mathbb{G}} = \mathbb{G}_\Delta \boxtimes \mathcal{P}$ is also pt-normalized

[7] The $-$ sign in the S_k operator does not appear in [Mor] due to the fact that in [Mor] the geometric realization is taken with respect to $-\mathbb{G}_\Delta$ instead of $\mathbb{G}_\Delta$.

it follows that $(\pi\times\mathrm{id})^*\mathbb{G}=\widetilde{\mathbb{G}}$ (in rational K-theory) and thus $\pi^*\circ\xi_{\mathbb{G}}=\xi_{\widetilde{\mathbb{G}}}$. Using that

$$\pi^*[M]^{\mathrm{vir}}=[\widetilde{M}]^{\mathrm{vir}}\coloneqq[M_\Delta]^{\mathrm{vir}}\times[\mathrm{Pic}^0(X)]$$

and Proposition 2.16 we get

$$\int_{[\widetilde{M}]^{\mathrm{vir}}}\xi_{\widetilde{\mathbb{G}}}\big((\mathsf{L}_k+\mathsf{S}_k^{\mathsf{pt}})(D)\big)=0 \tag{9}$$

by the push-pull formula and the fact that $\pi_*1=\deg(\pi)$, where $\mathsf{S}_k^{\mathsf{pt}}=-\frac{1}{r}\mathsf{R}_{-1}\mathrm{ch}_{k+1}(\mathsf{pt})$. Let $g=h^{0,1}(X)=h^{1,0}(X)$ and let $\{e_1,\dots,e_g\}\subseteq H^{0,1}(X)$ and $\{f_1,\dots,f_g\}\subseteq H^{1,0}(X)$ be basis; let $\{\hat{e}_1,\dots,\hat{e}_g\}\subseteq H^{m,m-1}(X)$, $\{\hat{f}_1,\dots,\hat{f}_g\}\subseteq H^{m-1,m}(X)$ be their dual basis, i.e. such that

$$\int_X e_i\hat{e}_j=\delta_{ij}=\int_X f_i\hat{f}_j$$

Let

$$\tau_j=\xi_{\mathcal{P}}(\mathrm{ch}_1^{\mathrm{H}}(\hat{e}_j))\in H^{1,0}(\mathrm{Pic}^0(X))\quad\text{and}\quad\rho_j=\xi_{\mathcal{P}}(\mathrm{ch}_0^{\mathrm{H}}(\hat{f}_j))\in H^{0,1}(\mathrm{Pic}^0(X))$$

so that

$$c_1(\mathcal{P})=\sum_{j=1}^g\tau_j\otimes e_j+\sum_{j=1}^g\rho_j\otimes f_j\in H^2(\mathrm{Pic}^0(X)\times X).$$

The Jacobian is topologically a real torus of dimension $2g$ and its cohomology is the exterior algebra generated by the classes $\{\tau_j,\rho_j\}_{j=1}^g$. Let

$$\omega=\sum_{j=1}^g\rho_j\tau_j\in H^2(\mathrm{Pic}^0(X)).$$

By rescaling the elements of the basis, we may assume that $\int_{\mathrm{Pic}^0(X)}\prod_{j=1}^g\rho_j\tau_j=\frac{1}{g!}$ so that $\omega^g\in H^{2g}(\mathrm{Pic}^0(X))$ is the class Poincaré dual to a point in $\mathrm{Pic}^0(X)$.

Since $\mathrm{ch}_0(\mathbb{G}_\Delta)=r$ and $\mathrm{ch}_1(\mathbb{G}_\Delta)=q^*c_1(\Delta)$, we have

$$\xi_{\widetilde{\mathbb{G}}}(\mathrm{ch}_1^{\mathrm{H}}(\hat{e}_j))=r\tau_j\quad\text{and}\quad\xi_{\widetilde{\mathbb{G}}}(\mathrm{ch}_0^{\mathrm{H}}(\hat{f}_j))=r\rho_j$$

in $H^\bullet\big(\widetilde{M}\big)=H^\bullet(M_\Delta\times\mathrm{Pic}^0(X))$ where we omit the obvious pullback. Let

$$W=\sum_{j=1}^g\mathrm{ch}_0^{\mathrm{H}}(\hat{f}_j)\mathrm{ch}_1^{\mathrm{H}}(\hat{e}_j)\in\mathbb{D}^X\quad\text{so that}\quad\xi_{\widetilde{\mathbb{G}}}(W)=r^2\omega.$$

We will apply equation (9) to descendents of the form W^gD for some $D\in\mathbb{D}^X$ and use that to deduce the Proposition. We have

$$\begin{aligned}0&=\int_{[\widetilde{M}]^{\mathrm{vir}}}\xi_{\widetilde{\mathbb{G}}}\big((\mathsf{L}_k+\mathsf{S}_k^{\mathsf{pt}})(W^gD)\big)\\&=r^{2g}\int_{[\widetilde{M}]^{\mathrm{vir}}}\omega^g\xi_{\widetilde{\mathbb{G}}}\big((\mathsf{L}_k+\mathsf{S}_k^{\mathsf{pt}})(D)\big)+\int_{[\widetilde{M}]^{\mathrm{vir}}}\xi_{\widetilde{\mathbb{G}}}\big(\mathsf{R}_k(W^g)D)\big)\\&=r^{2g}\int_{[M_\Delta]^{\mathrm{vir}}}\xi_{\mathbb{G}_\Delta}\big((\mathsf{L}_k+\mathsf{S}_k^{\mathsf{pt}})(D)\big)+gr^{2g-2}\int_{[\widetilde{M}]^{\mathrm{vir}}}\omega^{g-1}\xi_{\widetilde{\mathbb{G}}}\big(\mathsf{R}_k(W)D)\big).\end{aligned} \tag{10}$$

We now rewrite the second integral in terms of an integral in M_Δ. We have

$$\xi_{\widetilde{\mathbb{G}}}\big(\mathsf{R}_k(W)\big) = (k+1)!\sum_{j=1}^{g}\xi_{\widetilde{\mathbb{G}}}\big(\mathrm{ch}_0^{\mathrm{H}}(\hat{f}_j)\mathrm{ch}_{k+1}^{\mathrm{H}}(\hat{e}_j)\big) = r(k+1)!\sum_{j=1}^{g}\rho_j\xi_{\widetilde{\mathbb{G}}}\big(\mathrm{ch}_{k+1}^{\mathrm{H}}(\hat{e}_j)\big).$$

Claim 1. Let $h\colon \widetilde{M}\to M_\Delta$ be the projection and let $D\in\mathbb{D}^X$. Then we have

$$h_*\big(g\omega^{g-1}\rho_j\xi_{\widetilde{\mathbb{G}}}(D)\big) = \xi_{\mathbb{G}_\Delta}\big(\mathsf{R}_{-1}[e_j]D\big).$$

Proof of claim. Let $J\subseteq H^\bullet\big(\widetilde{M}\big)$ be the annihilator ideal of $\omega^{g-1}\rho_j$; in particular, $H^{\geqslant 2}(\mathrm{Pic}^0(X))\subseteq J$. By definition,

$$\begin{aligned}\xi_{\widetilde{\mathbb{G}}}\big(\mathrm{ch}_\bullet^{\mathrm{H}}(\gamma)\big) &= p_*\big(\mathrm{ch}(\mathbb{G}_\Delta)\mathrm{ch}(\mathcal{P})q^*\gamma\big) = p_*\big(\mathrm{ch}(\mathbb{G}_\Delta)\big(1+(p^*\tau_j)(q^*e_j)+\dots\big)q^*\gamma\big)\\ &= \xi_{\mathbb{G}_\Delta}\big(\mathrm{ch}_\bullet^{\mathrm{H}}(\gamma)\big)+\tau_j\xi_{\mathbb{G}_\Delta}\big(\mathrm{ch}_\bullet^{\mathrm{H}}(e_j\gamma)\big)+\dots\end{aligned}$$

where the terms in $\dots$ are all in the ideal J and where we omit all the pullbacks via the projections $\widetilde{M}\to M_\Delta$ and $\widetilde{M}\to\mathrm{Pic}^0(X)$. Thus,

$$\xi_{\widetilde{\mathbb{G}}}(D) = \xi_{\mathbb{G}_\Delta}(D)+\tau_j\xi_{\mathbb{G}_\Delta}\big(\mathsf{R}_{-1}[e_i]D\big)+\dots$$

Since

$$\int_{\mathrm{Pic}^0(X)}\omega^{g-1}\rho_j\tau_j = (g-1)!\int_X\prod_{l=1}^{g}\rho_l\tau_l = \frac{1}{g}$$

the claim is proven. ☐

Given the claim, (10) becomes

$$0 = r^{2g}\int_{M_\Delta}\xi_{\mathbb{G}_\Delta}\big((L_k+S_k^{\mathsf{pt}})(D)\big)+r^{2g-1}(k+1)!\int_{M_\Delta}\sum_{j=1}^{g}\xi_{\mathbb{G}_\Delta}\big(R_{-1}[e_j]\mathrm{ch}_{k+1}^{\mathrm{H}}(\hat{e}_j)D)\big)$$

Since $h^{0,q}=0$ for $q>1$, we have

$$\mathsf{S}_k = -r\mathsf{S}_k^{\mathsf{pt}}-\sum_{j=1}^{g}\mathsf{R}_{-1}[e_j]\mathrm{ch}_{k+1}^{\mathrm{H}}(\hat{e}_j)$$

so the Proposition follows. ☐

Remark 2.23. When S is a surface with $p_g>0$ we do not know a rigorous interpretation for the terms of the S_k with $\gamma_s^L\in H^{0,2}(S)$. Heuristically, these should be related to a diagram like (8) where the Picard group is interpreted as a derived stack, see [STV].

Example 2.24. Let $M=M_C(2,\Delta)$ be the moduli space of stable bundles on a curve C of genus g with rank 2 and fixed determinant $\Delta\in\mathrm{Pic}(C)$ of odd degree; this is a smooth moduli space of dimension $3g-3$. This moduli space has been studied a lot in the past and in particular its ring structure and descendent integrals are completely understood. We use this example to illustrate what kind of information the Virasoro constraints provide on descendent integrals. Let $\{e_1,\dots,e_g\}\subseteq H^{0,1}(C)$

and $\{f_1, \ldots, f_g\} \subseteq H^{1,0}(C)$ be dual basis. Newstead proved in [New, Theorem 1] that the cohomology $H^\bullet(M)$ is generated by the classes

$$\eta = -2\xi\big(\mathrm{ch}_1^{\mathrm{H}}(\mathbf{1})\big), \quad \theta = 4\xi\big(\mathrm{ch}_2^{\mathrm{H}}(\mathsf{pt})\big), \quad \xi_j = -\xi\big(\mathrm{ch}_1^{\mathrm{H}}(e_j)\big), \quad \psi_j = \xi\big(\mathrm{ch}_2^{\mathrm{H}}(f_j)\big).^8$$

The geometric realization ξ is taken with respect to the sheaf $\mathbb{G}\otimes\det(\mathbb{G})^{-1/2}$ (see Remark 2.17). Every descendent can be written explicitly in terms of the classes $\eta, \theta, \xi_j, \psi_j$. By [Tha, (24) Proposition], one can then write every descendent integral in terms of integrals of products of the classes η, θ and

$$\zeta = 2\sum_{j=1}^{g} \psi_j \xi_j.$$

These integrals are fully determined in [Tha, (30)]: for m, k, p such that $m + 2k + 3p = 3g - 3$ we have

$$\int_M \eta^m \theta^k \zeta^p = (-1)^{g-1-p} \frac{m!g!}{q!(g-p)!} 2^{2g-2-p}(2^q - 2)B_q \tag{11}$$

where $q = m + p - g + 1$ and B_q is a Bernoulli number. A careful combinatorial analysis shows that the Virasoro constraints given by Proposition 2.22 for M are equivalent to the relations

$$(g-p)\int_M \eta^m \theta^k \zeta^p = -2m \int_M \eta^{m-1}\theta^{k-1}\zeta^{p+1}$$

which of course follows from (11); it would be interesting to have a direct and simpler proof of this identity. Note that the Virasoro constraints do not capture the most interesting part of (11) which is the Bernoulli number. In some sense this is to be expected: since the Virasoro constraints hold in great generality (in particular are invariant under wall-crossing), it should not be expected that they can capture special information about particular moduli spaces.

3. Vertex operator algebra

To prove the conjectures from the previous sections, we will apply the wall-crossing machinery introduced by Joyce which relies on his geometric construction of vertex algebras. The resulting vertex algebras have been described in many cases as lattice vertex algebras and as such admit a natural family of conformal elements. This section focuses on developing the necessary vertex operator algebra language, including the definition of lattice vertex operator algebras in the generality that we need, Borcherds Lie algebra associated to a vertex algebra and the notion of primary states.

[8] In [Tha, New] the classes $\eta, \theta, \xi_j, \psi_j, \zeta$ are called, respectively, $\alpha, \beta, \psi_{j+g}, \psi_j, \gamma$.

3.1. **Vertex operator algebra.** There are many equivalent formulations of vertex algebras. We will follow the definitions and notation in [Kac]. In particular, vertex algebra for us means $\mathbb{Z}$-graded vertex superalgebra over $\mathbb{C}$.

Definition 3.1. A *vertex algebra* is a $\mathbb{Z}$-graded vector space $V_\bullet$ over $\mathbb{C}$ together with

(1) a *vacuum vector* $|0\rangle \in V_0$,
(2) a linear operator $T\colon V_\bullet \to V_{\bullet+2}$ called the *translation operator*,
(3) and a *state-field correspondence* which is a degree 0 linear map
$$Y\colon V_\bullet \longrightarrow \mathrm{End}(V_\bullet)[\![z, z^{-1}]\!]\,,$$
denoted by
$$Y(a,z) := \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}\,,$$
where $a_{(n)} : V_\bullet \to V_{\bullet+\deg(a)-2n-2}$ and $\deg(z) = -2$.

They are required to satisfy conditions which can be formulated in many different ways. We choose our favorite version:

(1) *(vacuum)* $T\,|0\rangle = 0\,,\, Y(|0\rangle, z) = \mathrm{id}\,,\, Y(a,z)|0\rangle = a + zV_\bullet[\![z]\!]\,,$
(2) *(translation covariance)* $[T, Y(a,z)] = \frac{d}{dz}Y(a,z)$ for any $a \in V_\bullet\,,$
(3) *(locality)* for any $a, b \in V_\bullet$, there is an $N \gg 0$ such that
$$(z-w)^N[Y(a,z), Y(b,w)] = 0\,,$$
where the supercommutator is defined on $\mathrm{End}(V_\bullet)$ by
$$[A,B] = A\circ B - (-1)^{|A||B|}B\circ A\,.$$

For later purposes it is useful to note that these axioms imply the two following identities which were used by Borcherds [Bor, §4] to originally define vertex algebras:

$$a_{(n)}b = \sum_{i\geqslant 0}(-1)^{|a||b|+i+n+1}\frac{T^i}{i!}b_{(n+i)}(a)\,,$$

$$\big(a_{(m)}b\big)_{(n)}c = \sum_{i\geqslant 0}(-1)^i\binom{m}{i}\Big[a_{(m-i)}\big(b_{(n+i)}c\big) - (-1)^{|a||b|+m}b_{(m+n-i)}\big(a_{(i)}c\big)\Big]\,. \tag{12}$$

They are a refinement of skew-symmetry and the Jacobi identity to the setting of vertex algebras. Additionally, we will also use

$$(Ta)_{(n)} = -n\cdot a_{(n-1)} \tag{13}$$

which follows from the more general reconstruction result. To understand it, one needs to make sense of a product of two fields $Y(v,z)$ and $Y(w,z)$ which naively could contain infinite sums for each coefficient. For this one uses the following trick.

Definition 3.2 ([Kac, (2.3.5)])**.** A *normal ordering* $: - :$ is defined by

$$: v_{(k)}w_{(l)} : = \begin{cases} (-1)^{|v||w|}w_{(l)}v_{(k)} & \text{if} \quad k \leqslant 0, l > 0\,,\\ v_{(k)}w_{(l)} & \text{otherwise}\,. \end{cases}$$

In general, this can be extended to any monomial in $v_{(k)}$ by iterating the above operation on the neighboring terms until all terms with non-positive index k are on the right.

Theorem 3.3 (Kac [Kac, Cor. 4.5])**.** Let $a^1, \cdots, a^n \in V_\bullet$ be a finite collection of elements and $k_1, \ldots, k_n \in \mathbb{Z}_{\geqslant 0}$, then a general field can be described as

$$Y\left(a^1_{(-k_1-1)}\cdots a^n_{(-k_n-1)}\left|0\right\rangle, z\right) = \frac{1}{k_1!k_2!\cdots k_n!} : \frac{d^{k_1}}{(dz)^{k_1}}Y(a^1, z)\cdots\frac{d^{k_n}}{(dz)^{k_n}}Y(a^n, z) : .$$

To get (13) simply use $Ta = a_{(-2)}\left|0\right\rangle$ and compare the coefficients on both sides.

We now recall the definition of an additional structure on vertex algebras called a conformal element or conformal vector. They generate Virasoro vertex subalgebras and in this paper give rise to a compact way to summarize all the information contained in geometric Virasoro constraints.

Definition 3.4. A *conformal element* ω on a vertex algebra $V_\bullet$ is an element of V_4 such that its associated fields $L_n = \omega_{(n+1)}$, defined by

$$Y(\omega, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}\,,$$

satisfy

(1) the Virasoro bracket
$$\big[L_n, L_m\big] = (n-m)L_{m+n} + \frac{n^3-n}{12}\delta_{n+m,0}\cdot C\,,$$
where $C \in \mathbb{C}$ is a constant called the *central charge* of ω,
(2) $L_{-1} = T$,
(3) and L_0 is diagonalizable.

A vertex algebra $V_\bullet$ together with a conformal element ω is called a *conformal vertex algebra* or *vertex operator algebra*. We denote by $V^\omega_\bullet$ the *conformal grading* on the underlying vector space, where V^ω_i is the $i \in \mathbb{C}$ eigenspace of L_0. We denote the *conformal degree* by

$$\deg_\omega(a) = i \quad \text{if} \quad a \in V^\omega_i\,,$$

to distinguish it from the original degree on $V_\bullet$.

3.2. **Lattice vertex (operator) algebras.** We next describe a particular construction of a vertex operator algebra which we will be working with, called *lattice vertex algebras*, while following closely the sections [Kac, §3.5, 3.6 and 5.5]. The next theorem gives the outline of the main statements that we need while we expand on them in the rest of the section.

Theorem 3.5 (Kac)**.** Assume that we have the following data:

(1) A $\mathbb{Z}_2$-graded $\mathbb{C}$-vector space $\Lambda = \Lambda_{\bar 0} \oplus \Lambda_{\bar 1}$ with a symmetric bilinear pairing $Q\colon \Lambda\times\Lambda \to \mathbb{C}$ which is a direct sum of its restrictions $Q_{\bar i} : \Lambda_{\bar i}\times\Lambda_{\bar i} \to \mathbb{C}$.

(2) The pairing Q is obtained as the symmetrization of a not necessarily symmetric pairing $q\colon \Lambda \times \Lambda \to \mathbb{Z}$, i.e.,

$$Q(v, w) = q(v, w) + q(w, v).$$

(3) An abelian group of a finite rank Λ_{sst} that admits an inclusion

$$\Lambda_{\mathrm{sst}} \otimes_{\mathbb{Z}} \mathbb{C} \hookrightarrow \Lambda_{\bar{0}}$$

such that the restriction of q to Λ_{sst} is integer valued. This makes $(\Lambda_{\mathrm{sst}}, Q)$ an even generalized integer lattice in the sense of [Gro, Section 3].

Then there is a uniquely defined vertex algebra $V_\bullet$ whose underlying vector space is

$$V_\bullet = \mathbb{C}[\Lambda_{\mathrm{sst}}] \otimes \mathbb{D}_\Lambda,$$

where

$$\mathbb{D}_\Lambda = \mathrm{SSym}\big[\mathrm{CH}_\Lambda\big] \quad \text{and} \quad \mathrm{CH}_\Lambda = \bigoplus_{k>0} \Lambda \cdot t^{-k}\,.$$

The state-to-field correspondence is determined by equations (19), (16) and (17) and the translation operator is defined by equation (14).

Suppose moreover that we have

(4) The pairing Q is non-degenerate.
(5) A decomposition of $\Lambda_{\bar{1}}$ as a sum of two maximal isotropic subspaces, i.e.,

$$\Lambda_{\bar{1}} = I \oplus \hat{I}.$$

Then $V_\bullet$ admits a conformal element ω defined as (22) whose central charge is

$$\mathrm{sdim}\,\Lambda = \dim \Lambda_{\bar{0}} - \dim \Lambda_{\bar{1}}.$$

We will use the notation $v_{-k} = v \cdot t^{-k} \in \mathbb{D}_\Lambda$ and $e^\alpha \in \mathbb{C}[\Lambda_{\mathrm{sst}}]$ for the elements associated to $v \in \Lambda$ and $\alpha \in \Lambda_{\mathrm{sst}}$, respectively. The $\mathbb{Z}$-grading of $V_\bullet$ is defined by the degree assignment

$$\deg\Big(e^\alpha \otimes v^1_{-k_1} \cdots v^n_{-k_n}\Big) = \sum_{i=1}^n (2k_i - |v_i|) + Q(\alpha, \alpha)\,.$$

A vacuum vector is defined as $|0\rangle := e^0 \otimes 1 \in V_0$.

For $v \in \Lambda$ and $k > 0$, the *creation operator* $v_{(-k)}$ on $V_\bullet$ is defined as a left multiplication by the element $v_{-k} \in \mathbb{D}_\Lambda$. This gives $V_\bullet$ the structure of a free $\mathbb{D}_\Lambda$-module with basis $\{e^\alpha\}_{\alpha \in \Lambda_{\mathrm{sst}}}$. Thus, specifying an operator $A\colon V_\bullet \to V_\bullet$ is equivalent to describing its commutators with creation operators $[A, v_{(-k)}]$ and its action on the basis $A(e^\alpha)$. The operators that appear are often derivations of the $\mathbb{D}_\Lambda$-module $V_\bullet$, and we will often define them by describing their action on the generators v_{-k} of $\mathbb{D}_\Lambda$ and on the basis e^α.

The translation operator T is a $\mathbb{C}$-linear even derivation of the $\mathbb{D}_\Lambda$-module $V_\bullet$ determined by

$$T(v_{-k}) = kv_{-k-1}\,, \qquad T(e^\alpha) = e^\alpha \otimes \alpha_{-1}\,. \tag{14}$$

For $k \geqslant 0$, the *annihilation operator* $v_{(k)}$ is defined as a derivation of the $\mathbb{D}_\Lambda$-module $V_\bullet$ as we explain below, following Kac [Kac, §3.6 and §5.4]. The field corresponding to v_{-1} is then obtained as a sum of creation and annihilation operators:

$$Y(v_{-1}, z) = \sum_{k\in\mathbb{Z}} v_{(k)} z^{-k-1}\,. \tag{15}$$

We separate the construction of the fields into two cases. If our lattice Λ is concentrated in even degree (i.e. $\Lambda_{\bar{1}} = \{0\}$) we use Kac's bosonic construction and if it is concentrated in odd degree (i.e. $\Lambda_{\bar{0}} = \{0\}$) we use a fermionic construction. In the general case, we take the tensor product of the two.

Definition 3.6. Suppose that

(1) $\Lambda_{\bar{1}} = \{0\}$, then the resulting vertex algebra $V_{\bar{0},\bullet} = \mathbb{C}[\Lambda_{\mathrm{sst}}] \otimes \mathbb{D}_{\Lambda_{\bar{0}}}$ is called a *bosonic* lattice vertex algebra. In this case, the action of the annihilation operators $\{v_{(k)}\}_{k\geqslant 0}$ for $v \in \Lambda_{\bar{0}}$ is an even derivation on the $\mathbb{D}_{\Lambda_{\bar{0}}}$-module $V_{\bar{0},\bullet}$ determined by

$$v_{(k)}(w_{-l}) = k\, Q(v,w)\, \delta_{k-l,0}\,, \qquad v_{(k)}(e^\alpha) = Q(v,\alpha)\, \delta_{k,0}\, e^\alpha\,, \quad k \geqslant 0, l > 0\,.$$

Kac [Kac, § 5.5][9] endows $V_{\bar{0},\bullet}$ with a vertex algebra structure such that

$$Y(v_{-1}, z) = \sum_{k\in\mathbb{Z}} v_{(k)} z^{-k-1}\,,$$

and

$$Y(e^\alpha, z) = (-1)^{q(\alpha,\beta)} z^{Q(\alpha,\beta)} e^\alpha \exp\Big[-\sum_{k<0} \frac{\alpha_{(k)}}{k} z^{-k}\Big] \exp\Big[-\sum_{k>0} \frac{\alpha_{(k)}}{k} z^{-k}\Big] \tag{16}$$

on $e^\beta \otimes \mathbb{D}_\Lambda$; in the formula, e^α stands for the operator sending $e^\beta \otimes w$ to $e^{\alpha+\beta} \otimes w$. Note that the signs $\varepsilon_{\alpha,\beta} = (-1)^{q(\alpha,\beta)}$ satisfy Equations 5.4.14 in [Kac]. The general state-field correspondence is set to be

$$\begin{aligned} &Y\big(e^\alpha \otimes v^1_{-k_1-1} \cdots v^n_{-k_n-1}, z\big) \\ &\qquad = \frac{1}{k_1! k_2! \cdots k_n!} :Y(e^\alpha, z) \frac{d^{k_1}}{(dz)^{k_1}} Y(v^1_{-1}, z) \cdots \frac{d^{k_n}}{(dz)^{k_n}} Y(v^n_{-1}, z) :\,. \end{aligned} \tag{17}$$

[9]The construction in [Kac, Section 5.4] is only for the case in which Λ_{sst} is a lattice (i.e. it is a free abelian group) and $\Lambda_{\mathrm{sst}} \otimes_{\mathbb{Z}} \mathbb{C} = \Lambda_{\bar{0}}$, but everything works just fine in this slightly more general setting. This modification was already considered by Gross in [Gro, Definition 3.5]. Note also that the case $\Lambda_{\mathrm{sst}} = \{0\}$ corresponds to what Kac calls the vertex algebra of free bosons, see [Kac, Section 3.5].

(2) $\Lambda_{\bar{0}} = \{0\}$, then the resulting vertex algebra $V_{\bar{1},\bullet} = \mathbb{D}_{\Lambda_{\bar{1}}}$ is called a *fermionic* lattice vertex algebra, see [Kac, Section 3.6.]. It is determined uniquely by setting the annihilation operators $\{v_{(k)}\}_{k\geqslant 0}$, for $v \in \Lambda_{\bar{1}}$, to be odd derivations on the supercommutative algebra $\mathbb{D}_{\Lambda_{\bar{1}}}$ such that

$$v_{(k)}(w_{-l}) = Q(v,w)\delta_{k-l+1,0}\,.$$

The remaining fields are obtained again by the reconstruction Theorem [Kac, Thm. 4.5], which states the analog of (17) without e^{α}.

