Virasoro constraints for sheaf moduli spaces via wall-crossing

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History of Virasoro constraints

- In 1990, Witten proposed a conjecture saying that integrals of ψ-classes in the moduli space of curves $\overline{\mathcal{M}}_{g,n}$ satisfy some relations which completely determine them:

$$L_k(Z) = 0 \quad \text{for } k \geqslant -1\,,$$

where Z is the generating function of these integrals and L_k are differential operators satisfying the Virasoro bracket

$$[L_k, L_\ell] = (\ell - k)L_{k+\ell}\,.$$

- Witten's conjecture was proven in 1992 by Kontsevich. Alternative proofs by Okounkov-Pandharipande and Mirzakhani were found later.
- Eguchi-Hori-Xiong propose in 1997 a generalization to the Gromov-Witten (GW) theory of a target variety X.

History of Virasoro constraints

- In 2006, Maulik-Nekrasov-Okounkov-Pandharipande (MNOP) propose a conjecture connecting Gromov-Witten invariants on 3-folds to Donaldson-Thomas (DT) invariants, defined using the moduli space of ideal sheaves.
- An analog of Virasoro constraints should exist in DT theory! Oblomkov-Okounkov-Pandharipande make a precise conjecture by calculations in $X = \mathbb{P}^3$.
- In 2020, with Oblomkov-Okounkov-Pandharipande we prove that the MNOP correspondence intertwines the GW Virasoro and the DT Virasoro constraints (in stationary regime).
- This proves Virasoro constraints for the DT theory of toric 3-folds with stationary descendents.

History of Virasoro constraints

- In 2020 I used the previous result to prove a version of Virasoro constraints for the Hilbert scheme of points on simply-connected surfaces.
- In 2021 D. van Bree conjectures a generalization of the Hilbert scheme result to moduli spaces of stable sheaves on surfaces.
- Much more general?...

Today

I will explain joint work with A. Bojko and W. Lim containing:

- General formulation of Virasoro for moduli spaces of sheaves and pairs.
- How the Virasoro constraints are naturally formulated using the vertex algebra that D. Joyce introduced to study wall-crossing.
- Virasoro constraints are compatible with wall-crossing.
- A proof of the Virasoro constraints for moduli spaces of stable sheaves on curves and surfaces with $h^{0,1} = h^{0,2} = 0$ (either torsion-free or dimension 1 sheaves).

Moduli spaces of sheaves

We consider moduli spaces M of stable sheaves on a smooth projective variety X (typically of small dimension) such that:

1. There are no strictly semistable sheaves in M.
2. There is an (in principle non-unique) universal sheaf $\mathbb{G}$ in $X \times M$; $\mathbb{G}$ is such that $\mathbb{G}_{|X\times\{[G]\}} \cong G$. Tensoring $\mathbb{G}$ by a line bundle pulled back from M gives another universal sheaf.
3. M admits a 2-term perfect obstruction theory with deformation theory at $[G] \in M$ given by

$$\mathrm{Tan} = \mathrm{Ext}^1(G,G)\,,\ \mathrm{Obs} = \mathrm{Ext}^2(G,G)\,,\ 0 = \mathrm{Ext}^{>2}(G,G)\,.$$

It follows that there is a virtual fundamental class $[M]^{\mathrm{vir}} \in H_\bullet(M)$.

Moduli spaces of pairs

Let V be a fixed sheaf. We also consider moduli spaces of pairs P parametrizing a sheaf F together with a map $V \to F$.

1. There are no strictly semistable pairs in P.
2. There is a unique (!) universal pair $q^*V \to \mathbb{F}$ in $X \times P$.
3. P admits a 2-term perfect obstruction theory with deformation theory at $(V \to F) \in P$ given by

$$\begin{aligned} \mathrm{Tan} &= \mathrm{Ext}^0([V \to F], F)\,,\ \mathrm{Obs} = \mathrm{Ext}^1([V \to F], F)\,, \\ 0 &= \mathrm{Ext}^{>1}([V \to F], F)\,. \end{aligned}$$

It follows that there is a virtual fundamental class $[P]^{\mathrm{vir}} \in H_\bullet(P)$.

Descendents

To get numerical invariants from M we integrate certain natural cohomology classes against the virtual fundamental class.

