Virasoro constraints for stable pairs and for Hilbert schemes of points of surfaces

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Introduction

Based on joint work with A. Oblomkov, A. Okounkov and R. Pandharipande.

- Theory of stable pairs (PT) provides a (sheaf theoretical) way to:
  – Compactify the space of nonsigular embedded curves on a 3-fold X.
  – Define numerical invariants ("curve counts").
- Conjecturally equivalent to other curve counting theories like Gromov-Witten (GW) and Donaldson-Thomas (DT).[1]
- Virasoro conjecture predicts universal constraints on Gromov-Witten invariants.[2]

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[1] Contributions by Maulik, Nekrasov, Okounkov, Pandharipande, Pixton, Oblomkov, Thomas, Stoppa, Bridgeland and many others.

[2] Contributions by Witten, Kontsevich, Eguchi, Hori, Xiong, Getzler, Givental, Teleman, Okounkov, Pandharipande and many others.

Introduction

- Expect:
  GW/PT correspondence + GW Virasoro $\Rightarrow$ PT Virasoro.
- Precise form of the conjecture for $X = \mathbb{P}^3$ was found in $\sim$ 2007 by Oblomkov, Okounkov, Pandharipande (OOP).
- Recent progress (together with OOP): precise formulation of the conjecture for any simply-connected 3-fold (at least when it doesn't have $(0, p)$ cohomology).
- (With OOP) Proof of the PT Virasoro for toric 3-folds in the stationary regime.
- Verification of the PT Virasoro for the cubic 3-fold in the curve class of lines.
- Proof of a certain specialization that gives a new set of relations satisfied by tautological classes in the Hilbert scheme of points of a surface.

Stable pairs

Let X be a smooth projective 3-fold over $\mathbb{C}$.

Definition (Pandharipande-Thomas)

A stable pair on X is a coherent sheaf F on X together with a section $\mathcal{O}_X \xrightarrow{s} F$ satisfying the following two stability conditions:

1. F is pure of dimension 1: every non-trivial coherent sub-sheaf of F has dimension 1.
2. The cokernel of s has dimension 0.

We associate two discrete invariants:

$$\beta = [\text{supp}(F)] \in H_2(X;\mathbb{Z}) \text{ and } n = \chi(X, F).$$

The space $P_n(X, \beta)$ parametrizing stable pairs with fixed discrete invariants is a projective fine moduli space.

Geometric locus

If $C \subseteq X$ is a smooth curve and D is an effective divisor on C

$$\mathcal{O}_X \xrightarrow{s} \iota_* \mathcal{O}_C(D)$$

is a stable pair with support C and .

$$\mathrm{coker}(s) = \mathcal{O}_D.$$

Roughly speaking: stable pair is a curve decorated with finite number of points contained in the curve (zeros of the section). In general $P_n(X, \beta)$ has more degenerate objects (supported in singular curves).

Geometric locus

Example

For $t \neq 0$ consider the embedded curve

$$C_t = \{x = z = 0\} \cup \{y = z - t = 0\} \subseteq \mathbb{C}^3.$$

As a stable pair:

$$\mathbb{C}[x, y, z] \xrightarrow{s} \mathbb{C}[x, y, z]/(x, z) \oplus \mathbb{C}[x, y, z]/(y, z - t).$$

In the limit $t \to 0$:

$$\mathbb{C}[x, y, z] \xrightarrow{s} \mathbb{C}[x, y, z]/(x, z) \oplus \mathbb{C}[x, y, z]/(y, z) \to \underbrace{\mathbb{C}/(x, y, z)}_{\text{coker}}.$$

Not surjective anymore.

Deformation theory

The moduli space $P_n(X,\beta)$ admits a 2-term perfect obstruction theory (Pandharipande-Thomas). Associate to a stable pair $\mathcal{O}_X \xrightarrow{s} F$ the 2-term complex

$$I^\bullet = \{\mathcal{O}_X \xrightarrow{s} F\} \in D^b(X).$$

The (fixed-determinant) obstruction theory on $D^b(X)$ provides a deformation theory on $P_n(X,\beta)$:

- Tangent space: $\mathrm{Ext}^1(I^\bullet, I^\bullet)_0$.
- Obstruction space: $\mathrm{Ext}^2(I^\bullet, I^\bullet)_0$.

Virtual fundamental class

Higher $\mathrm{Ext}^*(I^\bullet, I^\bullet)_0$ vanish $\rightsquigarrow$ 2-term perfect deformation theory
$\rightsquigarrow$ virtual fundamental class

$$[P_n(X,\beta)]^{\mathrm{vir}} \in A_{d_\beta}(X)$$

where d_β is the expected dimension:

$$d_\beta = -\chi(\mathrm{R\,Hom}(I^\bullet, I^\bullet)_0) = \int_\beta c_1(X).$$

Remark

For the vanishing of the higher Ext's we need X to be 3-dimensional.