In general, $V_\bullet$ is defined as the tensor product of the two lattice vertex algebras,

$$V_\bullet = V_{\bar{0},\bullet} \otimes V_{\bar{1},\bullet}$$

which are associated to $\Lambda_{\mathrm{sst}} \otimes_{\mathbb{Z}} \mathbb{C} \hookrightarrow \Lambda_{\bar{0}}$ and $\Lambda_{\bar{1}}$, respectively. The resulting fields are determine uniquely by

$$Y(a_{\bar{0}} \otimes a_{\bar{1}}, z) = Y(a_{\bar{0}}, z) \otimes Y(a_{\bar{1}}, z)$$

whenever $a_{\bar{i}} \in V_{\bar{i},\bullet}$.

We can summarize the above description by formulating the commutation relations of the operators $v_{(k)}$ as

$$[v_{(k)}, w_{(\ell)}] = k^{(1-|v|)}Q(v,w)\delta_{k+\ell+|v|,0}\,, \tag{18}$$

which holds for all $k,\ell \in \mathbb{Z}$. If we want to obtain a closed formula for (15), we may rewrite the annihilation operators in terms of derivatives to get

$$Y(v_{-1},z) = \sum_{k\geqslant 0} v_{(-k-1)}z^k + \sum_{k\geqslant 1-|v|}\sum_{w\in B} k^{(1-|v|)}Q(v,w)\frac{\partial}{\partial w_{-k-|v|}}z^{-k-1} + Q(v,\beta)z^{-1}\,, \tag{19}$$

on $e^{\beta} \otimes \mathbb{D}_{\Lambda} \subseteq V_\bullet$, where $B \subset \Lambda$ is a basis.

We will later identify Joyce's vertex algebra with a lattice vertex algebra. For that, the following proposition, which is a simple corollary of [Kac, Proposition 5.4], will be useful.

Proposition 3.7. Let $V_\bullet$ be a vertex algebra with the underlying vector space $\mathbb{C}[\Lambda_{\mathrm{sst}}] \otimes \mathbb{D}_{\Lambda}$ such that:

(1) The vacuum vector is $e^0 \otimes 1$;
(2) The fields $Y(e^0 \otimes v_{-1}, z)$ are given by (19);
(3) We have

$$[z^{Q(\alpha,\beta)}]Y(e^{\alpha},z)e^{\beta} = (-1)^{q(\alpha,\beta)}e^{\alpha+\beta}\,. \tag{20}$$

Then $V_\bullet$ is isomorphic to the lattice vertex algebra of Theorem 3.5.

Proof. By [Kac, Proposition 5.4], conditions (1) and (2) imply that $V_\bullet$ is uniquely determined by a choice of operators $c_\alpha \colon V_\bullet \to V_\bullet$ for each $\alpha \in \Lambda_{\mathrm{sst}}$ satisfying

$$c_0 = \mathrm{id}\,, \quad c_\alpha\,|0\rangle = |0\rangle\,, \quad [v_{(k)}, c_\alpha] = 0 \text{ for } v \in \Lambda, k \in \mathbb{Z}.$$

For such a choice of operators, the field $Y(e^\alpha, z)$ is given by

$$Y(e^\alpha, z) = z^{Q(\alpha,\beta)} e^\alpha \exp\Big[- \sum_{k<0} \frac{\alpha_{(k)}}{k} z^{-k}\Big] \exp\Big[- \sum_{k>0} \frac{\alpha_{(k)}}{k} z^{-k}\Big] c_\alpha$$

To show that $V_\bullet$ is the lattice vertex algebra we need to show that c_α acts as $(-1)^{q(\alpha,\beta)}$ on $e^\beta \otimes \mathbb{D}_\Lambda$. Since c_α commute with creation operators, it is enough to show that $c_\alpha(e^\beta) = (-1)^{q(\alpha,\beta)} e^\beta$ for all $\beta \in \Lambda_{\mathrm{sst}}$. We have

$$\begin{aligned} Y(e^\alpha, z)e^\beta &= z^{Q(\alpha,\beta)} e^\alpha \exp\Big[- \sum_{k<0} \frac{\alpha_{(k)}}{k} z^{-k}\Big] \exp\Big[- \sum_{k>0} \frac{\alpha_{(k)}}{k} z^{-k}\Big] c_\alpha(e^\beta) \\ &= z^{Q(\alpha,\beta)} e^\alpha \exp\Big[- \sum_{k<0} \frac{\alpha_{(k)}}{k} z^{-k}\Big] c_\alpha \exp\Big[- \sum_{k>0} \frac{\alpha_{(k)}}{k} z^{-k}\Big] e^\beta \\ &= z^{Q(\alpha,\beta)} e^\alpha \exp\Big[- \sum_{k<0} \frac{\alpha_{(k)}}{k} z^{-k}\Big] c_\alpha e^\beta \end{aligned}$$

Extracting the coefficient of $z^{Q(\alpha,\beta)}$ from both sides and using (3) the result follows. $\square$

3.3. **Kac's conformal element.** Let $V_\bullet = \mathbb{C}[\Lambda_{\mathrm{sst}}] \otimes \mathbb{D}_\Lambda$ be a lattice vertex algebra associated to a non-degenerate symmetric pairing

$$Q : \Lambda \times \Lambda \longrightarrow \mathbb{Z}.$$

Recall that $V_\bullet$ is expressed as a tensor product $V_\bullet = V_{\bar{0},\bullet} \otimes V_{\bar{1},\bullet}$. This allows us to address the construction of the conformal element as

$$\omega = \omega_{\bar{0}} + \omega_{\bar{1}}, \quad \omega_{\bar{i}} \in V_{\bar{i},4}.$$

Let us first fix a basis $B_{\bar{0}}$ of $\Lambda_{\bar{0}}$ and its dual $\hat{B}_{\bar{0}}$ with respect to Q; we denote by $\hat{v} \in \hat{B}_{\bar{0}}$ the dual of $v \in B_{\bar{0}}$, so that

$$Q(v, \hat{w}) = \delta_{v,w} \quad \text{for } v, w \in B_{\bar{0}}\,.$$

Then the bosonic part has a natural choice of a conformal element given by

$$\omega_{\bar{0}} = e^0 \otimes \frac{1}{2} \sum_{v \in B_{\bar{0}}} \hat{v}_{-1} v_{-1} \in V_{\bar{0},4}\,.$$

See [Kac, Proposition 5.5]. The central charge of $\omega_{\bar{0}}$ is $\dim(\Lambda_{\bar{0}})$.

We now consider the fermionic part. Recall that we have in the assumptions of Theorem 3.5 a splitting

$$\Lambda_{\bar{1}} = I \oplus \hat{I}\,. \tag{21}$$

into maximal isotropic subspaces. Given such a splitting, Kac [Kac, 3.6.14] constructs a family of conformal elements $\omega^\lambda_{\bar{1}}$ parameterized by $\lambda \in \mathbb{C}$. To give its explicit description, pick a basis $B_I \subset I$; then its dual basis is denoted by $B_{\hat{I}} \subset \hat{I}$ with elements $\hat{w}$ satisfying $Q(v, \hat{w}) = \delta_{v,w}$. One then sets

$$\omega^\lambda_{\bar{1}} = (1-\lambda) \sum_{v \in B_I} \hat{v}_{-2} v_{-1} + \lambda \sum_{v \in B_I} v_{-2} \hat{v}_{-1}\,.$$

Notice that the expression is independent of a choice of bases $B_I, B_{\hat{I}}$ and swapping I and $\hat{I}$ only interchanges λ and $(1-\lambda)$. We set $\lambda = 0$ and denote[10]

$$\omega_{\bar{1}} = \sum_{v\in B_I} \hat{v}_{-2}v_{-1}\,.$$

The central charge of $\omega_{\bar{1}}$ is

$$2\,\mathrm{sdim}(I) = \mathrm{sdim}(\Lambda_{\bar{1}}) = -\dim(\Lambda_{\bar{1}})\,,$$

by [Kac, 3.6.16] after plugging $\lambda = 0$. Therefore, the central charge of the full conformal element $\omega = \omega_{\bar{0}} + \omega_{\bar{1}}$ is given by

$$\dim(\Lambda_{\bar{0}}) - \dim(\Lambda_{\bar{1}}) = \mathrm{sdim}(\Lambda)\,.$$

Remark 3.8. The choice of the conformal element ω is equivalent to the choice of a splitting (21), so we will often denote the latter piece of data also by ω.

The corresponding conformal grading on $e^\alpha \otimes \mathbb{D}_\Lambda \subset V_\bullet$, depending on the choice of the splitting ω, is determined by the operator

$$\begin{aligned}
L_0 &= [z^{-2}]\big\{Y(\omega, z)\big\}\\
&= [z^{-2}]\Big\{\frac{1}{2}\sum_{v\in B_{\bar{0}}} :Y(\hat{v}_{-1}, z)Y(v_{-1}, z): + \sum_{v\in B_I} :\frac{\partial}{\partial z}Y(\hat{v}_{-1}, z)Y(v_{-1}, z):\Big\}\\
&= \frac{1}{2}\sum_{\substack{k\in\mathbb{Z}\\ v\in B_{\bar{0}}}} :\hat{v}_{(-k)}v_{(k)}: + \sum_{\substack{k\in\mathbb{Z}\\ v\in B_I}} k :\hat{v}_{(-k-1)}v_{(k)}:\\
&= \sum_{\substack{k>0\\ v\in B_{\bar{0}}}} k\, v_{(-k)}\frac{\partial}{\partial v_{-k}} + \sum_{\substack{k>0\\ v\in B_I}} \Big[(k-1)\,\hat{v}_{(-k)}\frac{\partial}{\partial \hat{v}_{-k}} + k\, v_{(-k)}\frac{\partial}{\partial v_{-k}}\Big] + \frac{1}{2}Q(\alpha,\alpha)\,.
\end{aligned}$$

The last equality is obtained by separating the sum into $k > 0$ or $k < 0$ terms and using (19) in the form

$$v_{(k-1)} = \frac{\partial}{\partial \hat{v}_{-k}}\,, \quad \hat{v}_{(k-1)} = \frac{\partial}{\partial v_{-k}}\,, \quad \text{for } v \in B_I,\ k > 0\,.$$

Specifically, the induced conformal grading is

$$\deg_\omega(v_{-k}) = \begin{cases} k & \text{if } v \in \Lambda_{\bar{0}}\,,\\ k & \text{if } v \in I \subseteq \Lambda_{\bar{1}}\,,\\ k-1 & \text{if } v \in \hat{I} \subseteq \Lambda_{\bar{1}}\,,\end{cases}$$

for each $k > 0$, while the $\mathbb{Z}$-grading on $V_\bullet$ that we started with is given by

$$\deg(v_{-k}) = \begin{cases} 2k & \text{if } v \in \Lambda_{\bar{0}}\,,\\ 2k-1 & \text{if } v \in I \subseteq \Lambda_{\bar{1}}\,,\\ 2k-1 & \text{if } v \in \hat{I} \subseteq \Lambda_{\bar{1}}\,.\end{cases}$$

[10]We suppress the dependence of $\omega_{\bar{1}}$ on the choice of I and $\hat{I}$ for simplicity of the notation.

The conformal grading $V^\omega_\bullet$ is useful when it comes to studying Virasoro operators as we explain now. Starting from the notation for the decomposition

$$v = v_I + v_{\hat{I}} \in \Lambda_{\bar{1}}\,, \quad \text{where} \quad v_I \in I, v_{\hat{I}} \in \hat{I}\,,$$

we label the conformal shift

$$v^\omega_{-1} := \begin{cases} v_{-1} & \text{if } v \in \Lambda_{\bar{0}}\,, \\ (v_I)_{-1} + (v_{\hat{I}})_{-2} & \text{if } v \in \Lambda_{\bar{1}}\,. \end{cases}$$

We also define a new pairing $Q^\omega(v,w)$ as

$$Q^\omega(v,w) := \begin{cases} Q(v,w) & \text{if } v,w \in \Lambda_{\bar{0}}\,, \\ Q(v_I, w_{\hat{I}}) - Q(v_{\hat{I}}, w_I) & \text{if } v,w \in \Lambda_{\bar{1}}\,, \\ 0 & \text{otherwise.} \end{cases}$$

This pairing is non-degenerate and *supersymmetric* because Q is non-degenerate and symmetric. Using the shift notation v^ω_{-1} and a new pairing Q^ω, we can rewrite the conformal element as

$$\begin{aligned} \omega &= \frac{1}{2}\sum_{v \in B_{\bar{0}}} \hat{v}_{-1} v_{-1} + \sum_{v \in B_I} \hat{v}_{-2} v_{-1} \\ &= \frac{1}{2}\sum_{v \in B_{\bar{0}}} \hat{v}_{-1} v_{-1} + \frac{1}{2}\sum_{v \in B_I} \hat{v}_{-2} v_{-1} - \frac{1}{2}\sum_{\hat{v} \in B_{\hat{I}}} v_{-1}\hat{v}_{-2} \\ &= \frac{1}{2}\sum_{v\in B} \tilde{v}^\omega_{-1} v^\omega_{-1}\,. \end{aligned} \tag{22}$$

In the last equality, we use the basis $B = B_{\bar{0}} \sqcup B_I \sqcup B_{\hat{I}}$ of Λ and $\{\tilde{v}\}$ denotes the dual basis to $B = \{v\}$ with respect to Q^ω, such that $Q^\omega(v, \tilde{w}) = \delta_{v,w}$. If we also shift the notation for the creation and annihilation operators by the formula

$$Y(v^\omega_{-1}, z) := \sum_{k \in \mathbb{Z}} v^\omega_{(k)}\, z^{-k-1}\,,$$

the Virasoro operators can be written as

$$L_n = \frac{1}{2}\sum_{\substack{i+j=n \\ v \in B}} :\tilde{v}^\omega_{(i)}\, v^\omega_{(j)}:\,, \quad n \in \mathbb{Z}\,. \tag{23}$$

We also note that shifted creation and annihilation operators are subject to the bracket relations that are similar to bosonic fields

$$\left[v^\omega_{(n)}, w^\omega_{(m)}\right] = n\, Q^\omega(v,w)\delta_{n+m,0}\,. \tag{24}$$

This can be shown by dividing into three cases;

$$\text{i) } v,w \in \Lambda_{\bar{0}}\,, \quad \text{ii) } v \in I,\ w \in \hat{I}\,, \quad \text{iii) } v \in \hat{I},\ w \in I\,,$$

and using the translation formula (13).

3.4. **Lie algebra of physical states.** Borcherds [Bor] associates to any vertex algebra $V_\bullet$ a graded Lie algebra

$$\check{V}_\bullet = V_{\bullet+2}\,/\,T(V_\bullet), \quad [\overline{a},\overline{b}] = \overline{a_{(0)}b}\,. \tag{25}$$

In this section, we study the Lie subalgebra of *primary* states (also known as *physical* states) that is closely related to Virasoro constraints. Algebraic statements in this section will translate into a compatibility between wall-crossing and Virasoro constraints for sheaves and pairs in the geometric setting.

Definition 3.9 (Borcherds [Bor], Kac [Kac, Cor. 4.10])**.** Let $\big(V_\bullet,\omega\big)$ be a conformal vertex algebra. The space of primary states of conformal weight $i \in \mathbb{Z}$ is defined as

$$P_i := \big\{a \in V_i^\omega \,\big|\, L_n(a) = 0 \text{ for all } n \geqslant 1\big\}\,.$$

The following assumption is used to construct Lie subalgebra of primary states.

Assumption 3.10. There is no nonzero element $a \in V_i^\omega$ with $i < 0$ and $T(a) = 0$.

This assumption is satisfied for the lattice vertex algebras with Kac's conformal element because we have

$$\ker(T) = \mathrm{span}_{\mathbb{C}}\big\{e^\alpha \otimes 1 \,\big|\, \text{torsion } \alpha\big\}\,.$$

Primary states yield a smaller Lie subalgebra by the proposition below.

Proposition 3.11 (Borcherds [Bor])**.** Let $\big(V_\bullet,\omega\big)$ be a conformal vertex algebra satisfying Assumption 3.10. Then $\check{P}_0 := P_1/T(P_0)$ defines a natural Lie subalgebra of $\check{V}_0^\omega := V_1^\omega/T(V_0^\omega)$.

Proof. We record the proof to be self-contained. For any $a \in V_\bullet$ and $n \in \mathbb{Z}$, we have

$$\begin{aligned} L_n(Ta) &= [L_n, T]a + T(L_n(a)) \\ &= (n+1)L_{n-1}(a) + T(L_n(a))\,. \end{aligned} \tag{26}$$

This implies that if $a \in P_0$ then $Ta \in P_1$, hence making sense of the quotient $\check{P}_0 = P_1/T(P_0)$. Furthermore, if $a \in V_0^\omega$ with $Ta \in P_1$ then (26) implies $a \in P_0$ by the induction argument and Assumption 3.10. This shows that $P_1 \subset V_1^\omega$ induces a subspace $\check{P}_0 \subset \check{V}_0^\omega$. Finally, this defines a Lie subalgebra because if $a, b \in P_1$ then

$$L_n(a_{(0)}b) = [L_n, a_{(0)}]b + u_{(0)}(L_n(b)) = 0\,, \quad n \geqslant 1$$

hence $a_{(0)}b \in P_1$. Here we used the fact that if $a \in P_1$ then the operator $a_{(0)}$ commutes with any Virasoro operators L_n [Bor, Section 5]. $\square$

We give an alternative way to define a Lie subalgebra of primary states that is related to the weight 0 Virasoro operator. For this new definition, we need a partial lift of the Borcherds' Lie bracket.

Lemma 3.12. There is a well-defined linear map [11]

$$[-,-]\colon \check{V}_i \times V_j \to V_{i+j}, \quad (\overline{a}, b) \mapsto a_{(0)}b\,, \tag{27}$$

which makes $V_\bullet$ a representation of a graded Lie algebra $\check{V}_\bullet$.

Proof. It suffices to check the factoring property of $a_{(0)}b$ in the first coordinate. This follows from (13) because $(Ta)_{(0)} = 0$. ☐

Proposition 3.13. Let $\big(V_\bullet, \omega\big)$ be a conformal vertex algebra satisfying Assumption 3.10. Then the linear map in Lemma 3.12 restricts to

$$[-,-]\colon \check{P}_0 \times P_i \to P_i$$

which makes P_i a subrepresentation of V_i^ω with respect to the Lie algebra $\check{P}_0$.

Proof. The proof is similar to that of Proposition 3.11 relying on the fact that if $a \in P_1$ then the operator $a_{(0)}$ commutes with any Virasoro operators L_n. ☐

Definition 3.14. Let $\big(V_\bullet, \omega\big)$ be a conformal vertex algebra. We define a Lie subalgebra

$$\check{K}_0 := \big\{\overline{a} \in \check{V}_0^\omega \,\big|\, [\overline{a}, \omega] = 0\big\} \subset \Big(\check{V}_0^\omega\,,\, [-,-]\Big)\,.$$

In the definition above, the operator $[-,\omega]$ is a linear map defined in Lemma 3.12. The fact that $\check{K}_0$ defines a Lie subalgebra follows from the representation property in Lemma 3.12. Connection of $\check{K}_0$ to the weight 0 Virasoro operator introduced in Definition 2.11 is explained below.

Lemma 3.15. Let $\big(V_\bullet, \omega\big)$ be a conformal vertex algebra. Then we have

$$[-,\omega] = \sum_{n \geqslant -1} \frac{(-1)^n}{(n+1)!} T^{n+1} \circ L_n : \check{V}_0^\omega \to V_2^\omega\,.$$

Proof. By (12), we have

$$[\overline{a}, \omega] = a_{(0)}\omega = \sum_{n \geqslant -1} \frac{(-1)^n}{(n+1)!} T^{n+1} \circ L_n(a)\,. \square$$

From Lemma 3.15, it is clear that $\check{P}_0 \subseteq \check{K}_0$. The next lemma states the converse when working with lattice vertex algebras.

Proposition 3.16. Let $(V_\bullet, \omega)$ be a lattice vertex operator algebra as in Theorem 3.5 with $V_\bullet = \mathbb{C}[\Lambda_{\mathrm{sst}}] \otimes \mathbb{D}_\Lambda$. Then we have $\check{P}_0 = \check{K}_0$ on the components $e^\alpha \otimes \mathbb{D}_\Lambda$ such that α is not torsion.

Proof. Let $\overline{a}$ be an element admitting a lift $a \in e^\alpha \otimes \mathbb{D}_\Lambda$ for some non torsion α. Suppose that $\overline{a} \in \check{K}_0$, i.e. $a_{(0)}\omega = 0$. We show that $\overline{a} \in \check{P}_0$ by constructing another lift of $\overline{a}$, not necessarily the same as a, that lies in P_1. Since the pairing Q is non-degenerate, there exists some $b \in \Lambda_{\overline{0}}$ such that $Q(\alpha, b) = 1$. In the proof of this

[11] We abuse the Lie bracket notation that was originally used for $[-,-] : \check{V}_i \times \check{V}_j \to \check{V}_{i+j}$.

lemma, we denote $e^0 \otimes b_{-1} \subset V_1^\omega$ simply by b; note that the creation/annihilation operators associated to $b \in \Lambda_{\overline{0}}$ and the fields induced by $b = e^0 \otimes b_{-1} \in V_\bullet$ are the same by definition, so the symbol $b_{(k)}$ is unambiguous. We claim that such b provides a desired lift of $\overline{a}$ defined as

$$\eta_b(\overline{a}) := -a_{(0)}b \in V_1^\omega .$$

Since (25) defines a Lie bracket, we know that

$$\eta_b(\overline{a}) \in b_{(0)}a + T(V_\bullet) .$$

Recall that the operator $b_{(0)}$ acts on $e^\alpha \otimes \mathbb{D}_\Lambda$ as multiplication by $Q(\alpha, b) = 1$. Therefore $\eta_b(\overline{a})$ is indeed another lift of $\overline{a}$.

To show that $\eta_b(\overline{a}) \in V_1^\omega$ is a primary state in P_1, we must show

$$\omega_{(n+1)}(a_{(0)}b) = 0 \quad \text{for all} \quad n \geqslant 1.$$

This follows from the assumption $a_{(0)}\omega = 0$ and the identity (12)

$$\omega_{(n+1)}(a_{(0)}b) = a_{(0)}(\omega_{(n+1)}b) - (a_{(0)}\omega)_{(n+1)}b ,$$

together with a basic fact that $b = e^0 \otimes b_{-1} \in P_1$ [Kac, page 81]. □

4. VOA from sheaf theory

The general treatment of vertex algebras in the previous section is now paralleled by their geometric construction formulated by Joyce [Joy1]. Beginning from the application to the moduli stack of pairs of perfect complexes, we compare the resulting vertex algebras to lattice vertex algebras following the work of Gross [Gro]. The later parts of the section are focused on describing the duals of the operators L_k as Virasoro operators for a natural conformal element.

4.1. **Joyce's vertex algebra construction.** We begin by describing the assumptions needed for the geometric construction of the vertex algebra for perfect complexes following Joyce's [Joy1]. We then follow it up with how to extend it to pairs of complexes which is the most natural setting to work in for conformal elements and the rank reduction arguments later on.

We use the notation

$$\pi_J \colon \prod_{i\in I} Z_i \longrightarrow \prod_{j\in J} Z_j$$

for projections to components whenever $J \subset I$ are finite sets, and denote for a K-theory class $\mathcal{K}$ on $\prod_{j\in J}\mathcal{M}_j$ the pullback by $\mathcal{K}_J = \pi_J^*(\mathcal{K})$.

Definition 4.1. (1) We work with a (higher) moduli stack $\mathcal{M}_X$ of perfect complexes on X constructed by Toën–Vaquié [TV] which admits a universal perfect complex $\mathcal{G}$ on $\mathcal{M}_X \times X$. The two structures we are interested in are the direct sum map $\Sigma\colon \mathcal{M}_X \times \mathcal{M}_X \to \mathcal{M}_X$, such that

$$(\Sigma \times \mathrm{id}_X)^*\mathcal{G} = \mathcal{G}_{1,3} \oplus \mathcal{G}_{2,3} ,$$

and an action $\rho\colon B\mathbb{G}_m \times \mathcal{M}_X \to \mathcal{M}_X$ determined by

$$(\rho \times \mathrm{id}_X)^*(\mathcal{G}) = \mathcal{Q}_1 \otimes \mathcal{G}_{2,3}$$

for the universal line bundle $\mathcal{Q}$ on $B\mathbb{G}_m$.

(2) The second major ingredient in constructing vertex algebras is a class

$$\mathrm{Ext} = (\pi_{1,2})_*\Big(\mathcal{G}_{1,3}^\vee \otimes \mathcal{G}_{2,3}\Big) \tag{28}$$

on $\mathcal{M}_X \times \mathcal{M}_X$. Denoting by $\sigma\colon \mathcal{M}_X \times \mathcal{M}_X \to \mathcal{M}_X \times \mathcal{M}_X$ the map swapping the factors we construct its symmetrization

$$\Theta = \mathrm{Ext}^\vee \oplus \sigma^*\mathrm{Ext}$$

which satisfies $\sigma^*\Theta \cong \Theta^\vee$.

(3) For any topological type $\alpha \in K^0_{\mathrm{sst}}(X) \simeq \pi_0(\mathcal{M}_X)$, we denote the corresponding connected component by $\mathcal{M}_\alpha \subseteq \mathcal{M}_X$. Any restriction of an object living on $\mathcal{M}_X$ to $\mathcal{M}_\alpha$ will be labeled by adjoining the subscript $(-)_\alpha$. This allows us to express

$$\chi_{\mathrm{sym}}(\alpha,\beta) := \mathrm{rk}\big(\Theta_{\alpha,\beta}\big) = \chi(\alpha,\beta) + \chi(\beta,\alpha)$$

for $\chi\colon K^0_{\mathrm{sst}}(X) \times K^0_{\mathrm{sst}}(X) \to \mathbb{Z}$ the usual Euler form.

By the construction of Blanc [Bla, §3.4], there exists a topological space $\mathcal{S}^{\mathrm{t}}$ defined up to homotopy which is assigned to each (higher) stack $\mathcal{S}$. The homology and cohomology of $\mathcal{S}$ are then defined by

$$H_\bullet(\mathcal{S}) := H_\bullet(\mathcal{S}^{\mathrm{t}}), \qquad H^\bullet(\mathcal{S}) := H^\bullet(\mathcal{S}^{\mathrm{t}}).$$

Each perfect complex $\mathcal{E}$ on $\mathcal{S}$ corresponds to a map $\mathcal{S} \xrightarrow{\mathcal{E}} \mathrm{Perf}_{\mathbb{C}}$ which induces $\mathcal{S}^{\mathrm{t}} \xrightarrow{\mathcal{E}^{\mathrm{t}}} BU \times \mathbb{Z}$ and the K-theory class

$$[\![\mathcal{E}]\!] = (\mathcal{E}^{\mathrm{t}})^*(\mathcal{U}) \in K^0(\mathcal{S}^{\mathrm{t}}).$$

The Chern classes of $\mathcal{E}$ are defined to be the Chern classes of $[\![\mathcal{E}]\!]$.