Definition (Descendent algebra)

Let $\mathbb{D}^X$ be the free (super)commutative $\mathbb{C}$-algebra generated by symbols

$$\mathrm{ch}_i^H(\gamma) \quad \text{for } i \geqslant 0, \gamma \in H^\bullet(X)\,.$$

Definition (Geometric realization of descendents)

Let M be a moduli of sheaves with a universal sheaf $\mathbb{G}$ in $X \times M$. Define the geometric realization morphism $\xi_{\mathbb{G}} \colon \mathbb{D}^X \to H^\bullet(M)$ by

$$\xi_{\mathbb{G}}\left(\mathrm{ch}_i^H(\gamma)\right) = p_*\left(\mathrm{ch}_{i+\dim(X)-s}(\mathbb{G})q^*\gamma\right) \in H^\bullet(M)$$

for $\gamma \in H^{s,t}(X)$. p, q are the projections of the product onto M and X, respectively.

Descendents for pairs

There is an analogous definition for pairs:

Definition (Pair descendent algebra)

Let $\mathbb{D}^{X,\mathrm{pa}} \simeq \mathbb{D}^X \otimes \mathbb{D}^X$ be the free (super)commutative $\mathbb{C}$-algebra generated by symbols

$$\mathrm{ch}_i^{H,\mathcal{V}}(\gamma),\ \mathrm{ch}_i^{H,\mathcal{F}}(\gamma) \quad \text{for } i \geqslant 0, \gamma \in H^\bullet(X)\,.$$

Definition (Geometric realization of pair descendents)

Let P be a moduli of sheaves with a universal pair $q^*V \to \mathbb{F}$ in $X \times P$. Define the geometric realization morphism by

$$\xi_{q^*V\to\mathbb{F}}\left(\mathrm{ch}_i^{H,\mathcal{F}}(\gamma)\right) = p_*\left(\mathrm{ch}_{i+\dim(X)-s}(\mathbb{F})q^*\gamma\right),$$

$$\xi_{q^*V\to\mathbb{F}}\left(\mathrm{ch}_i^{H,\mathcal{V}}(\gamma)\right) = p_*\left(\mathrm{ch}_{i+\dim(X)-s}(q^*V)q^*\gamma\right) = \delta_{i0}\int_X \mathrm{ch}(V)\gamma\,.$$

Virasoro operators

Definition

For $n \geqslant -1$ define the operators $\mathsf{L}_n \colon \mathbb{D}^X \to \mathbb{D}^X$ by $\mathsf{L}_n = \mathsf{R}_n + \mathsf{T}_n$ where:

1. The operator $\mathsf{R}_n \colon \mathbb{D}^X \to \mathbb{D}^X$ is a derivation defined on generators by
$$\mathsf{R}_n \mathrm{ch}_i^H(\gamma) = \left(\prod_{j=0}^{n}(i+j)\right) \mathrm{ch}_{i+n}^H(\gamma).$$

2. The operator $\mathsf{T}_k \colon \mathbb{D}^X \to \mathbb{D}^X$ is the multiplication by the element of $\mathbb{D}^X$ given by
$$\mathsf{T}_n = \sum_{i+j=n} i!j! \sum_s (-1)^{\dim X - p_s^L} \mathrm{ch}_i^H(\gamma_s^L)\mathrm{ch}_j^H(\gamma_s^R)\,,$$
where $\sum_s \gamma_s^L \otimes \gamma_s^R = \Delta_* \mathrm{td}(X)$.

Virasoro operators

They satisfy the Virasoro bracket:

$$[\mathsf{L}_n, \mathsf{L}_m] = (m-n)\mathsf{L}_{n+m}\,.$$

There is also a version $\mathsf{L}_n^{\mathrm{pa}} \colon \mathbb{D}^{X,\mathrm{pa}} \to \mathbb{D}^{X,\mathrm{pa}}$ for pairs. The main difference is in the T_n operator:

$$\mathsf{T}_n^{\mathrm{pa}} = \sum_{i+j=n} i!j! \sum_s (-1)^{\dim X - p_s^L} \mathrm{ch}_i^{H,\mathcal{F}-\mathcal{V}}(\gamma_s^L)\mathrm{ch}_j^{H,\mathcal{F}}(\gamma_s^R)\,.$$