Remark

The virtual dimension depends only on the support of the stable pair and not on the number of points decorating the curve.

Descendents

When X is Calabi-Yau the virtual dimension is 0. Can define the curve count

$$\langle 1 \rangle_{n,\beta}^{X,\mathrm{PT}} \stackrel{\mathrm{def}}{=} \int_{[P_n(X,\beta)]^{\mathrm{vir}}} 1 \in \mathbb{Z}.$$

If $d_\beta > 0$ one needs to impose constraints on the curve to get meaningful counts.

Definition

For $\gamma \in H^*(X)$, $k \geq 0$ define the descendents

$$\mathrm{ch}_k(\gamma) = (\pi_P)_* \left(\mathrm{ch}_k\left(\mathbb{F} - \mathcal{O}\right) \cdot \pi_X^*(\gamma)\right) \in H^*(P_n(X,\beta)).$$

$$\begin{array}{ccccc} & & \mathbb{F} & & \\ & & \downarrow & & \\ X & \xleftarrow{\pi_X} & X \times P_n(X,\beta) & \xrightarrow{\pi_P} & P_n(X,\beta) \end{array}$$

PT invariants

Remark

Since $\mathbb{F}$ is supported in codimension 2

$$\mathrm{ch}_0(\gamma) = -\int_X \gamma \in H^0(P) \text{ and } \mathrm{ch}_1(\gamma) = 0.$$

Given a product of descendent classes $D = \prod_{j=1}^m \mathrm{ch}_{k_j}(\gamma_j)$ we denote integration against the virtual fundamental class by

$$\langle D \rangle_{n,\beta}^{X,\mathrm{PT}} = \int_{[P_n(X,\beta)]^{\mathrm{vir}}} D \in \mathbb{Q}.$$

We assemble the information of all n in the partition function

$$\langle D \rangle_{\beta}^{X,\mathrm{PT}} = \sum_{n\in\mathbb{Z}} q^n \langle D \rangle_{n,\beta}^{X,\mathrm{PT}} \in \mathbb{Q}((q)).$$

Rationality and functional equation

Conjecture

Let $D = \prod_{j=1}^{m} \mathrm{ch}_{k_j}(\gamma_j)$*. Then* $\langle D \rangle_{\beta}^{X,\mathrm{PT}}$ *is the Laurent expansion of a rational function* $f(q)$ *satisfying the symmetry functional equation*

$$f(q^{-1}) = (-1)^{\sum_{j=1}^{m} k_j} q^{-d_\beta} f(q).$$

Evidence for the conjecture:

1. Both rationality and the functional equation hold for Calabi-Yau 3-folds (Bridgeland, Toda).
2. Rationality holds for toric 3-folds (Pandharipande-Pixton). The functional equation is known when $k_j = 2$.
3. Rationality holds for complete intersections in products of projective spaces for cohomology classes γ_i restricted from the ambient space (Pandharipande-Pixton).

Gromov-Witten compactification

On the Gromov-Witten side we compactify the moduli of embedded curves in a different way:

$$\overline{M}_{g,m}(X,\beta) = \{(C, p_1, \ldots, p_m, f)\}$$

parametrizes maps $f : C \to X$ from a nodal curve of genus g with m marked points to X such that $f_*[C] = \beta$.
(We take here a slight variation of the usual GW moduli space by allowing C to be disconnected without collapsed components of genus 0 and 1.)
This moduli space has a virtual fundamental class $[\overline{M}_{g,m}(X,\beta)]^{\mathrm{vir}}$ in virtual dimension

$$\mathrm{virdim} = d_\beta + m.$$

Gromov-Witten descendents

In Gromov-Witten theory descendents are defined by

$$\tau_k(\gamma) = \psi_i^k \mathrm{ev}_i^*(\gamma)$$

where

- $\psi_i = c_1(\mathbb{L}_i)$ and $\mathbb{L}_i$ is the cotangent line bundle associated to the i-th point. The fiber of $\mathbb{L}_i$ over $(C, p_1, \ldots, p_m, f)$ is $T_{p_i}^\vee C$.
- $\mathrm{ev}_i : \overline{M}_{g,m}(X, \beta) \to X$ is evaluation at the i-th point, $f(p_i)$.