Remark 4.2. We will only work with geometries where the natural map

$$K^i_{\mathrm{sst}}(X,\mathbb{Z}) \longrightarrow K^i(X,\mathbb{Z})$$

for the semi-topological K-theory $K^i_{\mathrm{sst}}(X,\mathbb{Z})$ of Blanc [Bla, §4.1] is an isomorphism for all $i > 0$ and an injection when $i = 0$. Gross [Gro] calls the class of such varieties class D; it includes curves, surfaces, rational 3-folds and rational 4-folds. From [Bla, Thm. 4.21] it follows that for such varieties

$$\pi_i(\mathcal{M}_X^{\mathrm{t}}) = K^i_{\mathrm{sst}}(X,\mathbb{Z}) = \begin{cases} K^i(X,\mathbb{Z}) & \text{for } i > 0 \\ K^0_{\mathrm{sst}}(X) & \text{for } i = 0\,. \end{cases}$$

The vertex algebra on the shifted homology

$$V_\bullet = \bigoplus_{\alpha \in K^0_{\mathrm{sst}}(X)} \widehat{H}_\bullet(\mathcal{M}_\alpha) \tag{29}$$

for the shift

$$\widehat{H}_\bullet\big(\mathcal{M}_\alpha\big) = H_{\bullet - 2\chi(\alpha,\alpha)}\big(\mathcal{M}_\alpha, \mathbb{C}\big)$$

was constructed by [Joy1] as follows:

Theorem 4.3 ([Joy1, Thm. 3.12])**.** There is a vertex algebra structure on $V_\bullet$ from (29) defined by

(1) letting $0\colon * \to \mathcal{M}_0$ be the inclusion of the zero object and setting

$$|0\rangle = 0_*(*) \in H_0(\mathcal{M}_0)\,.$$

(2) taking $t \in H_2\big(B\mathbb{G}_m\big)$ to be the dual of $c_1(\mathcal{Q}) \in H^2\big(B\mathbb{G}_m\big)$ and setting

$$T(u) = \rho_*\big(t \boxtimes u\big)\,.$$

(3) constructing the state-field correspondence by the formula

$$Y(v,z)u = (-1)^{\chi(\alpha,\beta)} z^{\chi_{\mathrm{sym}}(\alpha,\beta)} \Sigma_*\Big[e^{zT} \boxtimes \mathrm{id}\big(c_{z^{-1}}(\Theta) \cap (u \boxtimes v)\big)\Big] \tag{30}$$

for any $u \in \widehat{H}_\bullet(\mathcal{M}_\alpha)$ and $v \in \widehat{H}_\bullet(\mathcal{M}_\beta)$.

While moduli spaces of sheaves (recall Section 1.2) naturally define an element in the vertex algebra $V_\bullet$ (or the associated Lie algebra $\check{V}_\bullet$), this vertex algebra is not suitable to study wall-crossing in moduli spaces of pairs. Essentially this is due to the fact that the complex Ext in Definition 4.1 captures the deformation theory of sheaves but not of pairs.

Thus, when working with pairs we will work with a larger vertex algebra $V_\bullet^{\mathrm{pa}}$ that is constructed from the homology of a stack parametrizing a pair of complexes and replaces the class Ext by $\mathrm{Ext}^{\mathrm{pa}}$, more related to the deformation theory of pairs. It also turns out that the vertex algebra $V_\bullet^{\mathrm{pa}}$ is the adequate place to construct a conformal element that produces the Virasoro operators.

Definition 4.4. (1) Let $\mathcal{P}_X := \mathcal{M}_X \times \mathcal{M}_X$ and denote by $\mathcal{V} = \mathcal{G}_{1,3}$ and $\mathcal{F} = \mathcal{G}_{2,3}$ the pullbacks of $\mathcal{G}$ via the two possible projections $\mathcal{M}_X \times \mathcal{M}_X \times X \to \mathcal{M}_X \times X$. This stack has a natural direct sum map Σ^{pa} and a $B\mathbb{G}_m$-action ρ^{pa}

$$\begin{aligned}\Sigma^{\mathrm{pa}} &:= \big(\Sigma \times \Sigma\big) \circ \sigma_{2,3} : \mathcal{P}_X \times \mathcal{P}_X \longrightarrow \mathcal{P}_X\,,\\ \rho^{\mathrm{pa}} &:= \big(\rho \times \rho\big) \circ \big(\Delta_{B\mathbb{G}_m} \times \mathrm{id}_{\mathcal{P}_X}\big) : B\mathbb{G}_m \times \mathcal{P}_X \longrightarrow \mathcal{P}_X\,.\end{aligned}$$

where $\sigma_{2,3}$ swaps the second and third copy of $\mathcal{M}_X$ in $\mathcal{P}_X \times \mathcal{P}_X = \mathcal{M}_X^{\times 4}$.

(2) One extends Ext to a perfect complex

$$\mathrm{Ext}^{\mathrm{pa}} = (\pi_{1,2})_*\Big((\mathcal{F}_{1,3} - \mathcal{V}_{1,3})^\vee \otimes \mathcal{F}_{2,3}\Big) \tag{31}$$

in $\mathcal{P}_X \times \mathcal{P}_X$. We introduce its symmetrization

$$\Theta^{\mathrm{pa}} = (\mathrm{Ext}^{\mathrm{pa}})^\vee \oplus (\sigma^{\mathrm{pa}})^* \mathrm{Ext}^{\mathrm{pa}}\,,$$

where $\sigma^{\mathrm{pa}} : \mathcal{P}_X \times \mathcal{P}_X \to \mathcal{P}_X \times \mathcal{P}_X$ swaps the two factors.

(3) We now work with two copies of the K-theory with the connected components labeled by $\mathcal{P}_{\alpha^{\mathrm{pa}}}$ whenever $\alpha^{\mathrm{pa}} = (\alpha_1, \alpha_2) \in K^0_{\mathrm{sst}}(X)^{\oplus 2}$ and continue to denote by the subscript $(-)_{\alpha^{\mathrm{pa}}}$ the restriction of some object to each connected component. We define the pairing

$$\chi^{\mathrm{pa}}\big(\alpha^{\mathrm{pa}}, \beta^{\mathrm{pa}}\big) := \chi(\alpha_2 - \alpha_1, \beta_2) = \mathrm{rk}\big(\mathrm{Ext}^{\mathrm{pa}}_{\alpha^{\mathrm{pa}},\beta^{\mathrm{pa}}}\big)\,.$$

and its symmetrization

$$\chi^{\mathrm{pa}}_{\mathrm{sym}}\big(\alpha^{\mathrm{pa}}, \beta^{\mathrm{pa}}\big) := \chi\big(\alpha^{\mathrm{pa}}, \beta^{\mathrm{pa}}\big) + \chi\big(\beta^{\mathrm{pa}}, \alpha^{\mathrm{pa}}\big) = \mathrm{rk}\big(\Theta^{\mathrm{pa}}_{\alpha^{\mathrm{pa}},\beta^{\mathrm{pa}}}\big)\,.$$

The pair vertex algebra has the underlying graded vector space

$$V^{\mathrm{pa}}_\bullet = \hat{H}_\bullet(\mathcal{P}_X) = \bigoplus_{\alpha^{\mathrm{pa}}} \hat{H}_\bullet\big(\mathcal{P}_{\alpha^{\mathrm{pa}}}\big)\,,$$

where $\widehat{H}_\bullet\big(\mathcal{P}_{\alpha^{\mathrm{pa}}}\big) = H_{\bullet - 2\chi^{\mathrm{pa}}(\alpha^{\mathrm{pa}},\alpha^{\mathrm{pa}})}\big(\mathcal{P}_{\alpha^{\mathrm{pa}}}\big)$. The structure of a vertex algebra is constructed exactly as in Theorem 4.3. In particular, the state-to-field correspondence is given by

$$Y(v,z)u = (-1)^{\chi^{\mathrm{pa}}(\alpha^{\mathrm{pa}},\beta^{\mathrm{pa}})} z^{\chi^{\mathrm{pa}}_{\mathrm{sym}}(\alpha^{\mathrm{pa}},\beta^{\mathrm{pa}})} \Sigma^{\mathrm{pa}}_*\Big[e^{zT} \otimes \mathrm{id}\big(c_{z^{-1}}(\Theta^{\mathrm{pa}}) \cap u \boxtimes v\big)\Big] \tag{32}$$

for $u \in \widehat{H}_\bullet\big(\mathcal{P}_{\alpha^{\mathrm{pa}}}\big)$, $v \in \widehat{H}_\bullet\big(\mathcal{P}_{\beta^{\mathrm{pa}}}\big)$. Note that the inclusion of $\mathcal{M}_X \hookrightarrow \mathcal{P}_X$ sending $F \mapsto (0, F)$ realizes $V_\bullet$ as a vertex subalgebra of $V^{\mathrm{pa}}_\bullet$.

Remark 4.5. In Joyce's theory [Joy6, Section 8], wall-crossing for pairs plays an important role in the definition of the invariant classes $[M]^{\mathrm{inv}}$. However, the vertex algebra that he uses to formulate such wall-crossing formulas is not $V^{\mathrm{pa}}_\bullet$. For our purposes, it will be enough to consider the stack $\mathcal{N}_X$ parametrizing a vector space U and a sheaf F together with a morphism $U \otimes \mathcal{O}_X \to F$ (cf. [Joy6, Definition 8.2] with $L = \mathcal{O}_X$ or [Boj2, Definition 2.12] with $n = 0$). The stack $\mathcal{N}_X$ maps to $\mathcal{P}_X$ by sending

$$(U \otimes \mathcal{O}_X \to F) \mapsto (U \otimes \mathcal{O}_X, F).$$

This map induces $H_\bullet(\mathcal{N}_X) \to H_\bullet(\mathcal{P}_X)$ on homology; Joyce defines a vertex algebra structure on $\mathcal{N}_X$ that makes this map a vertex algebra homomorphism. Hence, the wall-crossing formulas for moduli of pairs proven with Joyce's theory in $\check{H}_\bullet(\mathcal{N}_X)$ will also hold in $\check{V}^{\mathrm{pa}}$.

4.2. **Joyce's vertex algebra as a lattice vertex algebra.** We will now give an explicit description of $V_\bullet$ and $V^{\mathrm{pa}}_\bullet$ as lattice vertex algebras (see Theorem 3.5) following Gross' work [Gro].

Let Λ be the $\mathbb{Z}_2$-graded vector space

$$\Lambda = \Lambda_{\bar{0}} \oplus \Lambda_{\bar{1}} = K^0(X) \oplus K^1(X) = K^\bullet(X).$$

Let also $\Lambda_{\mathrm{sst}} = K^0_{\mathrm{sst}}(X)$. Next we need to describe the bilinear forms Q, q on Λ extending the natural Euler pairing on $\Lambda_{\mathrm{sst}} = K^0_{\mathrm{sst}}(X)$. Recall that we have a natural

Chern character isomorphism $K^\bullet(X) \cong H^\bullet(X)$. Define the dual $(-)^\vee : K^\bullet(X) \to K^\bullet(X)$ by identifying with $H^\bullet(X)$ and setting

$$\mathrm{ch}(v^\vee) = (-1)^{\lfloor \frac{\deg \mathrm{ch}(v)}{2} \rfloor}\mathrm{ch}(v)\,,$$

where $\deg \mathrm{ch}(v)$ is the cohomological degree. This leads to

$$(v \cdot w)^\vee = (-1)^{|v||w|} v^\vee \cdot w^\vee\,. \tag{33}$$

We define the extension of the Euler pairing to $K^\bullet(X)$ as

$$\chi(v,w) = \int_X \mathrm{ch}(v^\vee)\cdot \mathrm{ch}(w)\cdot \mathrm{td}(X)\,, \qquad v,w \in K^\bullet(X)\,.$$

We will denote its symmetrization by χ_{sym}:

$$\chi_{\mathrm{sym}}(v,w) = \chi(v,w) + \chi(w,v)\,.$$

For the pair version, we let

$$\Lambda^{\mathrm{pa}}_{\bar{i}} = \Lambda^{\oplus 2}_{\bar{i}}\,, \quad \Lambda^{\mathrm{pa}} = \Lambda^{\oplus 2} = \Lambda^{\mathrm{pa}}_{\bar{0}} \oplus \Lambda^{\mathrm{pa}}_{\bar{1}}\,, \quad \Lambda^{\mathrm{pa}}_{\mathrm{sst}} = \Lambda^{\oplus 2}_{\mathrm{sst}}.$$

Given $v \in \Lambda$ we will denote by $v^{\mathcal{V}} = (v,0)$, $v^{\mathcal{F}} = (0,v) \in \Lambda^{\mathrm{pa}}$ the corresponding elements in the first and second copies of Λ, respectively. Given two elements

$$v^{\mathrm{pa}} = (v_1,v_2) = v_1^{\mathcal{V}} + v_2^{\mathcal{F}}\,, \quad w^{\mathrm{pa}} = (w_1,w_2) = w_1^{\mathcal{V}} + w_2^{\mathcal{F}}$$

their pairing is defined as

$$\chi^{\mathrm{pa}}\big(v^{\mathrm{pa}}, w^{\mathrm{pa}}\big) = \chi(v_2 - v_1, w_2)\,, \tag{34}$$

and as usual its symmetrization is

$$\chi^{\mathrm{pa}}_{\mathrm{sym}}\big(v^{\mathrm{pa}}, w^{\mathrm{pa}}\big) = \chi^{\mathrm{pa}}\big(v^{\mathrm{pa}}, w^{\mathrm{pa}}\big) + \chi^{\mathrm{pa}}\big(w^{\mathrm{pa}}, v^{\mathrm{pa}}\big)\,. \tag{35}$$

Note that the forms χ^{pa} and $\chi^{\mathrm{pa}}_{\mathrm{sym}}$ extend the ones in Definition 4.4.(3).

Lemma 4.6. The form $\chi^{\mathrm{pa}}_{\mathrm{sym}}$ is non-degenerate.

Proof. Using the decomposition $\Lambda^{\mathrm{pa}} = \Lambda \oplus \Lambda$ the symmetric form $\chi^{\mathrm{pa}}_{\mathrm{sym}}$ can be represented by the block matrix

$$\chi^{\mathrm{pa}}_{\mathrm{sym}} = \begin{bmatrix} 0 & -\chi \\ -\chi & \chi_{\mathrm{sym}} \end{bmatrix}.$$

Since clearly χ is non-degenerate it follows that $\chi^{\mathrm{pa}}_{\mathrm{sym}}$ is non-degenerate as well. □

The following is a necessary modification of [Gro, Thm. 5.7] which we write out in full detail to avoid imprecisions and to include the analog statement for the pair vertex algebra. We point out that while all the ideas already appear in loc. cit. the computation of the odd degree fields is not present there and their description has a degree shift to the correct one. Furthermore, unlike the Definition of the generalized lattice vertex algebra in [Gro], we do not rely on the construction of Abe [Abe].

Theorem 4.7. Let X be a variety in class D (cf. Remark 4.2). Then we have isomorphisms of vertex algebras

$$V_\bullet \cong \mathbb{C}\big[\Lambda_{\mathrm{sst}}\big] \otimes \mathbb{D}_\Lambda\,, \tag{36}$$

$$V_\bullet^{\mathrm{pa}} \cong \mathbb{C}\big[\Lambda_{\mathrm{sst}}^{\mathrm{pa}}\big] \otimes \mathbb{D}_{\Lambda^{\mathrm{pa}}}\,, \tag{37}$$

where: the left hand sides are the vertex algebras from Joyce's geometrical construction with the data of Definitions 4.1 and 4.4, respectively; the right hand sides are the lattice vertex algebras from Theorem 3.5 and the symmetric bilinear forms $q = \chi, Q = \chi_{\mathrm{sym}}$ and $q = \chi^{\mathrm{pa}}, Q = \chi^{\mathrm{pa}}_{\mathrm{sym}}$, respectively.

We begin the proof of the Theorem by explaining how to identify both sides as graded vector spaces. Using the universal sheaf $\mathcal{G}_\alpha$ on $X \times \mathcal{M}_\alpha$ we have for each $\alpha \in K^0_{\mathrm{sst}}(X)$ a geometric realization morphism (3)

$$\xi_{\mathcal{G}_\alpha} : \mathbb{D}^X_\alpha \to H^\bullet(\mathcal{M}_\alpha).$$

Lemma 4.8 (Theorem 4.15 in [Gro])**.** Let X be a variety in class D. Then the map $\xi_{\mathcal{G}_\alpha}$ is an isomorphism,

$$H^\bullet(\mathcal{M}_\alpha) \cong \mathbb{D}^X_\alpha.$$

Similarly,

$$H^\bullet(\mathcal{P}_{\alpha^{\mathrm{pa}}}) \cong \mathbb{D}^{X,\mathrm{pa}}_{\alpha^{\mathrm{pa}}} := \mathbb{D}^X_{\alpha_1} \otimes \mathbb{D}^X_{\alpha_2}.$$

Proof. Gross shows that $H^\bullet(\mathcal{M}_\alpha)$ is freely generated by the Kunneth components of $\mathrm{ch}_k(\mathcal{G}_\alpha) \in H^\bullet(\mathcal{M}_\alpha \times X)$. But these are precisely the geometric realization of descendents, see (44). The result for pairs follows from the sheaf version. ☐

We know from the previous lemma that

$$H^\bullet(\mathcal{M}_\alpha) \cong \mathbb{D}^X_\alpha = \mathrm{SSym}[\![\mathrm{CH}^X_\alpha]\!]\,,$$

and we define the pairing $\langle -, - \rangle \colon \mathrm{CH}^X_\alpha \times \mathrm{CH}_\Lambda \longrightarrow \mathbb{C}$ given by

$$\Big\langle \mathrm{ch}_k(\gamma), v_{-j} \Big\rangle = \int_X \gamma \cdot \mathrm{ch}(v) \frac{\delta_{k,j}}{(k-1)!}\,. \tag{38}$$

The reader can recall the definition of CH^X_α and CH_Λ in Definition 2.5 and Theorem 3.5, respectively. The pairing above is a perfect pairing, so it identifies the dual of the graded vector space CH^X_α with CH_Λ. Recalling the discussion from Section 2.1, we then get an identification between

$$H_\bullet(\mathcal{M}_\alpha) = H^\bullet(\mathcal{M}_\alpha)^\dagger = (\mathbb{D}^X_\alpha)^\dagger = \mathrm{SSym}[\![\mathrm{CH}^X_\alpha]\!]^\dagger = \mathrm{SSym}[\mathrm{CH}_\Lambda] = \mathbb{D}_\Lambda\,.$$

Moreover, by Definition 2.2 the pairing between $\mathbb{D}^X_\alpha$ and $\mathbb{D}_\Lambda$ can be promoted to a cap product

$$\cap : \mathbb{D}^X_\alpha \times \mathbb{D}_\Lambda \to \mathbb{D}_\Lambda$$

such that for a basis B of $K^\bullet(X)$, we have

$$\mathrm{ch}_k(\gamma) \cap (-) = \frac{1}{(k-1)!} \sum_{w \in B} \int_X \gamma \cdot \mathrm{ch}(w) \frac{\partial}{\partial w_{-k}}\,. \tag{39}$$

The isomorphisms $\mathbb{D}_\Lambda \cong H_\bullet(\mathcal{M}_\alpha)$ and $\mathbb{D}_\alpha^X \cong H^\bullet(\mathcal{M}_\alpha)$ identify this abstract cap product with the topological cap product, i.e.

$$\begin{tikzcd} \mathbb{D}_\alpha^X \times \mathbb{D}_\Lambda \arrow[r, "\cap"] \arrow[d, "\xi_{\mathcal{G}_\alpha} \times \xi^\dagger_{\mathcal{G}_\alpha}"] & \mathbb{D}_\Lambda \arrow[d, "\xi^\dagger_{\mathcal{G}_\alpha}"] \\ H^\bullet(\mathcal{M}_\alpha) \times H_\bullet(\mathcal{M}_\alpha) \arrow[r, "\cap"] & H_\bullet(\mathcal{M}_\alpha) \end{tikzcd} \tag{40}$$

commutes.

By assembling all the isomorphisms of graded vector spaces $H_\bullet(\mathcal{M}_\alpha) \cong \mathbb{D}_\Lambda$ we get

$$H_\bullet(\mathcal{M}_X) = \bigoplus_{\alpha\in K^0_{\mathrm{sst}}(X)} H_\bullet(\mathcal{M}_\alpha) \cong \bigoplus_{\alpha\in\Lambda_{\mathrm{sst}}} \mathbb{D}_\Lambda = \mathbb{C}[\Lambda_{\mathrm{sst}}] \otimes \mathbb{D}_\Lambda \tag{41}$$

and similarly

$$H_\bullet(\mathcal{P}_X) \cong \mathbb{C}[\Lambda^{\mathrm{pa}}_{\mathrm{sst}}] \otimes \mathbb{D}_{\Lambda^{\mathrm{pa}}}. \tag{42}$$

To prove Theorem 4.7 it remains to show that the vertex algebra that Joyce defined on the left hand side and the lattice vertex algebra on the right hand side are compatible under this isomorphism. We will do this using Proposition 3.7.

Before we analyze the fields required to show that the two vertex algebra structures are compatible, it will be useful to identify the translation operators on both sides. Recall Definition 2.9 of the operator $\mathsf{R}_{-1} : \mathbb{D}_\alpha^X \to \mathbb{D}_\alpha^X$.

Lemma 4.9. Under the identification of $\mathbb{D}_\alpha^X$ with $H^\bullet(\mathcal{M}_\alpha) \cong H_\bullet(\mathcal{M}_\alpha)^\dagger$, the translation operator $T\colon H_\bullet(\mathcal{M}_\alpha) \to H_{\bullet+2}(\mathcal{M}_\alpha)$ (defined in Theorem 4.3) is dual to R_{-1}. Moreover, the isomorphisms (41), (42) preserve the respective translation operators on both sides.

Proof. We recall the operator

$$\mathsf{E} = e^{\zeta \mathsf{R}_{-1}}$$

from Lemma 2.8. We claim that the following diagram commutes:

$$\begin{tikzcd} \mathbb{D}_\alpha^X \arrow[r, "\mathsf{E}"] \arrow[d, "\xi_{\mathcal{G}_\alpha}"] \arrow[dr, "\xi_{\mathcal{Q}\boxtimes\mathcal{G}_\alpha}"] & \mathbb{D}_\alpha^X[\![\zeta]\!] \arrow[dr, "\xi_{\mathcal{G}_\alpha}"] & \\ H^\bullet(\mathcal{M}_\alpha) \arrow[r, "\rho^*"] & H^\bullet(B\mathbb{G}_m \times \mathcal{M}_\alpha) \arrow[r, "\sim"] & H^\bullet(\mathcal{M}_\alpha)[\![\zeta]\!] \end{tikzcd}$$

The triangle on the left commutes because $\rho^*(\mathcal{G}) = \mathcal{Q} \boxtimes \mathcal{G}$. The parallelogram on the right commutes by Lemma 2.8. It follows that, under the identification $\mathbb{D}_\alpha^X \cong H^\bullet(\mathcal{M}_\alpha)$, $\mathsf{E} = \rho^*$ and thus $\mathsf{R}_{-1} = (t\backslash -) \circ \rho^*$ where the $(t\backslash -)$ stands for the slant product with the generator $t \in H_2(B\mathbb{G}_m)$ dual to $\zeta = c_1(\mathcal{Q}) \in H^2(B\mathbb{G}_m)$. This is clearly the dual to Joyce's translation operator defined by $T = \rho_* \circ (t \boxtimes -)$.