Virasoro constraints for pairs

Conjecture (Virasoro for pairs)

Let P be a moduli space of pairs with universal pair $q^*V \to \mathbb{F}$. For any $D \in \mathbb{D}^{X,\mathrm{pa}}$ and $n \geqslant 0$ we have

$$\int_{[P]^{\mathrm{vir}}} \xi_{q^*V\to\mathbb{F}}\left(\mathsf{L}_n^{\mathrm{pa}}(D)\right) = 0\,.$$

Virasoro constraints for sheaves

Let

$$\mathsf{L}_{\mathrm{inv}} = \sum_{n\geqslant -1} \frac{(-1)^n}{(n+1)!}\mathsf{L}_n\mathsf{R}_{-1}^{n+1}\,.$$

Fact

The geometric realization $\xi_{\mathbb{G}}(\mathsf{L}_{\mathrm{inv}}(D)) \in H^\bullet(M)$ does not depend on the choice of universal sheaf $\mathbb{G}$.

Conjecture (Virasoro for sheaves)

Let M be a moduli space of sheaves. For any $D \in \mathbb{D}^X$ we have

$$\int_{[M]^{\mathrm{vir}}} \mathsf{L}_{\mathrm{inv}}(D) = 0\,.$$

Example – rank 2 sheaves on a curve

Let $M = M_C(2,\Delta)$ be the moduli space of stable bundles on a curve C of genus g with rank 2 and fixed determinant Δ of odd degree; this is a smooth moduli space of dimension $3g-3$.
All integrals of descendents on M can be deduced from integrals of products of certain classes

$$\eta \in H^2(M), \quad \theta \in H^4(M), \quad \zeta \in H^6(M)\,.$$

Thaddeus proved:

$$\int_M \eta^m \theta^k \zeta^p = (-1)^{g-1-p} \frac{m!g!}{(g-p)!} 2^{2g-2-p} \frac{(2^q-2)B_q}{q!}\,,$$

where $m + 2k + 3p = 3g - 3$ and $q = m + p - g + 1$.
The Virasoro constraints for M are equivalent to

$$(g-p)\int_M \eta^m \theta^k \zeta^p = -2m \int_M \eta^{m-1}\theta^{k-1}\zeta^{p+1}\,.$$

Vertex algebras

A vertex algebra is a (graded) vector space $V_\bullet$ together with the following data:

1. A vacuum vector $|0\rangle \in V_\bullet$;
2. A translation operator $T\colon V_\bullet \to V_{\bullet+2}$;
3. A state to field correspondence $Y\colon V_\bullet \to \mathrm{End}(V_\bullet)[[z, z^{-1}]]$ denoted by

$$Y(v,z)u = \sum_{n\in\mathbb{Z}} v_n u\, z^{-n-1}\,.$$

   Equivalently, Y carries the information of the infinite collection of products $v \otimes u \mapsto v_n u$.

This data has to satisfy some compatibility axioms (vacuum, translation and locality axioms).

Conformal element

An element $\omega \in V_\bullet$ is called a conformal element if the corresponding fields $\mathrm{L}_n = \omega_{n+1} \in \mathrm{End}(V_\bullet)$ defined by

$$Y(\omega, z) = \sum_{n\in\mathbb{Z}} \mathrm{L}_n z^{-n-2}$$

satisfy

$$\mathrm{L}_{-1} = T$$

$$[\mathrm{L}_n, \mathrm{L}_m] = (n-m)\mathrm{L}_{m+n} + \delta_{m+n,0}\, c\, \frac{m^3-m}{12}\,\mathrm{id}$$

where $c \in \mathbb{C}$ is a constant called the central charge of ω.