Gromov-Witten invariants

Gromov-Witten invariants are defined by integrating against virtual fundamental class:

$$\left\langle \prod_{i=1}^{m} \tau_{k_i}(\gamma_i) \right\rangle_{g,\beta}^{X,\mathrm{GW}} = \int_{[\overline{M}_{g,m}(X,\beta)]^{\mathrm{vir}}} \prod_{i=1}^{m} \psi_i^{k_i} \mathrm{ev}_i^*(\gamma_i) \in \mathbb{Q}.$$

The associated partition function is

$$\left\langle \prod_{i=1}^{m} \tau_{k_i}(\gamma_i) \right\rangle_{\beta}^{X,\mathrm{GW}} = \sum_{g\in\mathbb{Z}} \left\langle \prod_{i=1}^{m} \tau_{k_i}(\gamma_i) \right\rangle_{g,\beta}^{X,\mathrm{GW}} u^{2g-2}.$$

GW/PT correspondence

Conjecturally, the collections of GW invariants and of PT invariants determine each other. This is easiest to state for primary fields:

$$(-q)^{-d_\beta/2}\langle \mathrm{ch}_2(\gamma_1)\dots\mathrm{ch}_2(\gamma_m)\rangle_\beta^{X,\mathrm{PT}} = (-\imath u)^{d_\beta}\langle \tau_0(\gamma_1)\dots\tau_0(\gamma_m)\rangle_\beta^{X,\mathrm{GW}}$$

after the change of variables $-q = e^{\imath u}$.

In general the correspondence is much more complicated. To state it let $\mathbb{D}^X_{\mathrm{PT}}, \mathbb{D}^X_{\mathrm{GW}}$ be the algebras generated by formal symbols

$$\mathrm{ch}_k(\gamma) \text{ and } \tau_k(\gamma), \text{ respectively.}$$

Conjecture (MNOP)

There is a universally defined invertible transformation $\mathfrak{C}^\bullet : \mathbb{D}^X_{\mathrm{PT}} \to \mathbb{D}^X_{\mathrm{GW}}$ such that

$$(-q)^{-d_\beta/2}\langle D\rangle_\beta^{X,\mathrm{PT}} = (-\imath u)^{d_\beta}\langle \mathfrak{C}^\bullet(D)\rangle_\beta^{X,\mathrm{GW}}$$

for every $D \in \mathbb{D}^X_{\mathrm{PT}}$ after the change of variable $-q = e^{\imath u}$.

Explicit GW/PT correspondence

Oblomkov-Okounkov-Pandharipande found explicit (partial) formulas for $\mathfrak{C}^\bullet$. To state them we introduce modified descendents:

$$\tilde{\mathrm{ch}}_k(\gamma) = \mathrm{ch}_k(\gamma) + \frac{1}{24}\mathrm{ch}_{k-2}(\gamma c_2).$$

$$\frac{(\imath u)^k \mathfrak{a}_{k+1}(\gamma)}{(k+1)!} = \tau_k(\gamma) + \left(\sum_{i=1}^{k} \frac{1}{i}\right)\tau_{k-1}(\gamma c_1) + \left(\sum_{1\leq i<j\leq k} \frac{1}{ij}\right)\tau_{k-2}(\gamma c_1^2).$$

Then the transformation has the form

$$\mathfrak{C}^\bullet\left(\tilde{\mathrm{ch}}_{k_1}(\gamma_1)\ldots\tilde{\mathrm{ch}}_{k_m}(\gamma_m)\right) = \sum_P \prod_{S\in P} \mathfrak{C}^\circ\left(\prod_{i\in S}\tilde{\mathrm{ch}}_{k_i}(\gamma_i)\right)$$

where the sum runs over partitions P of $\{1,\ldots,m\}$ and $\mathfrak{C}^\circ$ is...

Explicit GW/PT correspondence

$$\mathfrak{C}^\circ\left(\tilde{\mathrm{ch}}_{k+2}(\gamma)\right) = \frac{1}{(k+1)!}\mathfrak{a}_{k+1}(\gamma) + \frac{(\imath u)^{-1}}{k!}\sum_{|\mu|=k-1}\frac{\mathfrak{a}_{\mu_1}\mathfrak{a}_{\mu_2}(\gamma c_1)}{\mathrm{Aut}(\mu)}$$
$$+\frac{(\imath u)^{-2}}{k!}\sum_{|\mu|=k-2}\frac{\mathfrak{a}_{\mu_1}\mathfrak{a}_{\mu_2}(\gamma c_1^2)}{\mathrm{Aut}(\mu)} + \frac{(\imath u)^{-2}}{(k-1)!}\sum_{|\mu|=k-3}\frac{\mathfrak{a}_{\mu_1}\mathfrak{a}_{\mu_2}\mathfrak{a}_{\mu_3}(\gamma c_1^2)}{\mathrm{Aut}(\mu)} + \ldots$$