To show that Joyce's translation operator agrees with the one constructed in the lattice vertex algebra it is enough to check equations (14). This is straightforward with the identification of T with the dual of R_{-1}. $\square$

Before we reproduce the proof of Theorem 4.7, we introduce some notation. From now on we will omit the isomorphism $\xi_{\mathcal{G}_\alpha}$ and simply write $\mathrm{ch}_i(\gamma) \in H^\bullet(\mathcal{M}_\alpha)$. We also set $\mathrm{ch}_0(\gamma)$ to be $\int_X \gamma \cdot \mathrm{ch}(\alpha) \in H^0(\mathcal{M}_\alpha)$. We will use the notation $\mathrm{ch}_i^{\mathcal{V}}(\gamma) = \mathrm{ch}_i(\gamma) \otimes 1$ and $\mathrm{ch}_i^{\mathcal{F}}(\gamma) = 1 \otimes \mathrm{ch}_i(\gamma)$ for the generators of $\mathbb{D}^{X,\mathrm{pa}}_{\alpha^{\mathrm{pa}}} = \mathbb{D}^X_{\alpha_1} \otimes \mathbb{D}^X_{\alpha_2}$, as we did in Section 2.6. We also use the same notation for the images in $H^\bullet(\mathcal{P}_{\alpha^{\mathrm{pa}}})$. Given $\gamma \in H^\bullet(X)$ we let $\gamma^{\mathcal{V}} = (\gamma, 0), \gamma^{\mathcal{F}} = (0, \gamma) \in H^\bullet(X)^{\oplus 2}$. Given $\gamma^{\mathrm{pa}} \in H^\bullet(X)^{\oplus 2}$ we introduce the symbol $\mathrm{ch}_i(\gamma^{\mathrm{pa}}) \in \mathbb{D}^{X,\mathrm{pa}}_{\alpha^{\mathrm{pa}}}$ so that $\mathrm{ch}_i(\gamma^{\mathcal{V}}) = \mathrm{ch}_i^{\mathcal{V}}(\gamma)$, $\mathrm{ch}_i(\gamma^{\mathcal{F}}) = \mathrm{ch}_i^{\mathcal{F}}(\gamma)$. Given $\gamma^{\mathrm{pa}} \in H^\bullet(X)^{\oplus 2}$ and $w^{\mathrm{pa}} \in K^\bullet(X)^{\oplus 2}$ we define the pairing

$$\langle \gamma^{\mathrm{pa}}, w^{\mathrm{pa}} \rangle = \int_X \gamma_1 \cdot \mathrm{ch}(w_1) + \int_X \gamma_2 \cdot \mathrm{ch}(w_2).$$

If $\{w^{\mathrm{pa}}\}$ is a basis of $K^\bullet(X)^{\oplus 2}$ we then have the analogue of (39) for pairs:

$$\mathrm{ch}_k(\gamma^{\mathrm{pa}}) \cap (-) = \frac{1}{(k-1)!} \sum_{w^{\mathrm{pa}}} \langle \gamma^{\mathrm{pa}}, w^{\mathrm{pa}} \rangle \frac{\partial}{\partial w^{\mathrm{pa}}_{-k}} . \tag{43}$$

Proof of Theorem 4.7. We will only give the proof of the statement for pairs, since the sheaf version is easier and a direct consequence. Since we have already identified the underlying vector spaces, by Proposition 3.7 it is enough to check conditions (1)-(3) for the vertex algebra defined by Joyce. The vacuum condition (1) is immediate. Before we compute the fields necessary in (2), (3) we will obtain a formula for the Chern classes of Θ^{pa}. Fix a basis $\{\gamma\} \subseteq H^\bullet(X)$ and let $\{\overline{\gamma}\} \subseteq H^\bullet(X)$ be the dual basis, such that $\int_X \gamma_1 \cdot \overline{\gamma_2} = \delta_{\gamma_1, \gamma_2}$. Then the class of the diagonal in $X \times X$ is

$$\Delta_*(1) = \sum_\gamma \overline{\gamma} \otimes \gamma \in H^\bullet(X \times X) ,$$

where the sum is over the basis we fixed. We then have

$$\mathrm{ch}(\mathcal{F}) = (\pi_{1,2})_*\big((\mathrm{id}_{\mathcal{P}_X} \times \Delta)_*(\mathrm{ch}(\mathcal{F})\big) = \sum_{i \geqslant 0} \mathrm{ch}_i^{\mathcal{F}}(\overline{\gamma}) \boxtimes \gamma \tag{44}$$

and an analogous formula for $\mathcal{V}$. The term $\mathrm{ch}_i^{\mathcal{F}}(\overline{\gamma}) \otimes \gamma$ contributes to $\mathrm{ch}_k(\mathcal{F})$ for

$$k = \frac{2i - |\overline{\gamma}| + \deg \gamma}{2} = i + \left\lfloor \frac{\deg(\gamma)}{2} \right\rfloor .$$

Thus,

$$\mathrm{ch}(\mathcal{F}^\vee) = \sum_\gamma \sum_{i \geqslant 0} (-1)^{i + \lfloor \frac{\deg \gamma}{2} \rfloor} \mathrm{ch}_i^{\mathcal{F}}(\overline{\gamma}) \otimes \gamma .$$

Using Grothendieck-Riemann-Roch as in [Gro, Proposition 5.2] and being careful with signs obtained from commuting odd variables we find that

$$
\begin{aligned}
\mathrm{ch}(\mathrm{Ext}^{\mathrm{pa}}) &= \sum_{\gamma,\delta}\sum_{i,j\geqslant 0}(-1)^{i+|\gamma|}\chi(\gamma,\delta)\mathrm{ch}_i^{\mathcal{F}-\mathcal{V}}(\overline{\gamma})\boxtimes\mathrm{ch}_j^{\mathcal{F}}(\overline{\delta})\\
&= \sum_{\gamma^{\mathrm{pa}},\delta^{\mathrm{pa}}}\sum_{i,j\geqslant 0}(-1)^{i+|\gamma^{\mathrm{pa}}|}\chi^{\mathrm{pa}}(\gamma^{\mathrm{pa}},\delta^{\mathrm{pa}})\mathrm{ch}_i(\overline{\gamma}^{\mathrm{pa}})\boxtimes\mathrm{ch}_j(\overline{\delta}^{\mathrm{pa}})\in H^\bullet(\mathcal{P}_X\times\mathcal{P}_X)\,. \qquad (45)
\end{aligned}
$$

In the first line, we sum over γ,δ in our prescribed basis. In the second line, we sum over the basis

$$\{\gamma^{\mathrm{pa}}\}=\{\gamma^{\mathcal{V}}\}\cup\{\gamma^{\mathcal{F}}\}\subseteq H^\bullet(X)^{\oplus 2}$$

and use the definition of χ^{pa}. The pairings χ,χ^{pa} coincide with the ones for K-theory previously described under the Chern character isomorphism, i.e.

$$\chi(\gamma,\delta)=\int_X(-1)^{\lfloor\frac{\deg\gamma}{2}\rfloor}\gamma\cdot\delta\cdot\mathrm{td}(X).$$

Note that all the non-zero terms in (45) have $|\gamma|=|\delta|=|\overline{\gamma}|=|\overline{\delta}|$, so we replace the occurrence of any of those parities by $|\gamma|$. From (45) (and being again careful with the signs introduced by taking the dual and by $(\sigma^{\mathrm{pa}})^*$) we get the Chern character of the symmetrization

$$\mathrm{ch}(\Theta^{\mathrm{pa}})=\sum_{\gamma^{\mathrm{pa}},\delta^{\mathrm{pa}}}\sum_{i,j\geqslant 0}(-1)^j\chi^{\mathrm{pa}}_{\mathrm{sym}}(\gamma^{\mathrm{pa}},\delta^{\mathrm{pa}})\mathrm{ch}_i(\overline{\gamma}^{\mathrm{pa}})\boxtimes\mathrm{ch}_j(\overline{\delta}^{\mathrm{pa}})\,. \qquad (46)$$

By Newton's identities we have

$$
\begin{aligned}
c_{z^{-1}}(\Theta^{\mathrm{pa}}) &= \exp\Big[\sum_{k\geqslant 0}(-1)^k k!\,\mathrm{ch}_{k+1}(\Theta^{\mathrm{pa}})z^{-k-1}\Big] \qquad (47)\\
&= \exp\Big[\sum_{\gamma^{\mathrm{pa}},\delta^{\mathrm{pa}}}\sum_{i+j\geqslant|\gamma^{\mathrm{pa}}|+1}(-1)^{i-|\gamma^{\mathrm{pa}}|-1}(i+j-|\gamma^{\mathrm{pa}}|-1)!\,z^{-i-j+|\gamma^{\mathrm{pa}}|}\\
&\qquad\qquad \chi^{\mathrm{pa}}_{\mathrm{sym}}(\gamma^{\mathrm{pa}},\delta^{\mathrm{pa}})\mathrm{ch}_i(\overline{\gamma}^{\mathrm{pa}})\boxtimes\mathrm{ch}_j(\overline{\delta}^{\mathrm{pa}})\Big]\,.
\end{aligned}
$$

It suffices to consider the expansion of this exponential up to the linear terms in

$$Y(v^{\mathrm{pa}}_{-1},z)=\Sigma^{\mathrm{pa}}_*\big[e^{zT}\boxtimes 1\big(c_{z^{-1}}(\Theta^{\mathrm{pa}})\cap(v^{\mathrm{pa}}_{-1}\boxtimes -)\big)\big]\,,$$

because quadratic terms and beyond annihilate $(v^{\mathrm{pa}}_{-1}\boxtimes -)$ for degree reasons. The constant term of the exponential (47), namely 1, determines creation part of the field $Y(v^{\mathrm{pa}}_{-1},z)$ as

$$\Sigma^{\mathrm{pa}}_*\big[e^{zT}v^{\mathrm{pa}}_{-1}\boxtimes -\big]=\sum_{k\geqslant 0}v^{\mathrm{pa}}_{(-k-1)}z^k\,. \qquad (48)$$

This uses the fact that

$$\frac{T^k}{k!}v^{\mathrm{pa}}_{-1}=v^{\mathrm{pa}}_{-k-1}\,,\quad \Sigma^{\mathrm{pa}}_*\big(v^{\mathrm{pa}}_{-k-1}\boxtimes -\big)=v^{\mathrm{pa}}_{(-k-1)}$$

for $k\geqslant 0$, see [Gro, Lemmas 5.3, 5.5] (the first one also follows from Lemma 4.9). On the other hand, it suffices to consider the linear terms of (47) with $i=1$ and

$j-|\gamma^{\mathrm{pa}}|=k\geqslant 0$ for degree reasons. They determine the annihilation part of the field $Y(v^{\mathrm{pa}}_{-1},z)$ as

$$
\begin{aligned}
\Sigma^{\mathrm{pa}}_*\Big[\sum_{\gamma^{\mathrm{pa}},\delta^{\mathrm{pa}}}\sum_{k\geqslant 0}&(-1)^{|\gamma^{\mathrm{pa}}|}k!z^{-k-1}\chi^{\mathrm{pa}}_{\mathrm{sym}}(\gamma^{\mathrm{pa}},\delta^{\mathrm{pa}})\mathrm{ch}_1(\overline{\gamma}^{\mathrm{pa}})\boxtimes\mathrm{ch}_{k+|\gamma^{\mathrm{pa}}|}(\overline{\delta}^{\mathrm{pa}})\cap(v^{\mathrm{pa}}_{-1}\boxtimes -)\Big]\\
&=\sum_{\gamma^{\mathrm{pa}},\delta^{\mathrm{pa}}}\sum_{k\geqslant 0}k!z^{-k-1}\chi^{\mathrm{pa}}_{\mathrm{sym}}(\gamma^{\mathrm{pa}},\delta^{\mathrm{pa}})\langle\overline{\gamma}^{\mathrm{pa}},v^{\mathrm{pa}}\rangle\,\mathrm{ch}_{k+|\gamma^{\mathrm{pa}}|}(\overline{\delta}^{\mathrm{pa}})\cap\\
&=\sum_{\delta^{\mathrm{pa}}}\sum_{k\geqslant 0}k!z^{-k-1}(-1)^{|\delta^{\mathrm{pa}}|}\chi^{\mathrm{pa}}_{\mathrm{sym}}(v^{\mathrm{pa}},\delta^{\mathrm{pa}})\,\mathrm{ch}_{k+|\delta^{\mathrm{pa}}|}(\overline{\delta}^{\mathrm{pa}})\cap\ . \qquad (49)
\end{aligned}
$$

The sign disappears in the second line due to the interaction between the cap product and the tensor product [Dol, 12.17] and reappears in the last line by using

$$(-1)^{|\delta^{\mathrm{pa}}|}\sum_{\gamma^{\mathrm{pa}}}\langle\overline{\gamma}^{\mathrm{pa}},v^{\mathrm{pa}}\rangle\chi^{\mathrm{pa}}_{\mathrm{sym}}(\gamma^{\mathrm{pa}},\delta^{\mathrm{pa}})=\sum_{\gamma^{\mathrm{pa}}}\langle v^{\mathrm{pa}},\overline{\gamma}^{\mathrm{pa}}\rangle\chi^{\mathrm{pa}}_{\mathrm{sym}}(\gamma^{\mathrm{pa}},\delta^{\mathrm{pa}})=\chi^{\mathrm{pa}}_{\mathrm{sym}}(v^{\mathrm{pa}},\delta^{\mathrm{pa}})\,.$$

We can further simplify this expression by replacing descendent actions with derivatives. Let $\{w^{\mathrm{pa}}\}\subseteq K^\bullet(X)^{\oplus 2}=\Lambda^{\mathrm{pa}}$ be the basis obtained by applying the inverse of the Chern character isomorphism to $\{\delta^{\mathrm{pa}}\}$. By (43),

$$\mathrm{ch}_j(\overline{\delta}^{\mathrm{pa}})\cap - =\frac{(-1)^{|\delta^{\mathrm{pa}}|}}{(j-1)!}\frac{\partial}{\partial w^{\mathrm{pa}}_{-j}}\,,\quad \text{for } j\geqslant 1\,.$$

The constant descendent action by $\mathrm{ch}_0(\overline{\delta}^{\mathrm{pa}})$ is treated separately as a multiplication by $\langle\overline{\delta}^{\mathrm{pa}},\alpha^{\mathrm{pa}}\rangle$ on $H_\bullet(\mathcal{P}_{\alpha^{\mathrm{pa}}})$. Therefore, we have

$$(49)=\sum_{k\geqslant 1-|v^{\mathrm{pa}}|}\sum_{w^{\mathrm{pa}}}\chi^{\mathrm{pa}}_{\mathrm{sym}}(v^{\mathrm{pa}},w^{\mathrm{pa}})k^{1-|v^{\mathrm{pa}}|}\frac{\partial}{\partial w^{\mathrm{pa}}_{-k-|v^{\mathrm{pa}}|}}z^{-k-1}+\chi^{\mathrm{pa}}(v^{\mathrm{pa}},\alpha^{\mathrm{pa}})z^{-1}$$

on $H_\bullet(\mathcal{P}_{\alpha^{\mathrm{pa}}})$. Combining with the creation part (48), this matches exactly with (19), so we are done with (2).

For (3) we are left with computing $Y(e^{\alpha^{\mathrm{pa}}},z)e^{\beta^{\mathrm{pa}}}$:

$$
\begin{aligned}
Y(e^{\alpha^{\mathrm{pa}}},z)e^{\beta^{\mathrm{pa}}}&=(-1)^{\chi^{\mathrm{pa}}(\alpha^{\mathrm{pa}},\beta^{\mathrm{pa}})}z^{\chi^{\mathrm{pa}}_{\mathrm{sym}}(\alpha^{\mathrm{pa}},\beta^{\mathrm{pa}})}\Sigma_*\big[e^{zT}\boxtimes\mathrm{id}(c_{z^{-1}}(\Theta^{\mathrm{pa}})\cap(e^{\alpha^{\mathrm{pa}}}\boxtimes e^{\beta^{\mathrm{pa}}}))\big]\\
&=(-1)^{\chi^{\mathrm{pa}}(\alpha^{\mathrm{pa}},\beta^{\mathrm{pa}})}z^{\chi^{\mathrm{pa}}_{\mathrm{sym}}(\alpha^{\mathrm{pa}},\beta^{\mathrm{pa}})}\Sigma_*\big[e^{zT}(e^{\alpha^{\mathrm{pa}}})\boxtimes e^{\beta^{\mathrm{pa}}}\big]\,,
\end{aligned}
$$

where we used the fact that $\mathrm{ch}_i(\overline{\gamma}^{\mathrm{pa}})\cap -$ and $\mathrm{ch}_j(\overline{\delta}^{\mathrm{pa}})\cap -$ annihilate $e^{\alpha^{\mathrm{pa}}},e^{\beta^{\mathrm{pa}}}$ for any $i,j>0$. The $z^{\chi^{\mathrm{pa}}_{\mathrm{sym}}(\alpha^{\mathrm{pa}},\beta^{\mathrm{pa}})}$ coefficient is simply

$$(-1)^{\chi^{\mathrm{pa}}(\alpha^{\mathrm{pa}},\beta^{\mathrm{pa}})}e^{\alpha^{\mathrm{pa}}+\beta^{\mathrm{pa}}}\,,$$

as required in (3), thus finishing the proof. □

Recall from Section 3.4 that associated to a vertex algebra $V_\bullet$ we have a Lie algebra $\check{V}_\bullet=V_{\bullet+2}/TV_\bullet$. This quotient of $V_\bullet$ also has a geometric interpretation observed by [Joy1] and related to $\mathbb{D}^X_{\mathrm{wt}_0,\alpha}$ from Definition 2.9. We begin by recalling that the $B\mathbb{G}_m$ action ρ on $\mathcal{M}_X$ leads to a *rigidification* (see [AOV, §5]) which quotients out $\mathbb{G}_m \cdot \mathrm{id}$ from the stabilizer of each object in the moduli stack. We denote it by

$$\mathcal{M}_\alpha^{\mathrm{rig}} = \mathcal{M}_\alpha /\!\!/ B\mathbb{G}_m\,.$$

Lemma 4.10 ([Joy1])**.** Let X be a variety in class D. Let $\mathrm{ch}(\alpha) \neq 0$ and $\pi_\alpha^{\mathrm{rig}}$: $\mathcal{M}_\alpha \to \mathcal{M}_\alpha^{\mathrm{rig}}$ be the projection, then $(\pi_\alpha^{\mathrm{rig}})^* : H^\bullet(\mathcal{M}_\alpha^{\mathrm{rig}}) \to H^\bullet(\mathcal{M}_\alpha)$ is injective and the isomorphism of Lemma 4.8 induces

$$\mathbb{D}^X_{\mathrm{wt}_0,\alpha} \cong (\pi_\alpha^{\mathrm{rig}})^*\big(H^\bullet(\mathcal{M}_\alpha^{\mathrm{rig}})\big) \cong H^\bullet(\mathcal{M}_\alpha^{\mathrm{rig}})\,.$$

Equivalently, we have the isomorphism

$$\check{H}_{\bullet+2}(\mathcal{M}_\alpha^{\mathrm{rig}}) := H_{\bullet-2\chi(\alpha,\alpha)+2}(\mathcal{M}_\alpha^{\mathrm{rig}}) \cong \check{V}_{\alpha,\bullet}\,.$$

Proof. A big part of this proof is already due to Joyce [Joy1, Proposition 3.24], [Joy6, Theorem 4.8, Remark 4.10]. He proves that

$$H_\bullet(\mathcal{M}_\alpha^{\mathrm{rig}}) = H_\bullet(\mathcal{M}_\alpha)/T(H_{\bullet-2}(\mathcal{M}_\alpha))\,,$$

when $\mathrm{ch}(\alpha) \neq 0$. This is the last statement of the lemma. Dually, we have injectivity of the pull-back and

$$(\pi_\alpha^{\mathrm{rig}})^*\big(H^\bullet(\mathcal{M}_\alpha^{\mathrm{rig}})\big) = \{D \in H^\bullet(\mathcal{M}_\alpha)\colon \rho^* D = 1 \boxtimes D \text{ in } H^\bullet(B\mathbb{G}_m \times \mathcal{M}_\alpha)\}.$$

The isomorphism with $\mathbb{D}^X_{\mathrm{wt}_0,\alpha}$ finally follows from the the fact that $\rho^* = e^{\zeta \mathsf{R}_{-1}}$ under the identification $H^\bullet(\mathcal{M}_\alpha) \cong \mathbb{D}^X_\alpha$ as we showed in the proof of Lemma 4.9. $\square$

Example 4.11. We continue with the Example 2.10 proving it differently for the full stack. We define the virtual tangent bundle $T^{\mathrm{vir}}\mathcal{M}_\alpha := -\mathrm{RHom}_{\mathcal{M}_\alpha}(\mathcal{G},\mathcal{G})$ satisfying in K-theory

$$T^{\mathrm{vir}}\mathcal{M}_\alpha = (\pi_\alpha^{\mathrm{rig}})^*(T^{\mathrm{vir}}\mathcal{M}_\alpha^{\mathrm{rig}}) - \mathcal{O}_{\mathcal{M}_\alpha}\,,$$

where $T^{\mathrm{vir}}\mathcal{M}_\alpha^{\mathrm{rig}}$ is the virtual tangent bundle of $\mathcal{M}_\alpha^{\mathrm{rig}}$. We see from Lemma 4.10 that $\mathrm{ch}(T^{\mathrm{vir}}) \in \mathbb{D}^X_{\mathrm{wt}_0,\alpha}$. This clearly holds for any K-theory class pulled back from $\mathcal{M}_\alpha^{\mathrm{rig}}$.

4.3. **Virasoro operators from Kac's conformal element.** Construction of conformal element depends on Assumption 1.7 which we assume throughout this section. Recall that we need to make a choice of a maximal isotropic decomposition of $K^1(X)^{\oplus 2}$ with respect to a symmetric non-degenerate pairing $\chi^{\mathrm{pa}}_{\mathrm{sym}}$ (recall Lemma 4.6) to define Kac's conformal element. Hodge decomposition is a source of the decomposition that we use. Define the subspaces $K^{\bullet,\bullet+1}(X)$ and $K^{\bullet+1,\bullet}(X)$ of $K^1(X)$ such that via the chern character isomorphism we have

$$K^{\bullet,\bullet+1}(X) \xrightarrow{\sim} \bigoplus_p H^{p,p+1}(X), \quad K^{\bullet+1,\bullet}(X) \xrightarrow{\sim} \bigoplus_p H^{p+1,p}(X)\,.$$

We consider a decomposition of $K^1(X)^{\oplus 2} = I \oplus \hat{I}$ given by

$$I := K^{\bullet,\bullet+1}(X)^{\oplus 2}, \quad \hat{I} := K^{\bullet+1,\bullet}(X)^{\oplus 2}\,, \tag{50}$$

that is maximally isotropic due to Hodge degree consideration. This defines a conformal element ω inside $V_\bullet^{\mathrm{pa}} = \mathbb{C}[\Lambda_{\mathrm{sst}}^{\mathrm{pa}}]\otimes\mathbb{D}_{\Lambda^{\mathrm{pa}}}$ hence the vertex Virasoro operators L_n^{pa} for all $n \in \mathbb{Z}$. On the other hand, we defined in Section 2.6 descendent Virasoro operators $\mathsf{L}_n^{\mathrm{pa}}$ for $n \geqslant -1$ acting on the formal pair descendent algebra $\mathbb{D}^{X,\mathrm{pa}}$.

In this section, we prove duality between the two Virasoro operators L_n^{pa} and $\mathsf{L}_n^{\mathrm{pa}}$ for $n \geqslant -1$. To set up the stage where we state the duality, let $\alpha^{\mathrm{pa}} \in \Lambda_{\mathrm{sst}}^{\mathrm{pa}}$ and consider the realization homomorphism

$$p_{\alpha^{\mathrm{pa}}} : \mathbb{D}^{X,\mathrm{pa}} \to \mathbb{D}^{X,\mathrm{pa}}_{\alpha^{\mathrm{pa}}}\,,$$

as in Definition 2.5. By Assumption 1.7, this realization homomorphism can be checked to be surjective. Furthermore, descendent Virasoro operators $\mathsf{L}_n^{\mathrm{pa}}$ factor through this quotient and define the operators on $\mathbb{D}^{X,\mathrm{pa}}_{\alpha^{\mathrm{pa}}}$ for which we use same notation.

Theorem 4.12. For any $\alpha^{\mathrm{pa}} \in \Lambda_{\mathrm{sst}}^{\mathrm{pa}}$ and $n \geqslant -1$, Virasoro operators L_n^{pa} and $\mathsf{L}_n^{\mathrm{pa}}$ are dual to each other with respect to a perfect pairing

$$\mathbb{D}^X_{\alpha^{\mathrm{pa}}} \otimes \mathbb{D}_{\Lambda^{\mathrm{pa}}} \to \mathbb{C}\,.$$

Proof. On $K^\bullet(X)^{\oplus 2}$ we are given a symmetric non-degenerate pairing

$$\begin{aligned}\chi_{\mathrm{sym}}^{\mathrm{pa}}\big(v^{\mathrm{pa}}, w^{\mathrm{pa}}\big) &= \chi^{\mathrm{pa}}(v^{\mathrm{pa}}, w^{\mathrm{pa}}) + \chi^{\mathrm{pa}}(w^{\mathrm{pa}}, v^{\mathrm{pa}})\\ &= \chi(v_2 - v_1, w_2) + \chi(w_2 - w_1, v_2)\,.\end{aligned}$$

The maximal isotropic decomposition (50) determines the conformal element $\omega \in V_\bullet^{\mathrm{pa}}$ and a new supersymmetric non-degenerate pairing (see Section 3.3) $Q^\omega = \chi_{\mathrm{ssym}}^{\mathrm{H,pa}}$. We use the superscript H instead of ω to indicate the relevance to the holomorphic degree in the Hodge decomposition. This new pairing can be written as

$$\begin{aligned}\chi_{\mathrm{ssym}}^{\mathrm{H,pa}}\big(v^{\mathrm{pa}}, w^{\mathrm{pa}}\big) &= \chi^{\mathrm{H,pa}}(v^{\mathrm{pa}}, w^{\mathrm{pa}}) + (-1)^{|v^{\mathrm{pa}}||w^{\mathrm{pa}}|}\chi^{\mathrm{H,pa}}(w^{\mathrm{pa}}, v^{\mathrm{pa}})\\ &= \chi^{\mathrm{H}}(v_2 - v_1, w_2) + (-1)^{|v^{\mathrm{pa}}||w^{\mathrm{pa}}|}\chi^{\mathrm{H}}(w_2 - w_1, v_2)\,,\end{aligned}$$

where χ^{H} is a pairing on $K^\bullet(X)$ defined as

$$\chi^{\mathrm{H}}(v, w) := (-1)^p \int \mathrm{ch}(v)\mathrm{ch}(w)\mathrm{td}(X) \quad \text{if } \mathrm{ch}(v) \in H^{p,\bullet}(X)\,.$$

The supersymmetric pairing $\chi_{\mathrm{ssym}}^{\mathrm{H,pa}}$ allows a simple formulation of the conformally shifted field:

$$Y(v_{-1}^{\mathrm{H,pa}}, z) = \sum_{k\geqslant 0} v^{\mathrm{H,pa}}_{(-k-1)} z^k + \sum_{k\geqslant 0}\sum_{\delta^{\mathrm{pa}}} k!\, z^{-k-1}\chi_{\mathrm{ssym}}^{\mathrm{H,pa}}(\delta^{\mathrm{pa}}, v^{\mathrm{pa}})\, \mathrm{ch}_k^{\mathrm{H}}(\overline{\delta}^{\mathrm{pa}}) \cap\ . \tag{51}$$

This formula is proven by the non-shifted version for $Y(v_{-1}^{\mathrm{pa}}, z)$ as in (48), together with a case division analysis as in the proof of (24).

Recall that the vertex Virasoro operator are written as

$$L_n^{\mathrm{pa}} = \frac{1}{2}\sum_{\substack{i+j=n\\ v^{\mathrm{pa}}\in B^{\mathrm{pa}}}} : \tilde{v}^{\mathrm{H,pa}}_{(i)} v^{\mathrm{H,pa}}_{(j)} : \qquad n \in \mathbb{Z}\,.$$

When $n \geqslant -1$, we write $L_n^{\mathrm{pa}} = R_n^{\mathrm{pa}} + T_n^{\mathrm{pa}}$ where

$$R_n^{\mathrm{pa}} := \frac{1}{2} \sum_{\substack{i+j=n \\ i<0 \text{ or } j<0 \\ v^{\mathrm{pa}} \in B^{\mathrm{pa}}}} : \tilde{v}_{(i)}^{\mathrm{H,pa}} v_{(j)}^{\mathrm{H,pa}} : , \quad T_n^{\mathrm{pa}} := \frac{1}{2} \sum_{\substack{i+j=n \\ i,j \geqslant 0 \\ v \in B^{\mathrm{pa}}}} : \tilde{v}_{(i)}^{\mathrm{H,pa}} v_{(j)}^{\mathrm{H,pa}} : .$$

From the computation (51), we have

$$\begin{aligned}
T_n^{\mathrm{pa}} &= \frac{1}{2} \sum_{\substack{i+j=n \\ i,j \geqslant 0}} i!j! \sum_{v^{\mathrm{pa}} \in B^{\mathrm{pa}}} \sum_{\delta^{\mathrm{pa}}, \gamma^{\mathrm{pa}}} \chi_{\mathrm{ssym}}^{\mathrm{H,pa}}(\delta^{\mathrm{pa}}, \tilde{v}^{\mathrm{pa}}) \chi_{\mathrm{ssym}}^{\mathrm{H,pa}}(\gamma^{\mathrm{pa}}, v^{\mathrm{pa}}) \, \mathrm{ch}_i^{\mathrm{H}}(\overline{\delta}^{\mathrm{pa}}) \mathrm{ch}_j^{\mathrm{H}}(\overline{\gamma}^{\mathrm{pa}}) \cap - \\
&= \frac{1}{2} \sum_{\substack{i+j=n \\ i,j \geqslant 0}} i!j! \sum_{\delta^{\mathrm{pa}}, \gamma^{\mathrm{pa}}} \left(\chi^{\mathrm{H,pa}}(\gamma^{\mathrm{pa}}, \delta^{\mathrm{pa}}) + (-1)^{|\gamma^{\mathrm{pa}}||\delta^{\mathrm{pa}}|} \chi^{\mathrm{H,pa}}(\delta^{\mathrm{pa}}, \gamma^{\mathrm{pa}}) \right) \mathrm{ch}_i^{\mathrm{H}}(\overline{\delta}^{\mathrm{pa}}) \mathrm{ch}_j^{\mathrm{H}}(\overline{\gamma}^{\mathrm{pa}}) \cap - \\
&= \sum_{\substack{i+j=n \\ i,j \geqslant 0}} i!j! \sum_{\delta^{\mathrm{pa}}, \gamma^{\mathrm{pa}}} \chi^{\mathrm{H,pa}}(\gamma^{\mathrm{pa}}, \delta^{\mathrm{pa}}) \, \mathrm{ch}_i^{\mathrm{H}}(\overline{\delta}^{\mathrm{pa}}) \mathrm{ch}_j^{\mathrm{H}}(\overline{\gamma}^{\mathrm{pa}}) \cap - \\
&= \sum_{\substack{i+j=n \\ i,j \geqslant 0}} i!j! \sum_{\delta, \gamma} \chi^{\mathrm{H}}(\gamma, \delta) \, \mathrm{ch}_i^{\mathrm{H},\mathcal{F}}(\overline{\delta}) \mathrm{ch}_j^{\mathrm{H},\mathcal{F}-\mathcal{V}}(\overline{\gamma}) \cap - \\
&= \sum_{\substack{i+j=n \\ i,j \geqslant 0}} i!j! \sum_{\delta, \gamma} (-1)^{\dim(X) - p(\overline{\gamma})} \left(\int \delta \cdot \gamma \cdot \mathrm{td}(X) \right) \mathrm{ch}_i^{\mathrm{H},\mathcal{F}-\mathcal{V}}(\overline{\gamma}) \mathrm{ch}_j^{\mathrm{H},\mathcal{F}}(\overline{\delta}) \cap - \\
&= \sum_{\substack{i+j=n \\ i,j \geqslant 0}} i!j! \sum_{t} (-1)^{\dim(X) - p_t^L} \mathrm{ch}_i^{\mathrm{H},\mathcal{F}-\mathcal{V}}(\gamma_t^L) \mathrm{ch}_j^{\mathrm{H},\mathcal{F}}(\gamma_t^R) \cap - ,
\end{aligned}$$

where in the last equality the summation takes over

$$\Delta_*(\mathrm{td}(X)) = \sum_t \gamma_t^L \otimes \gamma_t^R .$$

This is exactly dual to the multiplication operator $\mathsf{T}_n^{\mathrm{pa}}$.

Now we prove that operators R_n^{pa} and $\mathsf{R}_n^{\mathrm{pa}}$ are dual to each other. Let $R_n^{\mathrm{pa},\dagger} \colon \mathbb{D}_{\alpha^{\mathrm{pa}}}^X \to \mathbb{D}_{\alpha^{\mathrm{pa}}}^X$ be the dual to R_n^{pa}. For $n \geqslant -1$, both $R_n^{\mathrm{pa},\dagger}$ and $\mathsf{R}_n^{\mathrm{pa}}$ annihilate 1 so it is enough to show that their commutators with right multiplication by descendents agree, i.e.

$$[R_n^{\mathrm{pa},\dagger}, \cdot \mathrm{ch}_k^{\mathrm{H}}(\gamma^{\mathrm{pa}})] = [\mathsf{R}_n^{\mathrm{pa}}, \cdot \mathrm{ch}_k^{\mathrm{H}}(\gamma^{\mathrm{pa}})] = \left(\prod_{j=0}^{n} (k+j) \right) \cdot \mathrm{ch}_{k+n}^{\mathrm{H}}(\gamma^{\mathrm{pa}}) .$$

Dually, this is equivalent to

$$\left[\mathrm{ch}_k^{\mathrm{H}}(\gamma^{\mathrm{pa}}) \cap , \, R_n^{\mathrm{pa}} \right] = \left(\prod_{j=0}^{n} (k+j) \right) \mathrm{ch}_{k+n}^{\mathrm{H}}(\gamma^{\mathrm{pa}}) \cap \; .$$

We finish the proof by showing the required commutator relation. Since $\chi_{\mathrm{ssym}}^{\mathrm{H,pa}}$ is a perfect pairing, there is a unique $w^{\mathrm{pa}} \in K(X)^{\mathrm{pa}}$ such that

$$\sum_{\delta^{\mathrm{pa}}} \chi_{\mathrm{ssym}}^{\mathrm{H,pa}}(\delta^{\mathrm{pa}}, w^{\mathrm{pa}}) \, \overline{\delta}^{\mathrm{pa}} = \gamma^{\mathrm{pa}} .$$

From (51), this implies that $w^{\mathrm{H,pa}}_{(k)} = k!\mathrm{ch}^{\mathrm{H}}_k(\gamma^{\mathrm{pa}})\cap$. On the other hand, we have

$$R^{\mathrm{pa}}_n = \frac{1}{2}\sum_{\substack{i+j=n\\ i<0\\ v^{\mathrm{pa}}\in B^{\mathrm{pa}}}} \tilde{v}^{\mathrm{H,pa}}_{(i)} v^{\mathrm{H,pa}}_{(j)} + \frac{1}{2}\sum_{\substack{i+j=n\\ j<0\\ v^{\mathrm{pa}}\in B^{\mathrm{pa}}}} (-1)^{|v^{\mathrm{pa}}|} v^{\mathrm{H,pa}}_{(j)} \tilde{v}^{\mathrm{H,pa}}_{(i)} = \sum_{\substack{i+j=n\\ i<0\\ v^{\mathrm{pa}}\in B^{\mathrm{pa}}}} \tilde{v}^{\mathrm{H,pa}}_{(i)} v^{\mathrm{H,pa}}_{(j)}\,,$$

since the dual $\tilde{v}^{\mathrm{pa}}$ is defined by supersymmetric pairing $\chi^{\mathrm{H,pa}}_{\mathrm{ssym}}$. Therefore, we obtain

$$\begin{aligned}
\left[\mathrm{ch}^{\mathrm{H}}_k(\gamma^{\mathrm{pa}})\cap\,,\, R^{\mathrm{pa}}_n\right] &= \frac{1}{k!}\sum_{\substack{i+j=n\\ i<0\\ v\in B^{\mathrm{pa}}}} \left[w^{\mathrm{H,pa}}_{(k)}\,, \tilde{v}^{\mathrm{H,pa}}_{(i)} v^{\mathrm{H,pa}}_{(j)}\right] \\
&= \frac{1}{k!}\sum_{\substack{i+j=n\\ i<0\\ v\in B^{\mathrm{pa}}}} \left[w^{\mathrm{H,pa}}_{(k)}\,, \tilde{v}^{\mathrm{H,pa}}_{(i)}\right] v^{\mathrm{H,pa}}_{(j)} + \tilde{v}^{\mathrm{H,pa}}_{(i)}\left[w^{\mathrm{H,pa}}_{(k)}\,, v^{\mathrm{H,pa}}_{(j)}\right] \\
&= \frac{1}{k!}\sum_{v\in B^{\mathrm{pa}}} k\cdot \chi^{\mathrm{H,pa}}_{\mathrm{ssym}}(w^{\mathrm{pa}}, \tilde{v}^{\mathrm{pa}}) v^{\mathrm{H,pa}}_{(k+n)} \\
&= \left(\prod_{j=0}^{n}(k+j)\right)\mathrm{ch}^{\mathrm{H}}_{k+n}(\gamma^{\mathrm{pa}})\cap\,,
\end{aligned}$$

where we used the bracket formula (24) with $k \geqslant 0$. □

Remark 4.13. To summarize the ideas leading up to this final statement, we explain where the obvious similarity between T_k and the virtual tangent bundle (see Example 2.6) comes from in general. It is best represented by the diagram

$$\begin{tikzcd}
& \chi^{\mathrm{pa}}_{\mathrm{ssym}}, \chi^{\mathrm{H,pa}}_{\mathrm{ssym}} \arrow[dr, bend left, "(iii)"] & \\
\mathrm{Ext}^{\mathrm{pa}}, \Theta^{\mathrm{pa}} \arrow[ur, bend left, "(ii)"] \arrow[d, "(i)"'] & & \omega \arrow[d, "(iv)"] \\
T^{\mathrm{vir}} \arrow[rr, dashed, "?"] & & \mathsf{L}^{\mathrm{pa}}_k\,.
\end{tikzcd}$$

Meaning of the arrows are explained below.

(i) represents pullback along the diagonal which restricts $\mathrm{Ext}^{\mathrm{pa}}$ to the virtual tangent bundle of any pair moduli space mapping to $\mathcal{P}_X$. This is the content of assumption [Joy6, Ass. 4.4] and is satisfied in larger generality than our Definition 4.4.

(ii) corresponds to taking ranks of the symmetrization Θ^{pa} of $\mathrm{Ext}^{\mathrm{pa}}$ and then fixing a choice of an isotropic splitting $\Lambda^{\mathrm{pa}}_{\bar{1}} = I \oplus \hat{I}$ of the odd part of Λ^{pa} as we did in (21). Out of it, we constructed a supersymmetric pairing which in the case

$$I = K^{\bullet,\bullet+1}(X)^{\oplus 2}, \quad \hat{I} = K^{\bullet+1,\bullet}(X)^{\oplus 2}\,,$$

led to $\chi^{\mathrm{H,pa}}_{\mathrm{ssym}}$. This can be generalized to any setting where the induced pairing is non-degenerate so that isotropic splitting can be chosen.

(iii) is assigning the conformal element for a given choice of an isotropic splitting in the procedure described in Section 3.3 (more explicitly see (22)) and works as long as Joyce's construction in Section 4.1 leads to a lattice vertex algebra.

(iv) is filled in by Theorem 4.12 that fundamentally depends on the comparison between $\mathbb{D}^{X,\mathrm{pa}}$ and $H^\bullet(\mathcal{P}_X)$ contained in the description of the lattice vertex algebra structure on $\widehat{H}_\bullet(\mathcal{P}_X)$ proved in Theorem 4.7. This would again work in a setting where a similar comparison can be made.

A direct geometric relation represented by the "?" between Virasoro constraints and the virtual tangent bundle is however unclear.

4.4. **Virasoro constraints and primary states.** Let $M = M_\alpha$, for $\mathrm{ch}(\alpha) \neq 0$, be a moduli space of sheaves as in Section 1.2 with a universal sheaf $\mathbb{G}$. By the universal property of the stack $\mathcal{M}_X$ there is a map $f_{\mathbb{G}}\colon M \to \mathcal{M}_X$ such that $(f_{\mathbb{G}}\times \mathrm{id}_X)^*\mathcal{G} = \mathbb{G}$ where $\mathcal{G}$ is the universal sheaf in $\mathcal{M}_X \times X$. Even without a universal sheaf $\mathbb{G}$, we always have an open embedding into the rigidified stack $\iota\colon M \hookrightarrow \mathcal{M}^{\mathrm{rig}}_X$. When a universal sheaf exists this embedding is the composition

$$\iota\colon M \xrightarrow{f_{\mathbb{G}}} \mathcal{M}_X \longrightarrow \mathcal{M}_X^{\mathrm{rig}}\,.$$

We define classes in $\mathcal{M}_X$, $\mathcal{M}_X^{\mathrm{rig}}$ by

$$\begin{aligned}[M]^{\mathrm{vir}} &:= \iota_*[M]^{\mathrm{vir}} \in H_\bullet(\mathcal{M}_X^{\mathrm{rig}})\,,\\ [M]^{\mathrm{vir}}_{\mathbb{G}} &:= (f_{\mathbb{G}})_*[M]^{\mathrm{vir}} \in H_\bullet(\mathcal{M}_X) = V_\bullet\,.\end{aligned}$$

By Lemma 4.10 we may regard the class $[M]^{\mathrm{vir}}$ as being in Borcherds Lie algebra $\check{V}_\bullet = V_{\bullet+2}/TV_\bullet$. Given any $\mathbb{G}$ the class $[M]^{\mathrm{vir}}_{\mathbb{G}}$ is a lift of $[M]^{\mathrm{vir}} \in V_{\bullet+2}/TV_\bullet$ to the vertex algebra $V_\bullet$ – quotienting by T removes the ambiguity in the choice of $\mathbb{G}$. The integrals of geometric realizations of descendents $D \in \mathbb{D}^X_\alpha$ can be expressed in terms of these classes by

$$\int_{[M]^{\mathrm{vir}}} \xi_{\mathbb{G}}(D) = \int_{[M]^{\mathrm{vir}}} (f_{\mathbb{G}})^*\big(\xi_{\mathcal{G}}(D)\big) = \int_{[M]^{\mathrm{vir}}_{\mathbb{G}}} \xi_{\mathcal{G}}(D)\,.$$

By Theorem 4.8, if X is a variety in class D then $\xi_{\mathcal{G}}$ is an isomorphism between $\mathbb{D}^X_\alpha$ and $H^\bullet(\mathcal{M}_\alpha)$, so knowing the class $[M]^{\mathrm{vir}}_{\mathbb{G}}$ is precisely the same as knowing all the descendent integrals. Similarly, by Lemma 4.10 the class $[M]^{\mathrm{vir}} \in \check{V}_\bullet$ contains precisely the information of integrals of weight 0 descendents $D \in \mathbb{D}^X_{\alpha,\mathrm{wt}_0}$.

An analogous situation happens for moduli spaces of pairs and the stack $\mathcal{P}_X$. Given a moduli of pairs as in Section 1.2 with universal sheaf $q^*V \to \mathbb{F}$, by the universal property of the stack $\mathcal{P}_X$ we have a map

$$f_{(q^*V,\mathbb{F})}\colon P \to \mathcal{P}_X\,,$$

such that

$$(f_{(q^*V,\mathbb{F})}\times \mathrm{id}_X)^*\mathcal{V}=q^*V\,,\quad (f_{(q^*V,\mathbb{F})}\times \mathrm{id}_X)^*\mathcal{F}=\mathbb{F}\,.$$

The classes

$$[P]^{\mathrm{vir}}_{(q^*V,\mathbb{F})}\coloneqq (f_{(q^*V,\mathbb{F})})_*[P]^{\mathrm{vir}}\in V^{\mathrm{pa}}_{\bullet}$$

contain exactly the information of the descendent integrals

$$\int_{[P]^{\mathrm{vir}}}\xi_{(q^*V,\mathbb{F})}(D)\quad \text{for } D\in \mathbb{D}^{X,\mathrm{pa}}_{\alpha^{\mathrm{pa}}}.$$

A class $[M]^{\mathrm{vir}}$ coming from a moduli of sheaves can also be considered in $\check{V}^{\mathrm{pa}}_{\bullet}$ via the embedding $\mathcal{M}_X\hookrightarrow \mathcal{P}_X$ sending $G\mapsto (0,G)$.

We now use Theorem 4.12, which states the duality between the Virasoro operators on the descendent algebra and and on the vertex algebra, to prove Corollary 1.8 saying that the Virasoro constraints holding for some moduli space of sheaves M or pairs P are equivalent to their respective classes on the Lie/vertex algebra being primary states.

Proof of Corollary 1.8. We start with part (2) which refers to pairs. Under Assumption 1.7 the morphism $p_{\alpha^{\mathrm{pa}}}\colon \mathbb{D}^{X,\mathrm{pa}}\to \mathbb{D}^{X,\mathrm{pa}}_{\alpha^{\mathrm{pa}}}$ is surjective, so Conjecture 2.18 holds if and only if

$$\int_{[P]^{\mathrm{vir}}}\xi_{(q^*V,\mathbb{F})}(\mathsf{L}^{\mathrm{pa}}_n(D))=0$$

for every $D\in \mathbb{D}^{X,\mathrm{pa}}_{\alpha^{\mathrm{pa}}}$ and $n\geqslant 0$. By Theorem 4.12 and the previous observations

$$\int_{[P]^{\mathrm{vir}}}\xi_{(q^*V,\mathbb{F})}(\mathsf{L}^{\mathrm{pa}}_n(D))=\int_{[P]^{\mathrm{vir}}_{(q^*V,\mathbb{F})}}\xi_{(\mathcal{V},\mathcal{F})}(\mathsf{L}^{\mathrm{pa}}_n(D))=\int_{L_n([P]^{\mathrm{vir}}_{(q^*V,\mathbb{F})})}\xi_{(\mathcal{V},\mathcal{F})}(D)\,.$$

Since $\xi_{(\mathcal{V},\mathcal{F})}$ defines an isomorphism between $\mathbb{D}^{X,\mathrm{pa}}_{\alpha^{\mathrm{pa}}}$ and the cohomology $H^\bullet(\mathcal{P}_{\alpha^{\mathrm{pa}}})$ the last integral vanishes for all D if and only if $L_n\big([P]^{\mathrm{vir}}_{(q^*V,\mathbb{F})}\big)=0$, i.e.

$$[P]^{\mathrm{vir}}_{(q^*V,\mathbb{F})}\in P^{\mathrm{pa}}_0\,.$$

The claim (1) for sheaves follows in a similar way by noting that the operator $\mathsf{L}_{\mathrm{wt}_0}$ is dual to $[-,\omega]$ by Theorem 4.12 and Lemma 3.15 and using the equivalente characterization of primary states in $\check{P}_0$ provided by Proposition 3.16. □

Joyce defines more generally (under certain conditions) classes $[M]^{\mathrm{inv}}\in \check{V}_{\bullet}$ even when M contains strictly semistable sheaves; when it does not contain strictly semistables, this class coincides with $[M]^{\mathrm{vir}}$. We say that M satisfies the Virasoro constraints if $[M]^{\mathrm{inv}}$ is a primary state.

5. Rank reduction via wall-crossing

In this section we will explain the main step in the proof of Theorem A, which consists in a rank reduction argument via wall-crossing as described in [Joy6, §8.6] for positive rank and generalized here to include case (3). We will treat the 3 cases (1), (2), (3) in a uniform way by fitting them into the general framework of Joyce [Joy6]. His theory relies on the existence of a continuous family of stability conditions interpolating between two moduli spaces. Then their virtual fundamental classes viewed as elements of $H_\bullet(\mathcal{M}_X^{\mathrm{rig}})$ are related in terms of the Lie bracket on $\check{V}_\bullet$.

Let X be a smooth projective variety of dimension $m = 1$ or $m = 2$. Let H be a fixed polarization of X. Given $1 \leqslant d \leqslant m$, we consider the moduli spaces of d-dimensional slope[12] semistable sheaves F (cf. [HL, Theorem 4.3.3])

$$M_\alpha = M_\alpha^{\mathrm{ss}}(\mu)$$

with respect to the slope stability μ, where $\alpha \in K^0_{\mathrm{sst}}(X)$ is the topological type of the sheaves we consider. Recall that μ is defined by

$$\mu(F) = \frac{\deg(F)}{r(F)} \in \mathbb{Q} \sqcup \{+\infty\}$$

where $\deg(F), r(F)$ are (normalized) coefficients of the Hilbert polynomial $P_F(z)$, that is:

$$\deg(F) = (d-1)![z^{d-1}]P_F(z) \quad \text{ and } r(F) = d![z^d]P_F(z).$$

When $d = m$, the number $r(F)$ is (up to a multiplication by a constant) the rank of F. In general, we regard the number $r(F)$ as a generalized rank; it is a non-negative integer for every sheaf of dimension at most d. The cases (1), (2), (3) in Theorem A correspond, respectively, to $(m, d) = (1, 1), (2, 2), (2, 1)$.

Recall that twisting by $\mathcal{O}_X(H)$ induces isomorphisms

$$M_\alpha \cong M_{\alpha(H)}.$$

It is a standard fact that, given a fixed α, for large n all the μ-semistable sheaves with topological type $\alpha(nH)$ are globally generated and have vanishing higher cohomology (e.g. [HL, Corollary 1.7.7]). Replacing α by $\alpha(nH)$ we shall often assume this to be the case.

Assumption 5.1. All the μ-semistable sheaves F of topological type α are globally generated and have vanishing higher cohomology $H^{>0}(F) = 0$.

[12]The entire argument could be written in terms of Gieseker stability. Alternatively, the Virasoro constraints for moduli spaces of Gieseker semistable sheaves follows from their slope counterpart and the wall-crossing formula between slope and Gieseker stabilities.

5.1. **Bradlow pairs.** We now describe the notion of Bradlow stability, depending on a parameter $t \in \mathbb{R}_{>0}$, on pairs (F, s) where $s \colon \mathcal{O}_X \to F$ is a section.

Definition 5.2. Let $t > 0$. A pair $s \colon \mathcal{O}_X \to F$ is μ^t-(semi)stable if it is non-zero and:

(1) For every subsheaf $G \hookrightarrow F$ we have

$$\mu(G)(\leqslant)\mu(F) + \frac{t}{r(F)}.$$

(2) For every proper subsheaf $G \hookrightarrow F$ through which the section s factors

$$s \colon \mathcal{O}_X \to G \to F\,,$$

we have

$$\mu(G) + \frac{t}{r(G)}(\leqslant)\mu(F) + \frac{t}{r(F)}\,.$$

The symbol $(\leqslant)$ stands for $\leqslant$ in the semistable case and $<$ in the stable case.

The coarse moduli space parametrizing such pairs (up to S-equivalence) is a projective scheme and can be constructed via GIT as in [Tha, Section 1], [Lin]. We denote by P^t_α the moduli space of μ^t-semistable pairs $\mathcal{O}_X \to F$ with topological type $[\![F]\!] = \alpha$. These are often called moduli spaces of Bradlow pairs. If t is such that $t \notin \frac{1}{r(\alpha)!}\mathbb{Z}$ then the moduli spaces P^t_α have no strictly semistable objects. When this is the case, P^t_α is a fine moduli space with a universal pair

$$\mathcal{O}_{P^t_\alpha \times X} \to \mathbb{F}\,,$$

admitting a virtual class $[P^t_\alpha]^{\mathrm{vir}}$ by the standard argument as recorded below.

Lemma 5.3. Assume that there are no strictly μ^t-semistable pairs in P^t_α. Then the moduli space P^t_α has a natural 2-term perfect obstruction theory given by $\mathrm{RHom}([\mathcal{O}_X \to F], F)$ when $(m, d) = (1,1), (2,2), (2,1)$.

Proof. Using the long exact sequence

$$\mathrm{Ext}^i(F,F) \to H^i(F) \to \mathrm{Ext}^i([\mathcal{O}_X \to F], F) \to \mathrm{Ext}^{i+1}(F,F) \to H^{i+1}(F)\,.$$

Mochizuki [Moc, Lemma 6.1.14] (see also Joyce [Joy6, Section 8.3.2]) showed the vanishing of Ext^i for $(m,d) = (1,1), (2,2)$ and $i \neq 0, 1$. When (m,d) is such that $d \leqslant 1$, then the terms vanish immediately for $i \neq -1, 0, 1$ because $H^2(F) = 0$. The vanishing for $i = -1$ would follow from the injectivity of

$$\mathrm{Ext}^0(F,F) \xrightarrow{-\circ s} H^0(F)\,.$$

Suppose for the contradiction that there exists a non-zero morphism $\phi \in \mathrm{Ext}^0(F,F)$ such that $\phi \circ s = 0$. Consider the induced short exact sequence

$$0 \to F_1 \to F \to F_2 \to 0\,,$$

where $F_1 = \ker(\phi)$ and $F_2 = \operatorname{coker}(\phi) \subsetneq F$. By μ^t-stability of $\mathcal{O}_X \xrightarrow{s} F$, we have

$$\mu(F_1) + \frac{t}{r(F_1)} < \mu(F) + \frac{t}{r(F)}\,, \quad \mu(F_2) < \mu(F) + \frac{t}{r(F)}\,.$$

Using the usual arithmetic of ratios, this gives the contradiction. ☐

These moduli spaces fit in the framework of Section 8 of [Joy6]. There, Joyce considers the abelian category $\grave{\mathcal{A}}$ of pairs $U \otimes \mathcal{O}_X \to F$ where U is a $\mathbb{C}$-vector space and $F \in \mathrm{Coh}(X)$, and introduces the stability function on such pairs given by

$$\mu^t(U \otimes \mathcal{O}_X \to F) = \frac{\deg(F) + t\dim(V)}{r(F)}.$$

The μ^t-(semi)stable pairs with $V = \mathbb{C}$ are precisely the μ^t-(semi)stable pairs in Definition 5.2. Conditions (1) and (2) in Definition 5.2 correspond to looking for destabilizing subpairs of the form $0 \to G$ and $\mathcal{O}_X \to G$, respectively.

The stack parametrizing objects in the category $\grave{\mathcal{A}}$ is the stack $\mathcal{N}_X$ from Remark 4.5. Hence the moduli spaces P_α^t admit a map

$$P_\alpha^t \hookrightarrow \mathcal{N}_X^{\mathrm{rig}} \to \mathcal{P}_X^{\mathrm{rig}}\,,$$

and thus define a class $[P_\alpha^t]^{\mathrm{vir}}$ in the Lie algebras $H_\bullet(\mathcal{N}_X^{\mathrm{rig}})$ or $H_\bullet(\mathcal{P}_X^{\mathrm{rig}})$ by pushing forward $[P_\alpha^t]^{\mathrm{vir}} \in H_\bullet(P_\alpha^t)$ along this map. Moreover, since we have a universal pair $\mathcal{O} \to \mathbb{F}$ there is actually a lift of this map to the non-rigidified stacks

$$f_{(\mathcal{O},\mathbb{F})} \colon P_\alpha^t \to \mathcal{N}_X \to \mathcal{P}_X\,.$$

This defines a lift of the class $[P_\alpha^t]^{\mathrm{vir}}$

$$[P_\alpha^t]^{\mathrm{vir}}_{(\mathcal{O},\mathbb{F})} \coloneqq (f_{(\mathcal{O},\mathbb{F})})_*[P_\alpha^t]^{\mathrm{vir}}$$

to the vertex algebras $H_\bullet(\mathcal{N}_X)$ and $H_\bullet(\mathcal{P}_X) = V_\bullet^{\mathrm{pa}}$ when P_α^t does not contain strictly semistable pairs. This class is in the connected component $\mathcal{P}_{(\llbracket \mathcal{O}_X \rrbracket,\alpha)}$; to alleviate notation, we will write

$$\mathcal{P}_{(1,\alpha)} \coloneqq \mathcal{P}_{(\llbracket \mathcal{O}_X \rrbracket,\alpha)}\,.$$

5.2. **Limits $t \to 0$ and $t \to \infty$.** Our rank reduction argument will be based on using the wall-crossing formula to compare the μ^t Bradlow pairs with t small and t large. We now identify the moduli spaces P_α^t in these two limits.

Proposition 5.4 ([Joy6, Thm. 8.13, Ex. 5.6])**.** Let $0 < t < 1/r(\alpha)!$ and F be a sheaf of topological type α. Then $s\colon \mathcal{O}_X \to F$ is μ^t-semistable if and only if it is μ^t-stable if and only if the following three conditions hold:

(1) F is semistable with respect to μ;
(2) $s \neq 0$;
(3) there is no $0 \neq G \subsetneq F$ with $\mu(G) = \mu(F)$ such that $\mathrm{im}(s) \subseteq G$.

Since the stable objects do not change for such small t we denote by $P_\alpha^{0+} = P_\alpha^t$ the moduli of μ^t-stable pairs for $0 < t \ll 1$ and by μ^{0+} the limit stability μ^t with $t \to 0$. The limit stability can be explicitly defined by

$$\mu^{0+}(U \otimes \mathcal{O}_X \to F) = (\mu(F), \dim(U)) \in (-\infty, \infty] \times \mathbb{Z}_{\geqslant 0}$$

where $(-\infty, \infty] \times \mathbb{Z}_{\geqslant 0}$ is given the lexicographic order.

Note in particular that if M_α has no strictly semistable sheaves then condition (3) is vacuous, so P_α^{0+} parametrizes stable sheaves $[F] \in M_\alpha$ together with a non-vanishing section $s \in H^0(F)$ (up to scaling of the section). Assuming 5.1,

$$P_\alpha^{0+} = \mathbb{P}_{M_\alpha}(p_* \mathbb{G})$$

is a projective bundle over M_α with fiber $\mathbb{P}(H^0(F)) \cong \mathbb{P}^{\chi(\alpha)-1}$ over $[F] \in M_\alpha$.