$$\mathfrak{C}^\circ\left(\tilde{\mathrm{ch}}_{k_1+2}(\gamma)\tilde{\mathrm{ch}}_{k_2+2}(\gamma')\right) =$$
$$-\frac{(\imath u)^{-1}}{k_1!k_2!}\mathfrak{a}_{k_1+k_2}(\gamma\gamma') - \frac{(\imath u)^{-2}}{k_1!k_2!}\mathfrak{a}_{k_1+k_2-1}(\gamma\gamma' c_1)$$
$$-\frac{(\imath u)^{-2}}{k_1!k_2!}\sum_{|\mu|=k_1+k_2-2}\max(k_1,k_2,\mu_1+1,\mu_2+1)\frac{\mathfrak{a}_{\mu_1}\mathfrak{a}_{\mu_2}}{\mathrm{Aut}(\mu)}(\gamma\gamma'\cdot c_1) + \ldots$$

$$\mathfrak{C}^{\circ}\left(\tilde{\mathrm{ch}}_{k_1+2}(\gamma)\tilde{\mathrm{ch}}_{k_2+2}(\gamma')\tilde{\mathrm{ch}}_{k_3+2}(\gamma'')\right) = \frac{(\imath u)^{-2}k}{k_1!k_2!k_3!}\mathfrak{a}_{k-1}(\gamma\gamma'\gamma'') + \ldots$$

for $k = k_1 + k_2 + k_3$. To control the entire transformation we would need the expression of $\mathfrak{C}^{\circ}$ for arbitrarily long monomials. However, if we restrict ourselves to the stationary descendents

$$\left\{\mathrm{ch}_k(\gamma) : k \geq 0, \gamma \in H^{\geq 2}(X)\right\}$$

the higher $\mathfrak{C}^{\circ}$ and the $\ldots$ terms vanish by degree reasons. Denote by $\mathbb{D}^{X+}_{\mathrm{PT}} \subseteq \mathbb{D}^{X}_{\mathrm{PT}}$ the stationary sub-algebra.

Upshot

We have a (very complicated) completely explicit way to write the GW/PT correspondence for stationary descendents.

Gromov-Witten Virasoro

The Virasoro constraints (first proposed by Eguchi, Hori and Xiong in '97) are a conjectured set of relations satisfied by GW invariants. For each $k \geq -1$ there is an operator $L_k^{\mathrm{GW}} : \mathbb{D}^X_{\mathrm{GW}} \to \mathbb{D}^X_{\mathrm{GW}}$. The Virasoro conjecture predicts:

$$\langle L_k^{\mathrm{GW}}(D)\rangle^{X,\mathrm{GW}}_{g,\beta} = 0 \text{ for } D \in \mathbb{D}^X_{\mathrm{GW}}.$$

The operators satisfy the Virasoro relation:

$$[L_k^{\mathrm{GW}}, L_m^{\mathrm{GW}}] = (k-m)L^{\mathrm{GW}}_{k+m}.$$

The first equation ($k=-1$) is the string equation:

$$\langle \tau_0(1)\tau_{k_1}(\gamma_1)\ldots\tau_{k_m}(\gamma_m)\rangle = \sum_{j=1}^{m}\langle \tau_{k_1}(\gamma_1)\ldots\tau_{k_j-1}(\gamma_j)\ldots\tau_{k_m}(\gamma_m)\rangle.$$

Stable pairs Virasoro

The stable pairs Virasoro have a similar form: it predicts

$$\langle L_k^{\mathrm{PT}}(D)\rangle_{n,\beta}^{X,\mathrm{PT}} = 0 \text{ for } D \in \mathbb{D}_{\mathrm{PT}}^X, k \geq -1$$

for certain operators $L_k^{\mathrm{PT}} : \mathbb{D}_{\mathrm{PT}}^X \to \mathbb{D}_{\mathrm{PT}}^X$.

The cases $k = -1, 0$ follow from the string and the divisor equations:

1. $\mathrm{ch}_2(1) = 0$. (string equation)
2. $\mathrm{ch}_2(D) = \int_\beta D$ for $D \in H^2(X)$. (divisor equation)
3. $\mathrm{ch}_3(1) = n - \frac{d_\beta}{2}$. (dilation equation)

Virasoro operators: R_k

To describe the operators L_k^{PT} we need several constructions:

- Define derivations $R_k : \mathbb{D}_{\mathrm{PT}}^X \to \mathbb{D}_{\mathrm{PT}}^X$ by their values on the generators:

$$R_k \mathrm{ch}_i(\gamma) = \left(\prod_{j=0}^{k}(i + p - 3 + j)\right) \mathrm{ch}_{k+i}(\gamma)$$

  for γ having Hodge type (p, q).
  In particular

$$R_{-1}\mathrm{ch}_i(\gamma) = \mathrm{ch}_{i-1}(\gamma).$$

Virasoro operators: T_k

- We use the abbreviation

$$\mathrm{ch}_a\mathrm{ch}_b(\gamma) = \sum_i \mathrm{ch}_a(\gamma_i^L)\mathrm{ch}_b(\gamma_i^R)$$

  where $\sum_i \gamma_i^L \otimes \gamma_i^R$ is the Kunneth decomposition of $\Delta_*\gamma \in H^*(X \times X)$.
- The notation

$$(-1)^{p^L p^R}(a + p^L - 3)!(b + p^R - 3)!\mathrm{ch}_a\mathrm{ch}_b(c_1)$$

  means

$$\sum_i (-1)^{p_i^L p_i^R}(a + p_i^L - 3)!(b + p_i^R - 3)!\mathrm{ch}_a(\gamma_i^L)\mathrm{ch}_b(\gamma_i^R)$$

  where $\sum_i \gamma_i^L \otimes \gamma_i^R$ is the Kunneth decomposition of $\Delta_* c_1 \in H^*(X \times X)$ and $\gamma_i^L \in H^{p_i^L, q_i^L}(X)$.

Virasoro operators: T_k

- Define the operator $T_k : \mathbb{D}^X_{\mathrm{PT}} \to \mathbb{D}^X_{\mathrm{PT}}$ as multiplication by

$$T_k = -\frac{1}{2} \sum_{a+b=k+2} (-1)^{p^L p^R} (a + p^L - 3)!(b + p^R - 3)! \mathrm{ch}_a \mathrm{ch}_b(c_1) + \frac{1}{24} \sum_{a+b=k} a!b! \mathrm{ch}_a \mathrm{ch}_b(c_1 c_2).$$

When X doesn't have any $(0, p)$ cohomology (for example: X toric, X cubic 3-fold) we can already say what L_k^{PT} is:

$$L_k^{\mathrm{PT}} = R_k + T_k + (k+1)! R_{-1} \mathrm{ch}_{k+1}(\mathrm{p})$$

Virasoro operators: S_k

In the general case (we think)

$$L_k^{\mathrm{PT}} = R_k + T_k + S_k.$$

- Given $\alpha \in H^{0,q}(X)$ define the derivation $R_{-1}[\alpha] : \mathbb{D}_{\mathrm{PT}}^X \to \mathbb{D}_{\mathrm{PT}}^X$ by its value on the generators:

$$R_{-1}[\alpha]\mathrm{ch}_i(\gamma) = \mathrm{ch}_{i-1}(\alpha\gamma).$$

- $S_k : \mathbb{D}_{\mathrm{PT}}^X \to \mathbb{D}_{\mathrm{PT}}^X$ is given by

$$S_k = (k+1)! \sum_{p_i^L = 0} R_{-1}[\gamma_i^L]\mathrm{ch}_{k+1}(\gamma_i^R).$$

The sum runs over the terms $\gamma_i^L \otimes \gamma_i^R$ in the Kunneth decomposition of the diagonal $\Delta_* 1$ such that $p_i^L = 0$.

Virasoro conjecture

Conjecture

For any X (simply-connected?), $n \in \mathbb{Z}$, $\beta \in H_2(X;\mathbb{Z})$ and $D \in \mathbb{D}^X_{\mathrm{PT}}$ we have

$$\langle L_k^{\mathrm{PT}}(D)\rangle^{X,\mathrm{PT}}_{n,\beta} = 0.$$

A striking feature of this conjecture is that, unlike the GW conjecture, the relations predicted are all defined in the same moduli space $P_n(X,\beta)$.

Vanishing of descendents of $(2,0)$, $(3,0)$ classes

Remark

By (Hodge) degree reasons if $\alpha \in H^{p,0}(X)$ then $\mathrm{ch}_2(\alpha) = 0$. An easy computation:

$$[L_k^{\mathrm{PT}}, \mathrm{ch}_2(\alpha)] = \frac{(p-1+k)!}{(p-2)!}\mathrm{ch}_{2+k}(\alpha).$$

Hence the conjecture implies the surprising vanishing

$$\langle \mathrm{ch}_k(\alpha) D\rangle_\beta^{X,\mathrm{PT}} = 0 \text{ for every } D \in \mathbb{D}_{\mathrm{PT}}^X.$$

Examples

- For $k=-1$, after setting $\mathrm{ch}_1=0$:

$$L_{-1}^{\mathrm{PT}} = \cancel{R_{-1}} + \cancel{\mathrm{ch}_0(\mathrm{p})R_{-1}}$$