When M_α has strictly semistable sheaves, P_α^{0+} plays a crucial role in Joyce's definition of the classes $[M_\alpha]^{\mathrm{inv}} \in H_\bullet(\mathcal{M}_X)$ in Section 9.1, as we will recall next. This idea is also present in Mochizuki's work [Moc].

Proposition 5.5 ([PT, Lemma 1.3])**.** *Let $t \gg 0$ be large enough. Then $s\colon \mathcal{O}_X \to F$ is μ^t-semistable if and only if it is μ^t-stable if and only if F is pure of dimension d and* coker(s) *is supported in dimension at most $d-1$.*

We denote by P_α^∞ the moduli of such pairs and by μ^∞ the limit stability. We proceed now to identify this moduli space in the three cases of interest to us, $(m, d) = (1, 1), (2, 2), (2, 1)$.

(1) Suppose that $(m, d) = (1, 1)$ and $\mathrm{rk}(\alpha) > 1$. Then $P_\alpha^\infty = \varnothing$ since

$$\mathrm{rk}(\mathrm{coker}(s)) \geqslant \mathrm{rk}(F) - \mathrm{rk}(\mathcal{O}_X) > 0$$

for any $s\colon \mathcal{O}_X \to F$ with $\mathrm{rank}(F) > 1$. Suppose now that $\mathrm{rk}(\alpha) = 1$; then the elements of P_α^∞ are non-zero pairs $s\colon \mathcal{O}_X \to F$ such that F is a torsion-free rank 1 sheaf. A torsion-free rank 1 sheaf on a curve C is necessarily a divisor. When $\mathrm{ch}(\alpha) = 1 + n \cdot \mathsf{pt}$ for $n > 0$ this implies that

$$P_\alpha^\infty = \{\mathcal{O}_X \to \mathcal{O}_X(E) \text{ such that } |E| = n\} = C^{[n]}$$

is the n-th symmetric power of C.

(2) Suppose that $(m, d) = (2, 2)$. As before, $P_\alpha^\infty = \varnothing$ whenever $\mathrm{rk}(\alpha) > 1$. Given a torsion-free rank 1 sheaf F on a surface S, we get an embedding into its double dual $F \hookrightarrow F^{\vee\vee}$, which must be a line bundle $\mathcal{O}_S(E)$ [HL, Example 1.1.16]. Twisting by $-E$ gives

$$\mathcal{O}_S(-E) \xrightarrow{s(-E)} F(-E) \lhook\joinrel\longrightarrow \mathcal{O}_S$$

so $F(-E) = I_Z$ is the ideal sheaf of a 0-dimensional subscheme $Z \subseteq E$. Thus, the moduli space P_α^∞ is isomorphic to the nested Hilbert scheme $S_\beta^{[0,n]}$

in [GSY] parametrizing a pair (E, Z) of a divisor E in class $\beta \in H_2(X, \mathbb{Z})$ and a zero dimensional subscheme $Z \subseteq E$ of length n, where

$$\mathrm{ch}(\alpha) = 1 + \beta + \frac{\beta^2}{2} - n \cdot \mathsf{pt}\,.$$

The isomorphism is given by sending

$$(E, Z) \mapsto \big(\mathcal{O}_S \to I_Z(E)\big)\,.$$

The obstruction theory, and hence virtual fundamental class, of P_α^∞ is easily seen to match the ones defined for $S_\beta^{[0,n]}$ in [GSY]. Indeed, the obstruction theory of the latter at (E, Z) (see Proposition 2.2 and the proof of Proposition 3.1 in loc. cit.) is given by

$$\begin{aligned}\mathrm{Cone}\big(&\mathrm{RHom}(\mathcal{O}_S(-E), I_Z) \to \mathrm{RHom}(I_Z, I_Z)\big)\\ &= \mathrm{RHom}(\mathcal{O}_S \to I_Z(E), I_Z(E))\,.\end{aligned}$$

(3) If $(m, d) = (2, 1)$ then P_α^∞ is shown in [GSY, Proposition 3.1.5] to also be isomorphic to the nested Hilbert scheme $S_\beta^{[0,n]}$ with $\mathrm{ch}(\alpha) = \beta - \frac{\beta^2}{2} + n \cdot \mathsf{pt}$. The isomorphism sends

$$(E, Z) \mapsto \big(\mathcal{O}_S \to \mathcal{O}_E(Z)\big)\,.$$

The virtual fundamental classes of P_α^∞ and $S_\beta^{[0,n]}$ are also shown to agree.

5.3. **Invariant classes $[M]^{\mathrm{inv}}$.** When there are strictly semistable sheaves in M_α, we cannot obtain a class in $\check{V}_\bullet$ by simply pushing forward a virtual fundamental class from $H_\bullet(M_\alpha)$. However, Joyce constructs classes $[M_\alpha]^{\mathrm{inv}}$ for every α such that $[M_\alpha]^{\mathrm{inv}} = [M_\alpha]^{\mathrm{vir}}$ when there are no strictly semistable sheaves. The classes $[M_\alpha]^{\mathrm{inv}}$ appear when one writes down wall-crossing formulas. We will now summarize – and slightly reformulate – the construction of these classes in [Joy6, Theorem 5.7, Section 9].

First, we observe that it is enough to define the classes $[M_\alpha]^{\mathrm{inv}}$ when α satisfies Assumption 5.1. The definition extends to all α by requiring that $[M_{\alpha(H)}]^{\mathrm{inv}}$ is obtained from $[M_\alpha]^{\mathrm{inv}}$ via the map $H_\bullet(\mathcal{M}_\alpha^{\mathrm{rig}}) \to H_\bullet(\mathcal{M}_{\alpha(H)}^{\mathrm{rig}})$ induced by tensoring with $\mathcal{O}_X(H)$. The argument that this definition is consistent is the core argument in Joyce's theory proved in [Joy6, Prop. 9.12].

Let $\Pi\colon \mathcal{P}_X \to \mathcal{M}_X^{\mathrm{rig}}$ be the composition of the projection $\mathcal{P}_X \to \mathcal{M}_X$ onto the second component with the rigidification map $\mathcal{M}_X \to \mathcal{M}_X^{\mathrm{rig}}$. We define the K-theory class $\mathrm{T}^{\mathrm{rel}}$ in $\mathcal{P}_X$ by

$$\mathrm{T}^{\mathrm{rel}} = Rp_*\mathcal{F} - \mathcal{O}_{\mathcal{P}_X}\,,$$

where $p\colon \mathcal{P}_X \times X \to \mathcal{P}_X$ is the projection. Then, one defines the class

$$\Upsilon_\alpha = \Pi_*\left(c_{\chi(\alpha)-1}(\mathrm{T}^{\mathrm{rel}}) \cap [P_\alpha^{0+}]^{\mathrm{vir}}_{(\mathcal{O},\mathbb{F})}\right) \in H^\bullet(\mathcal{M}_\alpha^{\mathrm{rig}}) \subseteq \check{V}_\bullet\,. \tag{52}$$

Alternatively, note that if we define Π_α as the composition

$$\Pi_\alpha\colon P_\alpha^{0+} \xrightarrow{f_{(\mathcal{O},\mathbb{F})}} \mathcal{P}_X \xrightarrow{\Pi} \mathcal{M}_X^{\mathrm{rig}}$$

then the relative tangent bundle T_{Π_α} is a vector bundle of rank $\chi(\alpha) - 1$ such that

$$\mathrm{T}_{\Pi_\alpha} = f^*_{(\mathcal{O},\mathbb{F})}\mathrm{T}^{\mathrm{rel}} .$$

Equation (52) can be rewritten as

$$\Upsilon_\alpha = (\Pi_\alpha)_* \left(c_{\mathrm{top}}(T_{\Pi_\alpha}) \cap [P^{0+}_\alpha]^{\mathrm{vir}}\right) \in \check{V}_\bullet \,, \tag{53}$$

which is the form in [Joy6, (5.29)].

Before we begin discussing wall-crossing, we set some notation. We will use $\underline{\alpha} = (\alpha_1, \cdots, \alpha_n)$ for the vector of K-theory classes and denote by $\underline{\alpha} \vdash \alpha$ the fact that it is a partition of α, i.e. $\alpha_1 + \cdots + \alpha_l = \alpha$, where l always denotes the length of α. When we write $\sum_{\underline{\alpha}\vdash\alpha}$ we mean a sum over all $\alpha_1, \ldots, \alpha_l$ such that $\alpha_0 + \ldots + \alpha_l = \alpha$.

The classes $[M_\alpha]^{\mathrm{inv}} \in \check{V}^\bullet$ are now defined by

$$\Upsilon_\alpha = \sum_{\substack{\underline{\alpha}\vdash\alpha \\ \mu(\alpha_i)=\mu(\alpha)}} \frac{(-1)^{l+1}\chi(\alpha_1)}{l!}\Big[\Big[\ldots\Big[[M_{\alpha_1}]^{\mathrm{inv}}, [M_{\alpha_2}]^{\mathrm{inv}}\Big], \ldots\Big], [M_{\alpha_l}]^{\mathrm{inv}}\Big] . \tag{54}$$

Equation (54) provides an inductive definition of the classes $[M_\alpha]^{\mathrm{inv}}$ by comparing

$$\Upsilon_\alpha = \chi(\alpha)[M_\alpha]^{\mathrm{inv}} + \ldots$$

where $\ldots$ is expressed in terms of classes $[M_{\alpha_i}]^{\mathrm{inv}}$ such that $r(\alpha_i) < r(\alpha)$. Note that Assumption 5.1 implies $\chi(\alpha) > 0$.

Remark 5.6. When $M^{\mathrm{ss}}_\alpha = M^{\mathrm{s}}_\alpha$ there are no decompositions $\alpha = \alpha_1 + \ldots + \alpha_l$ with $l > 1$, $\mu(\alpha_i) = \mu(\alpha)$ and $M_{\alpha_i} \neq \varnothing$. Hence the right hand side of (54) has only one non-zero term, so

$$\Upsilon_\alpha = \chi(\alpha)[M_\alpha]^{\mathrm{inv}}.$$

Recall that without strictly semistable sheaves $f\colon P^{0+}_\alpha \to M_\alpha$ is a projective bundle with fibers $\mathbb{P}(H^0(F)) \cong \mathbb{P}^{\chi(\alpha)-1}$ over $[F] \in M_\alpha$ by Proposition 5.4 and the discussion following; moreover, $[M_\alpha]^{\mathrm{inv}} = [M_\alpha]^{\mathrm{vir}}$ is actually the (pushforward to $H_\bullet(\mathcal{M}^{\mathrm{rig}}_X)$ of the) virtual fundamental class. Indeed, we have

$$f_*\left(c_{\mathrm{top}}(\mathrm{T}_f) \cap [P^{0+}_\alpha]^{\mathrm{vir}}\right) = \chi(\alpha)[M_\alpha]^{\mathrm{vir}}$$

by the virtual pullback formula [Man, Theorem 4.7]. The following heuristic is useful to keep in mind: the class Υ_α is constructed so that the intersection theories against $[P^{0+}_\alpha]^{\mathrm{vir}}$ and Υ_α are related in a way similar to a projective bundle; the wall-crossing type formula (54) defines $[M_\alpha]^{\mathrm{inv}}$ by "correcting" Υ_α.

5.4. **Wall-crossing formula for Bradlow stability.** When working with pair wall-crossing formulae, we will include α_0 into the partition $\underline{\alpha} \vdash \alpha$ by $\underline{\alpha} = (\alpha_0, \alpha_1, \cdots, \alpha_l)$ and $\alpha_0 + \alpha_1 + \cdots + \alpha_l = \alpha$. Joyce's wall-crossing formula (Theorem

5.9 in [Joy6]) between the (limit) stability conditions μ^{0+} and μ^{∞} then takes the form

$$[P_\alpha^{0+}]^{\mathrm{vir}} = \sum_{\underline{\alpha}\vdash\alpha} U(\underline{\alpha})\big[[M_{\alpha_1}]^{\mathrm{inv}}, \big[[M_{\alpha_2}]^{\mathrm{inv}}, \dots, \big[[M_{\alpha_l}]^{\mathrm{inv}}, [P_{\alpha_0}^{\infty}]^{\mathrm{vir}}\big], \dots\big]\big] \tag{55}$$

where

$$U(\underline{\alpha}) = U\big((0,\alpha_1), \dots, (0,\alpha_l), (1,\alpha_0); \mu^{0+}, \mu^{\infty}\big) \in \mathbb{Q}$$

are combinatorial coefficients defined in [Joy6, Section 3.2]. Equation (55) is proven as an equality in the Lie algebra $\check{H}_\bullet(\mathcal{N}_X)$ under some technical assumptions (Assumptions 5.1-5.3 in loc. cit.) on the category $\grave{\mathcal{A}}$ and on a set of permissible classes $C_{\mathrm{pe}}(\grave{\mathcal{A}})$. The necessary assumptions are all verified in Section 8 of loc. cit. for the cases $(m,d) = (1,1), (2,2)$. The case $(m,d) = (2,1)$ is not treated, but the first author plans to address this in a separate work focusing on proving them for all pair theories relevant to Joyce's wall-crossing.

Assumption 5.7. Let $\grave{\mathcal{A}}$ be the abelian category of pairs $U \otimes \mathcal{O}_X \to F$, where F is a sheaf with $\dim(F) \leqslant 1$ and U a vector space[13]. Then the assumptions in [Joy6, Ass. 5.1-5.3] hold for this category and a surface S such that $h^{0,2}(S) = 0$ with

$$\begin{aligned} C_{\mathrm{pe}}(\grave{\mathcal{A}}) &= \{(e, \llbracket F\rrbracket) : e = 0,1 \text{ and } \dim(F) = 1\} \subseteq \mathbb{Z} \times K^0_{\mathrm{sst}}(S)\,, \\ \mathscr{S} &= \{\mu^t : t \in \mathbb{R}_{>0}\}\,. \end{aligned}$$

Note that we have already shown [Joy6, Ass. 4.4] in Definition 4.4. Most of the assumptions 5.1 - 5.3 are satisfied by a simple adaptation of the arguments in [Joy6, §]. New ideas are only needed for assumptions 5.2(b) and 5.2(h) where one needs to show that the stacks of semistable pairs are finite type and the moduli spaces of stable quiver-pairs defined in [Joy6, Def. 5.5] are proper.

Under this assumption, formula (55) holds also in the case $(m,d) = (2,1)$. These assumptions can be summarized as follows without going into too much detail:

An important point in the rank reduction induction that we will use is that there are no contributions of rank 0 objects in the wall-crossing formula above.

Lemma 5.8. Let α be such that $r(\alpha) > 0$. If the coefficient $U(\underline{\alpha})$ is non-zero, then $r(\alpha_i) > 0$ for each $i = 0, 1, \dots, l$.

Proof. Let t be a wall and let $t_- < t < t_+$ define stability conditions on the two chambers adjacent to the wall. We have the wall-crossing formula between μ^{t_-}-stability and μ^{t_+}-stability:

$$[P_\alpha^{t_-}]^{\mathrm{vir}} = \sum_{\underline{\alpha}\vdash\alpha} U(\underline{\alpha}; \mu^{t_-}, \mu^{t_+}) \quad \big[[M_{\alpha_1}]^{\mathrm{inv}}, \big[[M_{\alpha_2}]^{\mathrm{inv}}, \dots, \big[[M_{\alpha_l}]^{\mathrm{inv}}, [P_{\alpha_0}^{t_+}]^{\mathrm{vir}}\big], \dots\big]\big]$$

[13]As opposed to $\acute{\mathcal{A}}$ which was used in [Joy6] to denote the category of all pairs without the restriction on dimension.

To prove the Lemma it is enough to show that the coefficients $U(\underline{\alpha};\mu^{t_-},\mu^{t_+})$ vanish unless $r(\alpha_i)\neq 0$ for every i, since (55) can be obtained by putting together the μ^{t_-}/μ^{t_+} wall-crossing formulas.

Since μ^{t_-}, μ^{t_+} are in the adjacent chambers to the wall defined by t, the stability μ^t dominates (cf. [Joy6, Definition 3.8]) both μ^{t_-} and μ^{t_+}. By [Joy6, Theorem 3.11], $U(\underline{\alpha};\mu^{t_-},\mu^{t_+})=0$ unless

$$\mu^t(1,\alpha_0)=\mu(\alpha_1)=\ldots=\mu(\alpha_l)=\mu^t(1,\alpha).$$

Since $r(\alpha)>0$, $\mu^t(1,\alpha)<\infty$ so it follows that $r(\alpha_i)>0$ for each $i=0,1,\ldots,l$. $\square$

Formula (55) holds a priori in the Lie algebra $H_\bullet(\mathcal{N}_X^{\mathrm{rig}})$ or $\check{H}_\bullet(\mathcal{N}_X)$. As explained in Remark 4.5, we can pushforward this identity to the Lie algebra $H_\bullet(\mathcal{P}_X^{\mathrm{rig}})$ or, equivalently, $\check{V}_\bullet^{\mathrm{pa}}$ (recall Lemma 4.10). However, our formulation of the Virasoro constraints for the pair moduli spaces P_α^t is not in terms of the class $[P_\alpha^t]^{\mathrm{vir}}\in\check{V}_\bullet^{\mathrm{pa}}$, but instead of its lift $[P_\alpha^t]^{\mathrm{vir}}_{(\mathcal{O},\mathbb{F})}\in V_\bullet^{\mathrm{pa}}$ to the vertex algebra (see Corollary 1.8). Hence it is desirable to lift the formula (55) to the vertex algebra. We use the following Lemma to do so:

Lemma 5.9. a) Suppose that $\overline{u}\in\check{V}_\bullet\subseteq\check{V}_\bullet^{\mathrm{pa}}$ and $v\in V_\bullet^{\mathrm{pa}}$ is such that

$$\mathrm{ch}_1^{\mathcal{V}}(\mathsf{pt})\cap v=0\,.$$

Then

$$\mathrm{ch}_1^{\mathcal{V}}(\mathsf{pt})\cap[\overline{u},v]=0$$

where the bracket is the partial lift to the vertex algebra from Lemma 3.12.

b) Let $u,v\in V^{\mathrm{pa}}_{\bullet,(\alpha_1,\alpha_2)}$ with $\mathrm{rk}(\alpha_1)>0$ be such that

$$\mathrm{ch}_1^{\mathcal{V}}(\mathsf{pt})\cap v=0=\mathrm{ch}_1^{\mathcal{V}}(\mathsf{pt})\cap u$$

and $\overline{u}=\overline{v}$ in $\check{V}_\bullet^{\mathrm{pa}}$. Then $u=v$ in $V_\bullet^{\mathrm{pa}}$.

Proof. Since $\chi^{\mathrm{pa}}_{\mathrm{sym}}$ is non-degenerate by Lemma 4.6, there is $w\in K^\bullet(X)^{\oplus 2}$ such that

$$\chi^{\mathrm{pa}}_{\mathrm{sym}}(w,-)=\langle\mathsf{pt}^{\mathcal{V}},-\rangle\,.$$

Comparing (19) and (43) it follows that $\mathrm{ch}_1^{\mathcal{V}}(\mathsf{pt})\cap -=w_1$. Using the identity (12) we have

$$\mathrm{ch}_1^{\mathcal{V}}(\mathsf{pt})\cap[\overline{u},v]=w_1(u_0v)=u_0(w_1v)+(w_0u)_1v-(w_1u)_0v.$$

By hypothesis $w_1v=0$. Since $u\in V_\bullet$ both $w_0u=w_1u=0$ as the pullbacks of $\mathrm{ch}_1^{\mathcal{V}}(\mathsf{pt}),\mathrm{ch}_0^{\mathcal{V}}(\mathsf{pt})$ to $H^\bullet(\mathcal{M}_X)$ both vanish.

For the second part, suppose that $u-v=T(x)$. Then

$$0=\mathrm{ch}_1^{\mathcal{V}}(\mathsf{pt})\cap T(x)=(\mathsf{R}_{-1}\mathrm{ch}_1^{\mathcal{V}}(\mathsf{pt}))\cap x=\mathrm{rk}(\alpha_1)x$$

so $x=0$ and $u=v$. We used Lemma 4.9 stating that T is dual to R_{-1}. $\square$

We can use the Lemma to lift the previous wall-crossing formula to an equality

$$[P^{0+}_\alpha]^{\mathrm{vir}}_{(\mathcal{O},\mathbb{F})} = \sum_{\underline{\alpha}\vdash\alpha} U(\underline{\alpha}) \quad \big[[M_{\alpha_1}]^{\mathrm{inv}}, \big[[M_{\alpha_2}]^{\mathrm{inv}}, \dots, \big[[M_{\alpha_l}]^{\mathrm{inv}}, [P^\infty_{\alpha_0}]^{\mathrm{vir}}_{(\mathcal{O},\mathbb{F})}\big], \dots\big]\big] \tag{56}$$

holding in $V^{\mathrm{pa}}_\bullet$, where the last bracket on the right hand side is the partial lift of the Lie bracket to the vertex algebra. To deduce this from the Lemma we note that

$$\mathrm{ch}^{\mathcal{V}}_1(\mathsf{pt}) \cap [P^t_{\alpha_0}]^{\mathrm{vir}}_{(\mathcal{O},\mathbb{F})} = 0$$

since the pullback of $\mathrm{ch}^{\mathcal{V}}_1(\mathsf{pt})$ to $P^t_{\alpha_0}$ is $\xi_{\mathcal{O}}(\mathrm{ch}_1(\mathsf{pt})) = 0$. By part a) of the Lemma the right hand side is also annihilated by $\mathrm{ch}^{\mathcal{V}}_1(\mathsf{pt})$ and by part b) we must have equality – both sides live in $V^{\mathrm{pa}}_{\bullet,(1,\alpha)}$ and their classes in $\check{V}^{\mathrm{pa}}_\bullet$ agree by (55).

5.5. **Rank reduction of Virasoro.** We can now explain the rank reduction argument for proving Virasoro on M_α assuming that it holds for the stable pair moduli space P^∞_α. We described these spaces explicitly in Section 5.2.

Theorem 5.10. Suppose that the pair Virasoro (Conjecture 2.18) holds for P^∞_α for every α with $r(\alpha) > 0$. Then the Virasoro conjecture holds for M_α, P^t_α for every α with $r(\alpha) > 0$ and $t \in [0+, \infty]$, i.e.,

$$[M_\alpha]^{\mathrm{inv}} \in \check{P}_0 \quad \text{and} \quad [P^t_\alpha]^{\mathrm{inv}} \in P^{\mathrm{pa}}_0 .$$

The strategy of the proof is quite simple: we will argue by induction on $r(\alpha)$ and we will prove (assuming the induction hypothesis) that

$$[P^\infty_\alpha]^{\mathrm{vir}}_{(\mathcal{O},\mathbb{F})} \in P^{\mathrm{pa}}_0 \overset{\text{(I)}}{\Longrightarrow} [P^{0+}_\alpha]^{\mathrm{vir}}_{(\mathcal{O},\mathbb{F})} \in P^{\mathrm{pa}}_0 \overset{\text{(II)}}{\Longrightarrow} \Upsilon_\alpha \in \check{P}_0 \overset{\text{(III)}}{\Longrightarrow} [M_\alpha]^{\mathrm{inv}} \in \check{P}_0$$

Implications (I) and (III) will follow from (56), (54) and the compatibility between wall-crossing and Virasoro constraints proven in Propositions 3.11 and 3.13.

The implication (II) is a projective bundle compatibility. We will postpone its proof until the next section, see Theorem 5.11, and prove Theorem 5.10 assuming it.

Proof of Theorem 5.10. We argue by induction on $r(\alpha)$. The base case is when $r(\alpha) > 0$ is minimal and is dealt essentially in the same way as the induction step. Assume then that $[M_{\alpha'}]^{\mathrm{inv}} \in \check{P}_0$ for every α' such that $0 < r(\alpha') < r(\alpha)$; note in particular that this holds vacuously if $r(\alpha)$ is minimal.

To prove implication (I) we consider the wall-crossing formula (56) and we look at each individual summand

$$\big[[M_{\alpha_1}]^{\mathrm{inv}}, \big[[M_{\alpha_2}]^{\mathrm{inv}}, \dots, \big[[M_{\alpha_l}]^{\mathrm{inv}}, [P^\infty_{\alpha_0}]^{\mathrm{vir}}_{(\mathcal{O},\mathbb{F})}\big], \dots\big]\big]$$

with $\sum_{i=0}^l \alpha_i = \alpha$ and non-vanishing coefficient $U(\underline{\alpha}) \neq 0$. By hypothesis, $[P^\infty_{\alpha_0}]^{\mathrm{vir}}_{(\mathcal{O},\mathbb{F})} \in P^{\mathrm{pa}}_0$. Moreover by Lemma 5.8 we have for $i = 1, \dots, l$ that

$$0 < r(\alpha_i) < r(\alpha_i) + r(\alpha_0) \leqslant r(\alpha) .$$

So the induction hypothesis applies and

$$[M_{\alpha_i}]^{\mathrm{inv}} \in \check{P}_0 \quad \text{for } i = 1, \dots, l\,.$$

By Propositions 3.11 and 3.13 we have

$$\big[[M_{\alpha_1}]^{\mathrm{inv}}, \big[[M_{\alpha_2}]^{\mathrm{inv}}, \dots, \big[[M_{\alpha_l}]^{\mathrm{inv}}, [P^{\infty}_{\alpha_0}]^{\mathrm{vir}}_{(\mathcal{O},\mathbb{F})}\big], \dots\big]\big] \in P_0^{\mathrm{pa}}$$

so $[P^{0+}_{\alpha}]^{\mathrm{vir}}_{(\mathcal{O},\mathbb{F})} \in P_0^{\mathrm{pa}}$. The exact same argument shows that $[P^{t}_{\alpha}]^{\mathrm{vir}}_{(\mathcal{O},\mathbb{F})} \in P_0^{\mathrm{pa}}$ for any $t > 0$: just replace the $0+/\infty$ wall-crossing formula (56) by the t/∞ wall-crossing.

For both implications (II) and (III) we will assume that α satisfies Assumption 5.1. This is enough to prove the result for every α since we may replace α by $\alpha(mH)$ for large enough m so that $\alpha(mH)$ satisfies the assumption. As explained in Section 5.3, Joyce classes $[M_\alpha]^{\mathrm{inv}}, [M_{\alpha(mH)}]^{\mathrm{inv}}$ are related by the automorphism $H_\bullet(\mathcal{M}_X) \to H_\bullet(\mathcal{M}_X)$ induced by $- \otimes \mathcal{O}_X(mH)$. This automorphism preserves physical states by Lemma 2.19.