- For $k=0$:

$$L_0^{\mathrm{PT}} = \cancel{R_0} + \cancel{\mathrm{ch}_0(\mathrm{p})\mathrm{ch}_2(c_1)} + \cancel{\frac{1}{24}\mathrm{ch}_0(c_1c_2)\mathrm{ch}_0(\mathrm{p})} + \cancel{\sum_{p_i^L=0}\mathrm{ch}_0(\gamma_i^L\gamma_i^R)}.$$

- Take $X=\mathbb{P}^3$, H, L the classes of hyperplanes and lines, respectively, $\beta=L$. Then $L_1\mathrm{ch}_4(L)$ predicts:

$$4\underbrace{\langle \mathrm{ch}_3(H)\mathrm{ch}_4(L)\rangle}_{\frac{5(q^4-3q^3+3q^2-q)}{4(q+1)}} + 12\underbrace{\langle \mathrm{ch}_5(L)\rangle}_{-\frac{q^4-9q^3+9q^2-q}{6(q+1)}} + 2\underbrace{\langle \mathrm{ch}_2(\mathrm{p})\mathrm{ch}_3(L)\rangle}_{\frac{3}{2}(q^3-q)} = 0$$

Evidence for the conjecture

Theorem (Oblomkov-Okounkov-Pandharipande-M)

If X is a toric 3-fold

$$\langle L_k^{\mathrm{PT}}(D)\rangle_\beta^{X,\mathrm{PT}} = 0$$

for every $D \in \mathbb{D}_{\mathrm{PT}}^{X+}$.

Theorem (M)

If X is the cubic 3-fold and β is the line class

$$\langle L_k^{\mathrm{PT}}(D)\rangle_\beta^{X,\mathrm{PT}} = 0$$

for every $D \in \mathbb{D}_{\mathrm{PT}}^{X}$.

Evidence for the conjecture

Theorem (M)

If S is a simply-connected surface then

$$\langle L_k^{\mathrm{PT}}(D)\rangle_{n,n[\mathbb{P}^1]}^{S\times\mathbb{P}^1,\mathrm{PT}} = 0$$

for every $D \in \mathbb{D}_{\mathrm{PT}}^{S\times\mathbb{P}^1}$.

Toric case

In the toric case the proof follows from 3 key ingredients:

- Virasoro for GW is known (Givental-Teleman theory).
- The stationary GW/PT correspondence is known (Pandharipande-Pixton, Oblomkov-Okounkov-Pandharipande).
- The GW and PT Virasoro operators are intertwined by the GW/PT correspondence:

Theorem (MOOP)

For $k \geq -1$ and $D \in \mathbb{D}_{\mathrm{PT}}^{X+}$ not containing descendents of $(0, p)$ classes we have

$$\mathfrak{C}^{\bullet} \circ L_k^{\mathrm{PT}}(D) = (\imath u)^{-k} L_k^{\mathrm{GW}} \circ \mathfrak{C}^{\bullet}(D).$$

A special case

From now on S is a simply-connected smooth projective surface. We denote by $S^{[n]}$ the Hilbert scheme of points on S parametrizing 0 dimensional subschemes of length n.
A stable pair supported in the curve class $\beta = n[\mathbb{P}^1]$ has Euler characteristic at least n. The stable pairs with minimal Euler characteristic have the form

$$\mathcal{O}_{S\times\mathbb{P}^1} \to \iota_*\mathcal{O}_{\xi\times\mathbb{P}^1}$$

for $\xi \in S^{[n]}$. So we have an identification

$$P_n(S\times\mathbb{P}^1, n[\mathbb{P}^1]) \cong S^{[n]}.$$

The virtual dimension agrees with the true dimension:

$$\int_{n[\mathbb{P}^1]} c_1(S\times\mathbb{P}^1) = 2n = \dim S^{[n]}.$$

Descendents

Definition

Let $\Sigma_n \subseteq S^{[n]} \times S$ be the universal subscheme.
We define descendents on the Hilbert scheme by

$$\mathrm{ch}_k(\gamma) = (\pi_2)_* \left(\mathrm{ch}_k\left(-\mathcal{I}_{\Sigma_n}\right) \cdot \pi_1^*\gamma\right) \in H^*(S^{[n]})$$

for $k \geq 0$, $\gamma \in H^*(S)$.

$$\begin{array}{ccccc} \Sigma_n & \hookrightarrow & S \times S^{[n]} & & \\ & \swarrow^{\pi_1} & & \searrow^{\pi_2} & \\ S & & & & S^{[n]} \end{array}$$

We have:

$$\mathrm{ch}_k^{\mathrm{PT}}(\gamma \times 1) = 0 \text{ and } \mathrm{ch}_k^{\mathrm{PT}}(\gamma \times \mathrm{p}) = \mathrm{ch}_k^{\mathrm{Hilb}}(\gamma).$$

Virasoro operators

Denote by $\mathbb{D}^S$ the algebra of descendents.

- Define derivations $R_k : \mathbb{D}^S \to \mathbb{D}^S$ by their values on the generators:

$$R_k \mathrm{ch}_i(\gamma) = \left(\prod_{j=0}^{k} (i + p - 2 + j) \right) \mathrm{ch}_{k+i}(\gamma)$$

  for γ having Hodge type (p, q).

- Define the operator $T_k : \mathbb{D}^S \to \mathbb{D}^S$ as multiplication by

$$T_k = -\frac{1}{2} \sum_{a+b=k+2} (-1)^{p^L p^R} (a + p^L - 2)!(b + p^R - 2)! \mathrm{ch}_a \mathrm{ch}_b(1) + \frac{1}{12} \sum_{a+b=k} a!b! \mathrm{ch}_a \mathrm{ch}_b (c_1^2 + c_2).$$

Virasoro operators

- $S_k : \mathbb{D}^S \to \mathbb{D}^S$ is given by

$$S_k = (k+1)! \sum_{p_i^L = 0} R_{-1}[\gamma_i^L]\mathrm{ch}_{k+1}(\gamma_i^R).$$

The sum runs over the terms $\gamma_i^L \otimes \gamma_i^R$ in the Kunneth decomposition of the diagonal $\Delta_* 1 \in H^*(S \times S)$ such that $p_i^L = 0$.

- Define

$$L_k^S = R_k + T_k + S_k.$$

Theorem (M)

Let S be simply-connected. For $D \in \mathbb{D}^S$, $k \geq -1$ we have

$$\int_{S^{[n]}} L_k^S D = 0.$$

$H^*(S^{[n]})$

A lot is known about $H^*(S^{[n]})$:

- The Betti numbers of $S^{[n]}$ were determined by Göttsche.
- Nakajima described $\bigoplus_{n\geq 0} H^*(S^{[n]})$ as a module over the Heisenberg algebra.
- The descendents $\mathrm{ch}_k(\gamma)$ generate $H^*(S^{[n]})$ (Li-Qin-Wang).
- Ring structure on $H^*(S^{[n]})$ can be algorithmically described (Ellingsrud-Göttsche-Lehn, Li-Qin-Wang).

Path of the proof

1. The integrals $\int_{S^{[n]}} L_k D$ admit universal formulas.
2. The conjecture behaves well with respect to disjoint unions.
3. If D only has (p, p) descendents then (disconnected) toric surfaces provide enough data to show that the universal formulas vanish.
4. If D has $(0, 2)$, $(2, 0)$ classes we add connected components and replace those classes by $(0, 0)$ and $(2, 2)$ classes.

Universal formulas for integrals

Theorem (EGL, LQW)

The integral

$$\int_{S^{[n]}} \mathrm{ch}_{k_1}(\gamma_1)\dots\mathrm{ch}_{k_m}(\gamma_m)$$

admits a universal formula depending only on $n, k_1, \dots, k_m$ *and (polynomially) on the integrals*

$$\int_S c_1^{\varepsilon_1} c_2^{\varepsilon_2} \prod_{i\in I} \gamma_i.$$

This is done by relating integrals in $S^{[n]}$ to integrals in S^n.

$$\begin{array}{ccc} S^{[n-1,n]} & \xrightarrow{\text{blowup } \Sigma_{n-1}} & S^{[n-1]} \times S \\ \big\downarrow{\scriptstyle n:1} & & \\ S^{[n]} & & \end{array}$$

Universal formulas for Virasoro integrals

Proposition

Let $\gamma_i \in H^{p_i,q_i}(S)$. The integral

$$\int_{S^{[n]}} L_k\left(\mathrm{ch}_{k_1}(\gamma_1)\ldots\mathrm{ch}_{k_m}(\gamma_m)\right)$$

admits a universal formula depending only on $n, k, k_1, \ldots, k_m, p_1, \ldots, p_m$ and (polynomially) on the integrals

$$\int_S c_1^{\varepsilon_1} c_2^{\varepsilon_2} \prod_{i\in I} \gamma_i.$$

Key observation:

$$\sum_{p_i^L=p} \int_S \gamma_i^L \gamma_i^R = \chi(S, \Omega^p) = \begin{cases} \frac{1}{12}\int_S (c_1^2 + c_2) & \text{if } p = 0, 2 \\ \frac{1}{6}\int_S (-c_1^2 + 5c_2) & \text{if } p = 1 \end{cases}$$