The implication (II) is precisely Theorem 5.11. So we are left with implication (III). For that, we use (54) and induction as in (I). The left hand side of (54) is $\Upsilon_\alpha \in \check{P}_0$. The right hand side is the sum of the leading term $\chi(\alpha)[M_\alpha]^{\mathrm{inv}}$ with terms of the form

$$\big[\big[\dots\big[[M_{\alpha_1}]^{\mathrm{inv}}, [M_{\alpha_2}]^{\mathrm{inv}}\big], \dots\big], [M_{\alpha_l}]^{\mathrm{inv}}\big]$$

with $l \geqslant 2$, $\underline{\alpha} \vdash \alpha$ and $\mu(\alpha_i) = \mu(\alpha)$. The latter condition implies that $r(\alpha_i) > 0$, and since $l \geqslant 2$ it follows that $0 < r(\alpha_i) < r(\alpha)$. Thus the induction hypothesis guarantees that $[M_{\alpha_i}]^{\mathrm{inv}} \in \check{P}_0$, so by Proposition 3.11 we get

$$\big[\big[\dots\big[[M_{\alpha_1}]^{\mathrm{inv}}, [M_{\alpha_2}]^{\mathrm{inv}}\big], \dots\big], [M_{\alpha_l}]^{\mathrm{inv}}\big] \in \check{P}_0\,.$$

Finally this implies that the leading term also satisfies Virasoro, i.e.

$$[M_\alpha]^{\mathrm{inv}} \in \check{P}_0$$

since $\chi(\alpha) > 0$ by Assumption 5.1. □

5.6. **Projective bundle compatibility.** We recall the reader of the notation $\Pi, \Pi_\alpha, \mathrm{T}^{\mathrm{rel}}, \mathrm{T}_{\Pi_\alpha}$ introduced in Section 5.3.

Theorem 5.11. Let α be a class satisfying Assumption 5.1. Suppose that P^{0+}_α satisfies the pair Virasoro constraints (Conjecture 2.18), i.e. $[P^{0+}_{\alpha}]^{\mathrm{vir}}_{(\mathcal{O},\mathbb{F})} \in P_0^{\mathrm{pa}}$. Then

$$\Upsilon_\alpha = \Pi_*\big(c_{\chi(\alpha)-1}(\mathrm{T}^{\mathrm{rel}}) \cap [P^{0+}_{\alpha}]^{\mathrm{vir}}_{(\mathcal{O},\mathbb{F})}\big) \in \check{P}_0$$

satisfies the sheaf Virasoro constraints.

Proof. We ought to show that $\int_{\Upsilon_\alpha} \mathsf{L}_{\mathrm{wt}_0}(D) = 0$ for any $D \in \mathbb{D}^X_\alpha$ where we use the suggestive integral notation to denote the pairing between a homology class $\Upsilon_\alpha \in H_\bullet(\mathcal{M}^{\mathrm{rig}}_\alpha)$ and a cohomology class $\mathsf{L}_{\mathrm{wt}_0}(D) \in \mathbb{D}^X_{\mathrm{wt}_0,\alpha} \cong H^\bullet(\mathcal{M}^{\mathrm{rig}}_\alpha)$.

We have the following commutative diagram:

$$\begin{tikzcd}
H^\bullet(P_\alpha^{0+}) & H^\bullet(\mathcal{P}_{(1,\alpha)}) \arrow[l, "(f_{(\mathcal{O},\mathbb{F})})^*"'] & H^\bullet(\mathcal{M}_\alpha^{\mathrm{rig}}) \arrow[l, "\Pi^*"'] \arrow[ll, bend right, "\Pi_\alpha^*"'] \\
& \mathbb{D}^{X,\mathrm{pa}}_{(1,\alpha)} \arrow[ul, "\xi_{(\mathcal{O},\mathbb{F})}"] \arrow[u, "\xi_{(\mathcal{V},\mathcal{F})}"', "\wr"] & \mathbb{D}^X_{\mathrm{wt}_0,\alpha} \arrow[l, hook'] \arrow[u, "\xi_{\mathcal{F}}"', "\wr"] \arrow[ull, bend left=40, "\xi_{\mathbb{F}}"]
\end{tikzcd}$$

Hence we can compute:

$$\begin{aligned}
\int_{\Upsilon_\alpha} \mathsf{L}_{\mathrm{wt}_0}(D) &= \int_{(\Pi_\alpha)_*\left(c_{\mathrm{top}}(\mathrm{T}_{\Pi_\alpha})\cap[P_\alpha^{0+}]^{\mathrm{vir}}\right)} \mathsf{L}_{\mathrm{wt}_0}(D) \\
&= \int_{[P_\alpha^{0+}]^{\mathrm{vir}}} \Pi_\alpha^*\big(\mathsf{L}_{\mathrm{wt}_0}(D)\big)c_{\mathrm{top}}(\mathrm{T}_{\Pi_\alpha}) \\
&= \int_{[P_\alpha^{0+}]^{\mathrm{vir}}} \xi_{\mathbb{F}}\left(\mathsf{L}_{\mathrm{wt}_0}(D)c_{\chi(\alpha)-1}\right) \\
&= \sum_{j\geqslant -1}\frac{(-1)^j}{(j+1)!}\int_{[P_\alpha^{0+}]^{\mathrm{vir}}} \xi_{\mathbb{F}}\left(\mathsf{L}_j(\mathsf{R}_{-1}^{j+1}D)c_{\chi(\alpha)-1}\right) \qquad (57)
\end{aligned}$$

We use $c_{\chi(\alpha)-1}\in\mathbb{D}_\alpha^X$ to denote the element in the algebra of descendents

$$c_{\chi(\alpha)-1} \coloneqq c_{\chi(\alpha)-1}(Rp_*\mathcal{F})\in H^\bullet(\mathcal{M}_\alpha)\cong\mathbb{D}_\alpha^X\,.$$

The penultimate equality is using that

$$\mathrm{T}_{\Pi_\alpha} = (f_{(\mathcal{O},\mathbb{F})})^*\mathrm{T}^{\mathrm{rel}} = (f_{(\mathcal{O},\mathbb{F})})^*\big(Rp_*\mathcal{F}-\mathcal{O}\big) = Rp_*\mathbb{F}-\mathcal{O}\,.$$

We can calculate $c_{\chi(\alpha)-1}$ more explicitly: by Grothendieck-Riemann-Roch we have

$$\mathrm{ch}\big(Rp_*\mathcal{F}\big) = p_*\big(\mathrm{ch}(\mathcal{F})\mathrm{td}(X)\big) = \xi_{\mathcal{F}}\big(\mathrm{ch}_\bullet(\mathrm{td}(X))\big)\,;$$

and by Newton identities it follows that

$$c(Rp_*\mathcal{F}) = \xi_{\mathcal{F}}\left(\exp\left(\sum_{\ell\geqslant 1}(-1)^{\ell-1}(\ell-1)!\mathrm{ch}_\ell(\mathrm{td}(X))\right)\right).$$

We denote by c the corresponding element in the algebra of descendents and we let c_i be the degree i part of c, i.e.,

$$\sum_{i\geqslant 0}c_i = c = \exp\left(\sum_{\ell\geqslant 1}(-1)^{\ell-1}(\ell-1)!\mathrm{ch}_\ell(\mathrm{td}(X))\right).$$

Let $n=\chi(\alpha)-1$. We shall now argue that the integral at the end of (57) vanishes assuming that P_α^{0+} satisfies the pair Virasoro constraints. For convenience, from now on we leave implicit the geometric realization map $\xi_{\mathbb{F}}$ in all the integrals against $[P_\alpha^{0+}]^{\mathrm{vir}}$.

We start with the $j=-1$ and $j=0$ terms in the last line of (57), which we treat together. Their sum vanishes by simple degree considerations:

$$\begin{aligned}\int_{[P_\alpha^{0+}]^{\mathrm{vir}}}\big(\mathsf{L}_0\mathsf{R}_{-1}(D)-\mathsf{R}_{-1}(D)\big)c_n &= \int_{[P_\alpha^{0+}]^{\mathrm{vir}}}\big(\mathsf{L}_0-\mathrm{id}-n\,\mathrm{id})(c_n\mathsf{R}_{-1}(D))\\ &= \int_{[P_\alpha^{0+}]^{\mathrm{vir}}}\mathsf{L}_0^{\mathcal{O}}(c_n\mathsf{R}_{-1}(D))=0\,.\end{aligned}\tag{58}$$

Note that we used

$$\xi_{\mathbb{F}}(\mathrm{ch}_0^{\mathrm{H}}(\mathrm{td}(X)))=\int_X\mathrm{ch}(\alpha)\mathrm{td}(X)=\chi(\alpha)=n+1\,.$$

Consider now $j\geqslant 1$. By the Virasoro constraints on P_α^{0+} we have

$$\begin{aligned}0=\int_{[P_\alpha^{0+}]^{\mathrm{vir}}}\mathsf{L}_j^{\mathcal{O}}\big(\mathsf{R}_{-1}^{j+1}(D)c_n\big) &= \int_{[P_\alpha^{0+}]^{\mathrm{vir}}}\mathsf{L}_j\big(\mathsf{R}_{-1}^{j+1}(D)\big)c_n\\ &+\int_{[P_\alpha^{0+}]^{\mathrm{vir}}}\mathsf{R}_{-1}^{j+1}(D)\mathsf{R}_j\big(c_n\big)-j!\int_{[P_\alpha^{0+}]^{\mathrm{vir}}}\mathrm{ch}_j(\mathrm{td}(X))\mathsf{R}_{-1}^{j+1}(D)c_n.\end{aligned}\tag{59}$$

We analyze the term where the derivation R_j applies to c_n; we may do so using the interaction between an exponential and a derivation:

$$\mathsf{R}_j(c)=\left(\sum_{\ell\geqslant 1}(-1)^{\ell-1}(\ell+j)!\mathrm{ch}_{\ell+j}(\mathrm{td}(X))\right)c\,,$$

so

$$\mathsf{R}_j(c_n)=\sum_{\substack{a\geqslant j+1,b\geqslant 0\\ a+b=n+j}}(-1)^{a-j-1}a!\mathrm{ch}_a(\mathrm{td}(X))c_b.$$

Using the Newton identities and the fact that T_{Π_α} is a vector bundle of rank n, the geometric realization of the above is

$$\begin{aligned}\xi_{\mathbb{F}}(\mathsf{R}_j(c_n)) &= \sum_{\substack{a\geqslant j+1,b\geqslant 0\\ a+b=n+j}}(-1)^{a-j-1}a!\mathrm{ch}_a(\mathrm{T}_{\Pi_\alpha})c_b(\mathrm{T}_{\Pi_\alpha})=j!\mathrm{ch}_j(\mathrm{T}_{\Pi_\alpha})c_n(\mathrm{T}_{\Pi_\alpha})\\ &= \xi_{\mathbb{F}}\big(j!\mathrm{ch}_j(\mathrm{td}(X))c_n\big)\,.\end{aligned}\tag{60}$$

It follows that the two last terms in (59) cancel out and we are left with

$$\int_{[P_\alpha^{0+}]^{\mathrm{vir}}}\mathsf{L}_j\big(R_{-1}^{j+1}(D)\big)c_n=0\tag{61}$$

for $j\geqslant 1$. Using (58) and (61) for every $j\geqslant 1$ we have shown that the last integral in (57) vanishes, and we are done. □

Remark 5.12. We can formulate the Lemma more abstractly as follows: let $u\in V^{\mathrm{pa}}_{\bullet,(1,\alpha)}$ be such that

(1) $\mathrm{ch}_i^{\mathcal{V}}(\gamma)\cap u=0$ for $i>0,\gamma\in H^\bullet(X)$;
(2) $c_b(Rp_*\mathcal{F})\cap u=0$ for $b\geqslant\chi(\alpha)$.

Then

$$u \in P_0^{\mathrm{pa}} \Rightarrow \Pi_*\big(c_{\chi(\alpha)-1}(\mathrm{T}^{\mathrm{rel}}) \cap u\big) \in \check{P}_0 \,.$$

The first condition is used when formulating the pair Virasoro constraints in terms of $\mathsf{L}_k^{\mathcal{O}}$ as in Conjecture 2.18. Condition (2) was used in (60); in the setting of the Lemma, it is a consequence of the fact that T_{Π_α} is a vector bundle of rank $\chi(\alpha)-1$.

6. Virasoro for P_α^∞ and virtual projective bundle compatibility

In this section we will finish the proof of Theorem A. By Theorem 5.10 it is sufficient to prove the Virasoro constraints for P_α^∞, which is what we now proceed to do. Recall that we explained in Section 5.2 that these moduli spaces are:

(1) symmetric powers $C^{[n]}$ for curves,
(2) nested Hilbert scheme $S_\beta^{[0,n]}$ (both in the torsion-free and torsion cases).

Both $C^{[n]}$ and $S_\beta^{[0,n]}$ can be regarded as pairs formed by either rank 1 sheaves or torsion sheaves. There is a sequence of equivalences of Virasoro constraints for different spaces:

(62)
$$\begin{tikzcd}
& M_{n\mathsf{pt}} \arrow[dl, bend right, "\mathrm{JS}"'] \\
C^{[n]} \arrow[ur, bend right, "\mathrm{RR}"'] \arrow[dr, bend right, "\mathrm{PB}"'] & \\
& \mathrm{Jac}(C) \arrow[ul, bend right, "\mathrm{PB}"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
& M_{\beta,n} \arrow[dl, bend right, "\mathrm{JS}"'] \\
S_\beta^{[0,n]} \arrow[ur, bend right, "\mathrm{RR}"'] \arrow[dr, bend right, "\mathrm{PB}"'] & \\
& S^{[n]} \arrow[ul, bend right, "\mathrm{PB}"']
\end{tikzcd}$$

The labels RR, JS and PB stand for rank reduction (Theorem 5.10), Joyce–Song wall-crossing and projective bundle compatibility, respectively. The rank reduction argument was explained previously. Joyce–Song wall-crossing expresses the virtual fundamental class of moduli of pairs in terms of the classes of moduli of sheaves. A general formulation of this is proved in [Boj1, Thm. A.4] (see (70) below); in the case of higher rank curves it appears already in [Bu, Theorem 2.8]. For the symmetric power of curves the consequence of (70) is

$$\big[C^{[n]}\big]_{(1,\mathcal{O}_\mathcal{E})} = \sum_{\underline{n}\vdash n} \frac{1}{l!}\,\big[[M_{n_1\mathsf{pt}}]^{\mathrm{inv}}, \big[\ldots, \big[[M_{n_l\mathsf{pt}}]^{\mathrm{inv}}, e^{(1,0)}\big]\ldots\big]\big] \tag{63}$$

where $e^{(1,0)} \in V_\bullet^{\mathrm{pa}}$ is the class of a point $\{(\mathcal{O}_X, 0)\}$ in the component $\mathcal{P}_{(1,0)}$. Virasoro constraints follow from Lemma 6.6, which shows that $[M_{n\mathsf{pt}}]^{\mathrm{inv}} \in \check{P}_0$. We also give an alternative direct proof of the Virasoro constraints for the symmetric product in §6.1 that has no interpretation in terms of the diagrams in (62).

When $\mathrm{rk}(\alpha) = 1$, $P_\alpha^\infty = P_\alpha^{0+}$ is a (virtual) projective bundle over M_α. This structure can be used to show the equivalence between Virasoro constraints on P_α^∞ or M_α. Indeed, the implication from P_α^∞ to M_α was already used in the general rank reduction argument (Theorem 5.11); the implication from M_α to P_α^∞ follows from (a special case of) Joyce–Song wall-crossing as explained in §6.4. For nested Hilbert schemes of points it leads to the expression

$$\left[S^{[0,n]}_\beta\right]_{(\mathcal{O}(-\mathcal{E}),I_{\mathcal{Z}})} = \left[\left[S^{[n]}\right], e^{(\llbracket L\rrbracket,0)}\right], \tag{64}$$

where $e^{(\llbracket L\rrbracket,0)} \in V^{\mathrm{pa}}_\bullet$ is the class of a point $\{(L,0)\}$ in the component $\mathcal{P}_{(\llbracket L\rrbracket,0)}$ for $\beta = -c_1(L)$. We give an independent proof of this formula in 6.3 using a direct proof of Joyce–Song wall-crossing for (virtual) projective bundles in 6.2.

Concluding this section is the application of Joyce–Song wall-crossing to Quot-schemes $\mathrm{Quot}_X(V,n)$ parameterizing 0-dimensional quotients $V \twoheadrightarrow F$ for a torsion-free sheaf V on $X = C, S$. Because of

$$\mathrm{Quot}_C(\mathcal{O}_C, n) = C^{[n]}\,,$$

Proposition 6.1 is a consequence of this more general result.

6.1. **Symmetric powers of curves.** Symmetric powers $C^{[n]}$ parametrize divisors $E \subseteq C$ of degree n or, equivalently, pairs of the form $\mathcal{O}_C \to \mathcal{O}_C(E)$. They come equipped with a universal pair

$$\mathcal{O}_{C^{[n]}\times C} \to \mathcal{O}_{C^{[n]}\times C}(\mathcal{E})$$

where $\mathcal{E} \subseteq C^{[n]} \times C$ is the universal divisor.

Let

$$\{e_j\}_{j=1}^g \subseteq H^{0,1}(C)\,, \quad \{f_j\}_{j=1}^g \subseteq H^{1,0}(C)$$

be basis such that

$$\int_X f_j e_i = -\int_X e_i f_j = \delta_{ij}.$$

Proposition 6.1. Let C be a curve and $n \geqslant 1$. Then the pair Virasoro conjecture (cf. Conjecture 2.18) holds for the symmetric powers $C^{[n]}$, i.e. $\left[C^{[n]}\right]_{(\mathcal{O},\mathcal{O}(\mathcal{E}))} \in V^{\mathrm{pa}}_\bullet$.

Proof. Let $f\colon C^n \to C^{[n]}$ be the projection from the n-fold product $C^n = C^{\times n}$ to the symmetric product; f is an étale morphism of degree $n!$. The pullback via f of the universal pair

$$\mathcal{O}_{C\times C^{[n]}} \to \mathcal{O}_{C\times C^{[n]}}(\mathcal{E})$$

to $C \times C^n$ is

$$\mathcal{O}_{C^n\times C} \to \mathcal{O}_{C^n\times C}(\Delta)$$

where $\Delta = \sum_{i=1}^n \Delta_i$ and $\Delta_i \subseteq C^n \times C$ is the pullback of the class of the diagonal $C \subseteq C \times C$ via the projection onto coordinates i and $n+1$. By the push-pull formula we have

$$\int_{C^n} \xi_{\mathcal{O}(\Delta)}(D) = n! \int_{C^{[n]}} \xi_{\mathcal{O}(\mathcal{E})}(D).$$

Thus Virasoro for symmetric powers may be formulated entirely as a relation among integrals in C^n. Let us denote by $\alpha_i \in H^\bullet(C^n)$ the pullback of a class

$\alpha \in H^\bullet(C)$ via projection onto the i-th coordinate with $i = 1, \ldots, n$. We compute descendents in $H^\bullet(C^n)$; in the formulas below and from now on we omit the geometric realization morphism $\xi_{\mathcal{O}(\Delta)}$:

$$\begin{aligned}
\mathrm{ch}_k^{\mathrm{H}}(\mathsf{pt}) &= \sum_{I \subseteq [n]\,,\, |I|=k} \prod_{i \in I} \mathsf{pt}_i = \frac{1}{k!}\eta^k \\
\mathrm{ch}_k^{\mathrm{H}}(1) &= n\mathrm{ch}_k^{\mathrm{H}}(\mathsf{pt}) - \theta \mathrm{ch}_{k-1}^{\mathrm{H}}(\mathsf{pt}) = n\frac{\eta^k}{k!} - \frac{\theta\eta^{k-1}}{(k-1)!} \\
\mathrm{ch}_k^{\mathrm{H}}(e_j) &= \mathrm{ch}_0^{\mathrm{H}}(e_j)\mathrm{ch}_k^{\mathrm{H}}(\mathsf{pt}) = \mathrm{ch}_0^{\mathrm{H}}(e_j)\frac{\eta^k}{k!} \\
\mathrm{ch}_k^{\mathrm{H}}(f_j) &= \mathrm{ch}_1^{\mathrm{H}}(f_j)\mathrm{ch}_{k-1}^{\mathrm{H}}(\mathsf{pt}) = \mathrm{ch}_1^{\mathrm{H}}(f_j)\frac{\eta^{k-1}}{(k-1)!}
\end{aligned}$$

where

$$\mathrm{ch}_0^{\mathrm{H}}(e_j) = \sum_{i=1}^n e_{ji}\,, \quad \mathrm{ch}_1^{\mathrm{H}}(f_j) = \sum_{i=1}^n f_{ji}\,,$$
$$\theta = \sum_{j=1}^g \mathrm{ch}_1^{\mathrm{H}}(f_j)\mathrm{ch}_0^{\mathrm{H}}(e_j)\,, \quad \eta = \mathrm{ch}_1^{\mathrm{H}}(\mathsf{pt}) = \sum_{i=1}^n \mathsf{pt}_i\,.$$

The formulas above show that the geometric realization map factors through the ring

$$\widetilde{\mathbb{D}}^C = \mathbb{C}[\eta, \{\mathrm{ch}_0^{\mathrm{H}}(e_j)\}_{j=1}^g, \{\mathrm{ch}_1^{\mathrm{H}}(f_j)\}_{j=1}^g],$$

formally generated by symbols $\eta, \mathrm{ch}_0^{\mathrm{H}}(e_j), \mathrm{ch}_1^{\mathrm{H}}(f_j)$. Moreover the Virasoro operators are well defined on $\widetilde{\mathbb{D}}^C$. Indeed, define

$$\widetilde{\mathsf{L}}_k = \widetilde{\mathsf{R}}_k + \widetilde{\mathsf{T}}_k^{\mathcal{O}} \colon \widetilde{\mathbb{D}}^C \to \widetilde{\mathbb{D}}^C$$

as follows:

(1) $\widetilde{\mathsf{R}}_k$ is a derivation on $\widetilde{\mathbb{D}}^C$ defined on generators by

$$\widetilde{\mathsf{R}}_k(\eta) = \eta^{k+1}\,, \quad \widetilde{\mathsf{R}}_k(\mathrm{ch}_0^{\mathrm{H}}(e_j)) = 0\,, \quad \widetilde{\mathsf{R}}_k(\mathrm{ch}_1^{\mathrm{H}}(f_j)) = (k+1)\eta^k\mathrm{ch}_1^{\mathrm{H}}(f_j)\,.$$

(2) $\widetilde{\mathsf{T}}_k^{\mathcal{O}}$ is multiplication by the element

$$(1-g)k\eta^k - n\eta^k + k\theta\eta^{k-1} \in \widetilde{\mathbb{D}}^C\,.$$

Claim 2. The following square commutes:

$$\begin{tikzcd}
\mathbb{D}^C \arrow[r] \arrow[d, "\mathsf{L}_k^{\mathcal{O}}"] & \widetilde{\mathbb{D}}^C \arrow[d, "\widetilde{\mathsf{L}}_k^{\mathcal{O}}"] & \\
\mathbb{D}^C \arrow[r] & \widetilde{\mathbb{D}}^C \arrow[r] & H^\bullet(C^n)
\end{tikzcd}$$

Proof. The proof is a straightforward computation. □

We now take an element

$$D = \eta^{\ell}\prod_{j=1}^{g}\mathrm{ch}_1^{\mathrm{H}}(f_j)^{a_j}\mathrm{ch}_0^{\mathrm{H}}(e_j)^{b_j} \in \widetilde{\mathbb{D}}^{C}\,.$$

Then we have

$$\widetilde{\mathsf{L}}_k^{\mathcal{O}}(D) = (\ell + (k+1)a)\eta^k D + (1-g)k\eta^k D - m\eta^k D + k\theta\eta^{k-1}D$$

where $a = \sum_{j=1}^g a_j$. By degree reasons, the integral of $\mathsf{L}_k^{\mathcal{O}}(D)$ vanishes unless $k + a + \ell = n$; when that is the case, it simplifies to

$$\widetilde{\mathsf{L}}_k^{\mathcal{O}}(D) = k(a-g)\eta^k D + k\theta\eta^{k-1}D.$$

To finish the proof we are required to show that

$$(g-a)\int_{C^n}\eta^k D = \int_{C^n}\eta^{k-1}\theta D. \tag{65}$$

We use the following easy claim:

Claim 3. The integral

$$\int_{C^n}\eta^{k+\ell}\prod_{i=1}^{g}\mathrm{ch}_1^{\mathrm{H}}(f_j)^{a_j}\mathrm{ch}_0^{\mathrm{H}}(e_j)^{b_j}$$

vanishes unless $a_j = b_j \in \{0,1\}$ for every $j = 1,\ldots,g$ and $k+\ell+\sum_{j=1}^g a_j = m$. In that case, the integral is equal to

$$\int_{C^n}\eta^n = n!$$

By the claim we may assume that $a_j = b_j \in \{0,1\}$, otherwise both sides of (65) vanish. Letting $J = \{1 \leqslant j \leqslant g\colon a_j = 1\}$ we have

$$\begin{aligned}
&\int_{C^n}\eta^{k+\ell-1}\theta\prod_{j\in J}\mathrm{ch}_1^{\mathrm{H}}(f_j)\mathrm{ch}_0^{\mathrm{H}}(e_j)\\
&\qquad = \sum_{t=1}^{g}\int_{C^n}\eta^{k+\ell-1}\mathrm{ch}_1^{\mathrm{H}}(f_t)\mathrm{ch}_0^{\mathrm{H}}(e_t)\prod_{j\in J}\mathrm{ch}_1^{\mathrm{H}}(f_j)\mathrm{ch}_0^{\mathrm{H}}(e_j)\\
&\qquad = \sum_{t\in[g]\backslash J}\int_{C^n}\eta^{k+\ell-1}\mathrm{ch}_1^{\mathrm{H}}(f_t)\mathrm{ch}_0^{\mathrm{H}}(e_t)\prod_{j\in J}\mathrm{ch}_1^{\mathrm{H}}(f_j)\mathrm{ch}_0^{\mathrm{H}}(e_j)\\
&\qquad = (g-|J|)n! = (g-a)\int_{C^n}\eta^{k+\ell}\prod_{j\in J}\mathrm{ch}_1^{\mathrm{H}}(f_j)\mathrm{ch}_0^{\mathrm{H}}(e_j)
\end{aligned}$$

showing (65) and concluding the proof. □

We have opted for giving a direct proof of the constraints in $C^{[n]}$ since this is easy enough and does not rely on wall-crossing or even on the vertex algebra language. However, as explained in the beginning of the section,there are two alternative ways of proving this result. By using Joyce–Song wall-crossing formula (63) as is done in Proposition 6.7 or by using the (virtual) projective bundle $C^{[n]} \to \mathrm{Jac}(C)$ – the Virasoro constraints on the Jacobian are almost trivial – and Corollary 6.3. Note that $\big[C^{[n]}\big]_{(\mathcal{O},\mathcal{O}_{\mathcal{E}})} \in V_\bullet^{\mathrm{pa}}$ and $\big[C^{[n]}\big]_{(\mathcal{O},\mathcal{O}(\mathcal{E}))} \in V_\bullet^{\mathrm{pa}}$ can be shown to be equivalent as in the proof of Proposition 6.5.

6.2. **Virtual projective bundle compatibility.** Let $M = M_\alpha$ be a moduli space with universal sheaf $\mathbb{G}$ as in Section 1.2, without strictly semistable sheaves. If $H^{\geqslant 1}(G) = 0$ for every sheaf $[G] \in M$, then $Rp_*\mathbb{G} = p_*\mathbb{G}$ is a vector bundle of rank $\chi(\alpha)$ and we may form the projective bundle

$$f\colon P = \mathbb{P}_M(Rp_*\mathbb{G}) \to M\,.$$

This projective bundle is naturally a moduli space of pairs: it parametrizes non-zero pairs of the form $\mathcal{O}_X \to F$ such that $[F] \in M$.