Disconnected surfaces

The Virasoro operators are still well defined with disconnected surfaces.If $S = S_1 \sqcup S_2$ then

$$\mathbb{D}^S = \mathbb{D}^{S_1} \otimes \mathbb{D}^{S_2}$$

$$L_k^S = \mathrm{id}_{\mathbb{D}^{S_1}} \otimes L_k^{S_2} + L_k^{S_1} \otimes \mathrm{id}_{\mathbb{D}^{S_2}}$$

$$\int_{S^{[n]}} L_k^S(D_1 \otimes D_2) = \sum_{n_1+n_2=n} \left(\int_{S_1^{[n_1]}} D_1\right)\left(\int_{S_2^{[n_2]}} L_k^{S_2}(D_2)\right) + \left(\int_{S_1^{[n_1]}} L_k^{S_1}(D_1)\right)\left(\int_{S_2^{[n_2]}} D_2\right).$$

Thus: if the Virasoro holds for S_1 and S_2 it also holds for S.

$(1,1)$-classes

Suppose that D has no $(0,2)$ and no $(2,0)$ classes:

$$D = \prod_{i=1}^{s} \mathrm{ch}_{k_i}(1) \prod_{i=1}^{t} \mathrm{ch}_{\ell_i}(\mathrm{p}) \prod_{i=1}^{m} \mathrm{ch}_{m_i}(\gamma_i)$$

where $\gamma_i \in H^{1,1}(X)$.
Then the integral

$$\int_{S^{[n]}} L_k^S(D)$$

depends only on $n, k, s, t, m, k_i, \ell_i, m_i$ and on the data $\left(\binom{m+1}{2} + m + 2\right)$-tuple of rational numbers

$$\left\{\int_S \gamma_i\gamma_j\right\}_{1\leq i\leq j\leq m} \cup \left\{\int_S \gamma_i c_1\right\}_{1\leq i\leq m} \cup \left\{\int_S c_1^2, \int_S c_2\right\}.$$

Zariski density

We know that the previous integral vanishes if S is toric, so it's enough to prove that toric surfaces give enough data points:

Proposition

By varying the (possibly disconnected) toric surface and classes $\gamma_j \in H^2(S)$, the set of possible $\left(\binom{m+1}{2} + m + 2\right)$-tuples

$$\left\{\int_S \gamma_i\gamma_j\right\}_{1\leq i\leq j\leq m} \cup \left\{\int_S \gamma_i c_1\right\}_{1\leq i\leq m} \cup \left\{\int_S c_1^2, \int_S c_2\right\}.$$

is Zariski dense in $\mathbb{Q}^{\binom{m+1}{2}+m+2}$.

Zariski density

Proof.

Start with N disjoint copies of $\mathbb{P}^1 \times \mathbb{P}^1$.Perform M successive toric blow-ups of points in one of the copies in a way that the last m blow-ups have disjoint exceptional divisors $D_1, \dots, D_m$.Pick D_0 in another copy of $\mathbb{P}^1 \times \mathbb{P}^1$. Set

$$\gamma_i = \sum_{j=0}^{m} a_{ij} D_j$$

and vary $a_{ij} \in \mathbb{Q}$.

$$\int_S c_2 = 4N + M \text{ and } \int_S c_1^2 = 8N - M$$
$$\int_S \gamma_i \gamma_j = (-AA^t)_{ij}.$$

□

$(0,2)$ and $(2,0)$-classes

Pick a basis $\alpha_1, \ldots, \alpha_{h^{0,2}} \in H^{0,2}(S)$ and $\beta_1, \ldots, \beta_{h^{0,2}} \in H^{2,0}(S)$ such that

$$\int_S \alpha_i \beta_j = \delta_{ij}.$$

We add new connected components to S

$$T = S \sqcup E_1 \sqcup \ldots \sqcup E_N$$

and replace appearances of α_j, β_j by $(0,0)$ and $(2,2)$ classes supported in the new connected components such that all the integrals appearing in the universal formula agree. Let $\omega = e^{2\pi i/N}$ and

$$\alpha = \sum_{i=0}^{N-1} \omega^i 1_i \in H^0(T;\mathbb{C}) \text{ and } \beta = \frac{1}{N}\sum_{i=0}^{N-1} \omega^{-i} \mathrm{p}_i \in H^4(T;\mathbb{C})$$

satisfy for example

$$\int_S \alpha^j \beta = \delta_{j1} \text{ and } \alpha\gamma = \beta\gamma = 0 \text{ for all } \gamma \in H^{1,1}(S).$$

Thank you for your attention!