More generally, H. Park considers in [Par, Section 4] the situation in which $H^{\geqslant 2}(G) = 0$. In this case, we have a *virtual projective bundle* $f\colon P \to M$ where

$$P = \mathbb{P}_M(Rp_*\mathbb{G}) := \mathrm{Proj}\ \mathrm{Sym}^\bullet(p_*\mathbb{G}^\vee)\,.$$

The morphism f comes equipped with a natural relative perfect obstruction theory. By [Man], there is a virtual pullback $f^!\colon A_\bullet(M) \to A_\bullet(P)$ between Chow homologies. It is easily seen that the sheaf obstruction theory on M, the pair obstruction theory on P and the relative obstruction theory on f are compatible in the sense of [Man, Corollary 4.9], hence

$$[P]^{\mathrm{vir}} = f^![M]^{\mathrm{vir}}\,.$$

The moduli space P comes equipped with a unique universal pair

$$\mathcal{O}_{P\times X} \to \mathbb{F} := f^*\mathbb{G}(1)\,.$$

Note that $\mathbb{F}$ does not depend on the choice of $\mathbb{G}$.

The virtual pullback relation between the virtual fundamental classes can be translated to Joyce's vertex algebra framework as follows:

Proposition 6.2. Let $f\colon P \to M$ be a virtual projective bundle as described before. Then we have:

$$[P]^{\mathrm{vir}}_{(\mathcal{O},\mathbb{F})} = \big[[M]^{\mathrm{vir}}, e^{(1,0)}\big]\,, \tag{66}$$

$$\chi(\alpha)[M]^{\mathrm{vir}} = \Pi_*\left(c_{\chi(\alpha)-1}(\mathrm{T}^{\mathrm{rel}}) \cap [P]^{\mathrm{vir}}_{(\mathcal{O},\mathbb{F})}\right)\,. \tag{67}$$

In the first formula, the bracket is the partial lift to the vertex algebra in Lemma 3.12 and $e^{(1,0)}$ is the class of the point $\{(\mathcal{O}_X, 0)\}$ in $H_0(\mathcal{P}_{(1,0)}) \subseteq V_\bullet^{\mathrm{pa}}$.

Proof. The second statement (67) is a consequence of [Par, Theorem 0.5(2)] with $E = \mathcal{O}_X$. The first formula is (the lift to the vertex algebra of) a special case of a Joyce–Song type formula due to Joyce that can be found in [Bu, Theorem 2.8] for curves. Since this case is much easier, we give a direct proof straight from the definition of the bracket. We do so by evaluating both sides against descendents

$$D \in \mathbb{D}^{X,\mathrm{pa}}_{(1,\alpha)} \cong H^\bullet(\mathcal{P}_{(1,\alpha)})\,.$$

By a similar argument to the one in Lemma 5.9.a) it is enough to consider $D \in \mathbb{D}^X_\alpha \subseteq \mathbb{D}^{X,\mathrm{pa}}_{(1,\alpha)}$ since otherwise both sides would vanish.

We start with the left side:

$$\int_{[P]^{\mathrm{vir}}_{(\mathcal{O},\mathbb{F})}} D = \int_{[P]^{\mathrm{vir}}} \xi_{\mathbb{F}}(D) = \int_{f^![M]^{\mathrm{vir}}} \xi_{f^*\mathbb{G}(1)}(D) = \deg\Big(f_*\big(\xi_{f^*\mathbb{G}(1)}(D) \cap f^![M]^{\mathrm{vir}}\big)\Big)\,.$$

By Lemma 2.8 we have

$$\xi_{f^*\mathbb{G}(1)}(D) = \sum_{j\geqslant 0} \frac{1}{j!} f^*\xi_{\mathbb{G}}(\mathsf{R}^j_{-1}D)c_1(\mathcal{O}(1))^j$$

and the argument in the proof of Proposition 4.2 in [Par] shows that

$$f_*\big(c_1(\mathcal{O}(1))^j \cap f^![M]^{\mathrm{vir}}\big) = s_{j-\chi(\alpha)+1}(Rp_*\mathbb{G}) \cap [M]^{\mathrm{vir}}\,,$$

where $s_i(Rp_*\mathbb{G}) = c_i(-Rp_*\mathbb{G})$ are the Segre classes of $Rp_*\mathbb{G}$. Putting everything together, we find

$$\int_{[P]^{\mathrm{vir}}_{(\mathcal{O},\mathbb{F})}} D = \sum_{j\geqslant 0} \frac{1}{j!} \int_{[M]^{\mathrm{vir}}} \xi_{\mathbb{G}}(\mathsf{R}^j_{-1}D)s_{j-\chi(\alpha)+1}(Rp_*\mathbb{G})\,. \tag{68}$$

The analogous formula for the pairing with the right hand side can be deduced directly from Joyce's definition of the fields (32). Since $[M]^{\mathrm{vir}}_{\mathbb{G}}$ is a lift of $[M]^{\mathrm{vir}}$ we can compute the bracket by

$$\big[[M]^{\mathrm{vir}}, e^{(1,0)}\big] = \mathrm{Res}_{z=0}\, Y([M]^{\mathrm{vir}}_{\mathbb{G}}, z)e^{(1,0)}\,.$$

Recall that R_{-1} is dual to T and note that the pullback of Θ^{pa} to M via the map

$$M \cong \{(\mathcal{O}_X, 0)\} \times M \to \mathcal{P}_{(1,0)} \times \mathcal{P}_{(0,\alpha)}$$

is precisely $-Rp_*\mathbb{G}$. Using these two facts one checks that

$$\int_{Y([M]^{\mathrm{vir}}_{\mathbb{G}},z)e^{(1,0)}} D = \sum_{j,i\geqslant 0} \frac{z^{j-i-\chi(\alpha)}}{j!} \int_{[M]^{\mathrm{vir}}} \xi_{\mathbb{G}}(\mathsf{R}^j_{-1}D)c_i(-Rp_*\mathbb{G})\,. \tag{69}$$

Clearly taking the residue in (69) gives (68), finishing the proof of (66). □

As a result, we get compatibility of the Virasoro constraints with respect to (virtual) projective bundles.

Corollary 6.3. Let $f\colon P \to M$ be a virtual projective bundle as described before. Then the sheaf Virasoro constraints on M imply the pair Virasoro constraints on P, i.e.

$$[M]^{\mathrm{vir}} \in \check{V}_0 \Rightarrow [P]^{\mathrm{vir}}_{(\mathcal{O},\mathbb{F})} \in V_0^{\mathrm{pa}}\,.$$

If $f\colon P \to M$ is actually a smooth projective bundle (i.e. $R^1p_*\mathbb{G} = 0$) we have the converse implication

$$[P]^{\mathrm{vir}}_{(\mathcal{O},\mathbb{F})} \in V_0^{\mathrm{pa}} \Rightarrow [M]^{\mathrm{vir}} \in \check{V}_0\,.$$

Proof. The first implication follows from the first formula in Theorem 6.2, Proposition 3.13 and the straightforward fact that $e^{(1,0)} \in P_0^{\mathrm{pa}}$. The converse implication follows from Theorem 5.11. □

Remark 6.4. For the second implication, we really need f to be a smooth projective bundle. This is due to the fact that in the proof of Theorem 5.11 we used that T_{Π_α} is a vector bundle of rank $\chi(\alpha) - 1$, see Remark 5.12. Indeed, if the moduli space M is such that $p_*\mathbb{G} = 0$ (e.g. if $M = M_{\alpha(mH)}$ for m sufficiently negative) then P is empty but the Virasoro constraints on M are non-trivial.

6.3. **Nested Hilbert scheme.** We now treat the base cases for parts (2), (3) of Theorem A. Let S be a surface with $h^{0,1} = h^{0,2} = 0$. Let $S_\beta^{[0,n]}$ be the nested Hilbert scheme as in [GSY]. It parametrizes a pair of subschemes

$$Z \subseteq E \subseteq S$$

where E is a divisor in class β and $Z \subseteq E$ is a 0 dimensional subscheme of length n. We have universal subschemes

$$\mathcal{Z} \subseteq \mathcal{E} \subseteq \mathcal{S}\,,$$

where we use $\mathcal{S} = S \times S_\beta^{[0,n]}$. As explained in Section 5.2, the nested Hilbert scheme $S_\beta^{[0,n]}$ can be seen as a moduli of Bradlow pairs in 2 ways, by looking at a point $(E, Z) \in S_\beta^{[0,n]}$ either as

$$\mathcal{O}_S \to I_Z(E) \quad \text{or} \quad \mathcal{O}_S \to \mathcal{O}_E(Z)\,.$$

That is,

$$\begin{aligned} S_\beta^{[0,n]} &\cong P^\infty_{(1,\beta,-n+\beta^2/2)} \\ &\cong P^\infty_{(0,\beta,n-\beta^2/2)}\,. \end{aligned}$$

Each description comes with a natural universal pair, namely

$$\mathcal{O}_{\mathcal{S}} \to I_{\mathcal{Z}}(\mathcal{E}) \quad \text{and} \quad \mathcal{O}_{\mathcal{S}} \to \mathcal{O}_{\mathcal{E}}(\mathcal{Z})\,.$$

The first description allows us to describe $S_\beta^{[0,n]}$ as a virtual projective bundle over the Hilbert scheme of points on S. Let α be such that $\mathrm{ch}(\alpha) = (1, \beta, -n + \beta^2/2)$. Since α does not decompose as $\alpha_1 + \alpha_2$ with $r(\alpha_i) > 0$, a pair $\mathcal{O}_S \to F$ is μ^t-(semi)stable if and only if F is torsion free if and only if F is stable, so the moduli space P_α^t does not change with t and we have a map

$$f\colon P_\alpha^\infty = P_\alpha^{0+} \to M_\alpha\,.$$

Since $h^{0,1} = 0$ there exists a unique line bundle L_β with $c_1(L_\beta) = \beta$. Hence,

$$M_\alpha = \{I_Z \otimes L_\beta \colon Z \subseteq X \text{ is 0 dimensional of lenfth } n\} \cong M_{(1,0,-n)} = S^{[n]}\,.$$

We claim that if $S_\beta^{[0,n]}$ is not empty then the map f is a virtual projective bundle as described in the previous section. For this we need to show that if $F = I_Z \otimes L_\beta \in M_\alpha$

then $H^2(F) = 0$. If $S_\beta^{[0,n]}$ is not empty then there must exist a divisor $E \subseteq S$ in class β. Considering the long exact sequence on cohomology obtained from

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(E) \cong L_\beta \to \mathcal{O}_E(E) \to 0$$

and using that $H^2(\mathcal{O}_S) = 0$ it follows that $H^2(L_\beta) = 0$. Then the long exact sequence on cohomology associated to

$$0 \to I_Z \otimes L_\beta \to L_\beta \to \mathcal{O}_Z \otimes L_\beta \to 0$$

shows that $H^2(I_Z \otimes L_\beta) = 0$.

Proposition 6.5. The nested Hilbert scheme $S_\beta^{[0,n]}$ satisfies the Virasoro constraints with either of the two descriptions as a pair moduli space, that is,

$$\left[S_\beta^{[0,n]}\right]^{\mathrm{vir}}_{(\mathcal{O}, I_{\mathcal{Z}}(\mathcal{E}))} \in P_0^{\mathrm{pa}} \quad \text{and} \quad \left[S_\beta^{[0,n]}\right]^{\mathrm{vir}}_{(\mathcal{O}, \mathcal{O}_{\mathcal{E}}(\mathcal{Z}))} \in P_0^{\mathrm{pa}}\,.$$

Proof. We begin with the first statement. It was proven in [Mor, Theorem 5] that the Hilbert scheme $S^{[n]} \cong M_{(1,0,-n)}$ satisfies Virasoro constraints; see Remark 2.4 and Proposition 2.16 for a comparison between the formulation in loc. cit. and ours. By Lemma 2.19, it follows that Virasoro constraints hold for M_α for any α of rank 1. By Corollary 6.3 and the discussion preceeding this Proposition,

$$\left[S_\beta^{[0,n]}\right]^{\mathrm{vir}}_{(\mathcal{O}, I_{\mathcal{Z}}(\mathcal{E}))} = [P_\alpha^\infty]^{\mathrm{vir}}_{(\mathcal{O},\mathbb{F})} = [P_\alpha^{0+}]^{\mathrm{vir}}_{(\mathcal{O},\mathbb{F})} \in P_0^{\mathrm{pa}}\,.$$

We now deduce the second statement from the first. The dual of $\mathcal{O}_{\mathcal{E}}(\mathcal{Z})$ in K-theory can be computed to be

$$\mathcal{O}_{\mathcal{E}}(\mathcal{Z})^\vee = -\mathcal{O}_{\mathcal{E}}(-\mathcal{Z}) \otimes \mathcal{O}_{\mathcal{S}}(\mathcal{E}) = -I_{\mathcal{Z}}(\mathcal{E}) + \mathcal{O}_{\mathcal{S}}\,.$$

In the first equality we used [Huy, Example 3.41] and in the second we used

$$\mathcal{O}_{\mathcal{E}}(-\mathcal{Z}) = \mathcal{O}_{\mathcal{E}} - \mathcal{O}_{\mathcal{Z}} = -(\mathcal{O}_{\mathcal{S}}(-\mathcal{E}) - I_Z)\,.$$

As a consequence,

$$(\mathcal{O}_{\mathcal{E}}(\mathcal{Z}) - \mathcal{O}_{\mathcal{S}})^\vee = -I_{\mathcal{Z}}(\mathcal{E})\,.$$

Define the involution $\mathsf{I}\colon \mathbb{D}^{X,\mathrm{pa}} \to \mathbb{D}^{X,\mathrm{pa}}$ by

$$\begin{aligned} \mathsf{I}\big(\mathrm{ch}_i^{H,\mathcal{F}}(\gamma)\big) &= -(-1)^{i-p}\mathrm{ch}_i^{H,\mathcal{F}-\mathcal{V}}(\gamma)\,,\\ \mathsf{I}\big(\mathrm{ch}_i^{H,\mathcal{F}-\mathcal{V}}(\gamma)\big) &= -(-1)^{i-p}\mathrm{ch}_i^{H,\mathcal{F}}(\gamma)\,. \end{aligned}$$

By the previous computation of duals, we have

$$\xi_{(\mathcal{O},\mathcal{O}_{\mathcal{E}}(\mathcal{Z}))} = \xi_{(\mathcal{O},I_{\mathcal{Z}}(\mathcal{E}))} \circ \mathsf{I}\,.$$

We can see straight from the definition of the pair Virasoro operators $\mathsf{L}_k^{\mathrm{pa}}$ (see Section 2.6) that

$$\mathsf{I} \circ \mathsf{L}_k^{\mathrm{pa}} = (-1)^k \mathsf{L}_k^{\mathrm{pa}} \circ \mathsf{I}\,.$$

With all these observations, the equivalence between the two statements becomes clear. Indeed,

$$\begin{aligned}\int_{\left[S_\beta^{[0,n]}\right]^{\mathrm{vir}}_{(\mathcal{O},\mathcal{O}_{\mathcal{E}}(\mathcal{Z}))}} \mathsf{L}_k^{\mathrm{pa}}(D) &= \int_{\left[S_\beta^{[0,n]}\right]^{\mathrm{vir}}} \xi_{(\mathcal{O},\mathcal{O}_{\mathcal{E}}(\mathcal{Z}))}(\mathsf{L}_k^{\mathrm{pa}}(D))\\ &= \int_{\left[S_\beta^{[0,n]}\right]^{\mathrm{vir}}} \xi_{(\mathcal{O},I_{\mathcal{Z}}(\mathcal{E}))}(\mathsf{I}(\mathsf{L}_k^{\mathrm{pa}}(D)))\\ &= (-1)^k \int_{\left[S_\beta^{[0,n]}\right]^{\mathrm{vir}}} \xi_{(\mathcal{O},I_{\mathcal{Z}}(\mathcal{E}))}(\mathsf{L}_k^{\mathrm{pa}}(\mathsf{I}(D)))\\ &= (-1)^k \int_{\left[S_\beta^{[0,n]}\right]^{\mathrm{vir}}_{(\mathcal{O},I_{\mathcal{Z}}(\mathcal{E}))}} \mathsf{L}_k^{\mathrm{pa}}(\mathsf{I}(D)) = 0\,.\end{aligned}$$

This shows that $\left[S_\beta^{[0,n]}\right]^{\mathrm{vir}}_{(\mathcal{O},\mathcal{O}_{\mathcal{E}}(\mathcal{Z}))} \in P_0^{\mathrm{pa}}$ as well and finishes the proof. ☐

6.4. **Joyce–Song wall-crossing.** Joyce–Song stable pairs have their name after their first appearance in the work of Joyce and Song [JS, §12.1]. Denote by $p(F)$ the reduced Gieseker polynomial for a sheaf F, then they are defined as pairs

$$\mathcal{O}_X \xrightarrow{s} F$$

satisfying Assumption 1.7 for a fixed monic polynomial $r = r(F)$ and

(1) F is Gieseker-semistable;
(2) $s \neq 0$;
(3) there is no $0 \neq G \subsetneq F$ with $r(G) = r$ such that $\mathrm{im}(s) \subseteq G$.

Notice that unlike the result in Proposition 5.4, where the stability is given in terms of μ-stability, we now work with Gieseker stability because of a technicality explained in [Boj1, Rem. A.3]. This makes no difference in our applications, because we either work with sheaves supported in dimension $\leqslant 1$ or ideal sheaves, but for the sake of consistency of notation we denote the resulting moduli spaces by $P_{1,\alpha}^{0+}$.

The necessary assumptions for wall-crossing formulae to hold equivalent to the ones alluded to in Assumption 5.7 were explicitly checked in [Boj1, App. A] in larger generality that we now explain. In the definition of $P_{1,\alpha}^{0+}$, one can replace $\mathcal{O}_X$ with any sufficiently negative torsion-free sheaf V so that

$$\mathrm{Ext}^{\geqslant 2}(V, F) = 0$$

for any semistable sheaf F of class α. We denote the resulting moduli space by $P_{V,\alpha}^{0+}$ with the obstruction theory at each $[V \to F] \in P_{V,\alpha}^{0+}$ given by

$$\mathrm{RHom}([V \to F], F)\,.$$

Denote by $[P^{0+}_{V,\alpha}]^{\mathrm{vir}}$ the resulting virtual fundamental class. Then by [Boj1, Thm. A.4] we have the wall-crossing formula for any $X = C, S$

$$[P^{0+}_{V,\alpha}]^{\mathrm{vir}} = \sum_{\substack{\underline{\alpha}\vdash\alpha \\ r_{\alpha_i}=r_\alpha}} \frac{1}{l!}\left[[M_{\alpha_1}]^{\mathrm{inv}}, \left[\dots, \left[[M_{\alpha_l}]^{\mathrm{inv}}, e^{([\![V]\!],0)}\right]\dots\right]\right],$$

where $e^{([\![V]\!],0)} \in V^{\mathrm{pa}}_\bullet$ is the class of a point $\{(V,0)\}$ in the component $\mathcal{P}_{([\![V]\!],0)}$. As it is point-normalized, i.e., $\mathrm{ch}_1(\mathsf{pt}^{\mathcal{V}}) \cap e^{[\![V]\!],0} = 0$, the same holds for $[P^{0+}_{V,\alpha}]^{\mathrm{vir}}_{(p^*V,\mathbb{F})}$ which satisfy

$$[P^{0+}_{V,\alpha}]^{\mathrm{vir}}_{(p^*V,\mathbb{F})} = \sum_{\substack{\underline{\alpha}\vdash\alpha \\ r_{\alpha_i}=r_\alpha}} \frac{1}{l!}\left[[M_{\alpha_1}]^{\mathrm{inv}}, \left[\dots, \left[[M_{\alpha_l}]^{\mathrm{inv}}, e^{([\![V]\!],0)}\right]\dots\right]\right], \tag{70}$$

where $[-,-]$ is the lift from Lemma 3.12.

The first immediate implication of this result is the proof of (64) and therefore using [Mor, Thm. 5] also the proof of

$$\left[S^{[0,n]}_\beta\right]^{\mathrm{vir}} \in P_0$$

by the arguments in §6.3.

One other example that (70) covers that we currently can not address by hand are the punctual quot-schemes $\mathrm{Quot}_X(V,n)$ for $X = C, S$. These parameterize surjective morphisms $V \to F$ from a torsion-free sheaf V to a zero-dimensional sheaf F. When V is a vector bundle the virtual fundamental classes

$$\left[\mathrm{Quot}_X(V,n)\right]^{\mathrm{vir}}$$

were constructed by Marian–Oprea–Pandharipande [MOP, Lem. 1.1] (see Stark [Sta, Prop. 5] for a more detailed proof) in the case that V is a vector bundle. It was remarked in §[Boj1, §1.1] that the same obstruction theory given at each $[V \to F] \in \mathrm{Quot}_X(V,n)$ by

$$\mathrm{RHom}([V \to F], F)$$

is perfect of tor-amplitude $[-1,0]$ whenever V is more generally torsion-free. To apply (70), we use the identifications of moduli spaces and virtual fundamental classes

$$P^{0+}_{V,n} = \mathrm{Quot}(V,n)\,, \qquad [P^{0+}_{V,n}]^{\mathrm{vir}} = \left[\mathrm{Quot}(V,n)\right]^{\mathrm{vir}}$$

following immediately from their descriptions above. To prove Virasoro constraints for Quot schemes, we first show that zero-dimensional sheaves satisfy them.

Lemma 6.6. For any $n > 0$ and any $X = C, S$, the class $[M_n]^{\mathrm{inv}}$ is physical, i.e.

$$[M_n]^{\mathrm{inv}} \in \check{P}_0\,.$$

Proof. By Proposition 3.16, it is sufficient to prove that

$$\int_{\overline{[M_n]^{\mathrm{inv}}}} (\mathsf{L}_k - \delta_{k,0})(D) = 0 \quad \text{for all} \quad k \geqslant 0,\ D \in \mathbb{D}^X\,,$$

where $[M_n]^{\overline{\mathrm{inv}}}$ is the 1-normalized lift satisfying $\mathrm{ch}_1^{\mathrm{H}}(1) \cap [M_n]^{\overline{\mathrm{inv}}}$. Since L_0 acts as the multiplication by the conformal degree 1, the case $k = 0$ follows. On the other hand $\mathsf{R}_1(D)$ annihilates $[M_n]^{\overline{\mathrm{inv}}}$ by degree reasons. In conclusion, it suffices to prove that $\int_{[M_n]^{\overline{\mathrm{inv}}}} \mathsf{T}_1 = 0$.

Recall the definition of T_1 from Section 2.3. By simple modification, we have

$$\mathsf{T}_1 = \sum_s (-1)^{\dim(X)} \Big[(-1)^{p_s^L} + (-1)^{p_s^R}\Big] \mathrm{ch}_0^{\mathrm{H}}(\gamma_s^L)\mathrm{ch}_1^{\mathrm{H}}(\gamma_s^R)$$

where $\Delta_*(\mathrm{td}(X)) = \sum_s \gamma_s^L \otimes \gamma_s^R$. Therefore it suffices to consider the Kunneth components satisfying $|p_s^L| = |p_s^R|$ which is further restricted by $p_s^L + p_s^R \geqslant \dim(X)$. On the other hand, $\mathrm{ch}_0^{\mathrm{H}}(-)$ has a property that (after realization)

$$\mathrm{ch}_0^{\mathrm{H}}(\gamma_s^L) = \begin{cases} \int_X \gamma_s^L \cup n\mathsf{pt} & \text{if } \ p_s^L = q_s^L = 0\,, \\ 0 & \text{if } \ p_s^L = q_s^L > 0 \ \text{ or } \ p_s^L > q_s^R\,. \end{cases}$$

These vanishing properties are enough to prove that $\int_{[M_n]^{\overline{\mathrm{inv}}}} \mathsf{T}_1 = 0$ when $X = C$.

When $X = S$, we additionally need that

$$\mathrm{ch}_0^{\mathrm{H}}(\gamma^{1,2}) \cap [M_n]^{\overline{\mathrm{inv}}} = \mathrm{ch}_1^{\mathrm{H}}(\gamma^{2,\bullet}) \cap [M_n]^{\overline{\mathrm{inv}}} = 0 \tag{71}$$

for all $\gamma^{1,2} \in H^{1,2}(X)$ and $\gamma^{2,\bullet} \in H^{2,\bullet}(X)$. This follows from [Boj1, Lemma 4.2], but we make the argument used to prove it explicit in terms of descendants. Note that both $\mathrm{ch}_0^{\mathrm{H}}(\gamma^{1,2})$ and $\mathrm{ch}_1^{\mathrm{H}}(\gamma^{2,\bullet})$ use the first chern character $\mathrm{ch}_1(\mathcal{F})$ of the universal complex over $\mathcal{M}_{n\mathsf{pt}} \times X$. By the construction of invariant classes, they lie in the image of the pushforward map

$$\iota_* : H_\bullet(\mathcal{N}_{n\mathsf{pt}}) \to H_\bullet(\mathcal{M}_{n\mathsf{pt}})\,,$$

where $\iota : \mathcal{N}_{n\mathsf{pt}} = \mathcal{N}_{(0,n\mathsf{pt})} \hookrightarrow \mathcal{M}_{n\mathsf{pt}}$ denotes the open immersion from the stack of zero dimensional coherent sheaves of length n. Then the extra vanishings (71) follow from the geometric fact that $(\iota \times \mathrm{id}_X)^*\mathcal{F}$ is the universal zero dimensional sheaves on $\mathcal{N}_{n\mathsf{pt}} \times X$ hence $(\iota \times \mathrm{id}_X)^*\mathrm{ch}_1(\mathcal{F}) = 0$. □

We now conclude the precise version of Theorem 1.10.

Theorem 6.7. If $X = C$ or $X = S$ with $h^{2,0}(S) = 0$, punctual Quot schemes $\mathrm{Quot}_X(V, n)$ satisfy pair Virasoro constraints, i.e.

$$\big[\mathrm{Quot}_X(V, n)\big]^{\mathrm{vir}}_{(p^*V,\mathbb{F})} \in P_0^{\mathrm{pa}}\,.$$

Proof. As a corollary of (70), we obtain the wall-crossing formula for Quot scheme involving invariant classes of zero dimensional sheaves:

$$\big[\mathrm{Quot}_X(V, n)\big]^{\mathrm{vir}}_{(p^*V,\mathbb{F})} = \sum_{\underline{n} \vdash n} \frac{1}{l!} \Big[[M_{n_1}]^{\mathrm{inv}}, \big[\dots, \big[[M_{n_l}]^{\mathrm{inv}}, e^{(\llbracket V \rrbracket,0)}\big]\dots\big]\Big]\,.$$

Using Lemmas 6.6 and 3.13, we conclude the result. □

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ETH Zürich, Department of Mathematics
Email address: arkadij.bojko@math.ethz.ch

ETH Zürich, Department of Mathematics
Email address: woonam.lim@math.ethz.ch

ETH Zürich, Department of Mathematics
Email address: miguel.moreira@math.ethz.ch