Borcherds Lie algebra

Given a Lie algebra, Borcherds showed that the quotient

$$\check{V}_\bullet = V_\bullet / T(V_\bullet)$$

has the structure of a Lie algebra with bracket given by

$$[\overline{v}, \overline{u}] = \overline{v_0 u} \,.$$

This bracket lifts to $\check{V}_\bullet \otimes V_\bullet \to V_\bullet$ for which we still use the same notation:

$$[\overline{u}, v] = u_0 v \,.$$

Physical states

There is a vertex algebra notion of physical states that roughly corresponds to elements of $V_\bullet$ or $\check{V}_\bullet$ that satisfy Virasoro constraints:

$$P_i = \{v \in V_\bullet : \mathrm{L}_n(v) = \delta_{n0} iv \,, n \geqslant 0\} \subseteq V_\bullet \,,$$
$$\check{P}_0 = P_1 / T(P_0) \subseteq \check{V}_\bullet \,.$$

Proposition

Under some conditions, $\overline{u} \in \check{P}_0$ if and only if

$$0 = [\overline{u}, \omega] = \sum_{n \geqslant -1} \frac{(-1)^n}{(n+1)!} T^{n+1} \mathrm{L}_n(u) \,.$$

Wall-crossing compatibility

Proposition

1. *The subspace $\check{P}_0 \subseteq \check{V}_\bullet$ is a Lie subalgebra, i.e.*

$$\overline{u}, \overline{v} \in \check{P}_0 \Rightarrow [\overline{u}, \overline{v}] \in \check{P}_0 \,.$$

2. *The subspace $P_0 \subseteq V_\bullet$ is a Lie algebra subrepresentation of $\check{P}_0 \subseteq \check{V}_\bullet$, i.e.*

$$\overline{u} \in \check{P}_0, \ v \in P_0 htarrow [\overline{u}, v] \in P_0 \,.$$

This proposition will translate to a compatibility between the Virasoro constraints and wall-crossing in moduli spaces of sheaves!

Joyce's geometric vertex algebra

Let $\mathcal{M}_X$ be the (higher) stack parametrizing objects in $D^b(X)$, i.e. perfect complexes. If this is scary replace it by the topological version

$$\mathcal{M}_X^{\mathrm{top}} = \mathrm{Maps}_{C^0}(X^{\mathrm{an}}, BU \times \mathbb{Z})\,.$$

Joyce defined a vertex algebra structure on the homology

$$V_\bullet = H_\bullet(\mathcal{M}_X)\,.$$

The translation operator T is related to the $B\mathbb{G}_m$ action $B\mathbb{G}_m \times \mathcal{M}_X \to \mathcal{M}_X$. The state-to-field is given explicitly by

$$Y(v,z)u = (-1)^{\chi(\alpha,\beta)} z^{\chi_{\mathrm{sym}}(\alpha,\beta)} \Sigma_*\Big((e^{zT} \boxtimes \mathrm{id})\big(c_{z^{-1}}(\Theta) \cap v \boxtimes u\big)\Big)\,.$$

Θ is a complex on $\mathcal{M}_X \times \mathcal{M}_X$ related to the deformation theory of sheaves.

Joyce's geometric vertex algebra

There is a pair version. Let $\mathcal{P}_X = \mathcal{M}_X \times \mathcal{M}_X$ and

$$V_\bullet^{\mathrm{pa}} = H_\bullet(\mathcal{P}_X)\,.$$

The state-to-field is defined by modifying

$$Y(v,z)u = (-1)^{\chi^{\mathrm{pa}}(\alpha,\beta)} z^{\chi^{\mathrm{pa}}_{\mathsf{sym}}(\alpha,\beta)} \Sigma_*\Big((e^{zT} \boxtimes \mathrm{id})\big(c_{z^{-1}}(\Theta^{\mathrm{pa}}) \cap v \boxtimes u\big)\Big)\,.$$

Θ^{pa} is a complex on $\mathcal{P}_X \times \mathcal{P}_X$ related to the deformation theory of pairs.

Joyce invariant classes

If M is a moduli of sheaves with universal sheaf $\mathbb{G}$, by the universal property of $\mathcal{M}_X$ there is a map $f_{\mathbb{G}}\colon M \to \mathcal{M}_X$. Define

$$[M]^{\mathrm{vir}}_{\mathbb{G}} = (f_{\mathbb{G}})_*[M]^{\mathrm{vir}} \in V_\bullet = H_\bullet(\mathcal{M}_X)\,,$$

$$[M]^{\mathrm{vir}} = \overline{[M]^{\mathrm{vir}}_{\mathbb{G}}} \in \widehat{V}_\bullet = V_\bullet / T(V_\bullet)\,.$$

Theorem (J. Gross)

The cohomology $H^\bullet(\mathcal{M}_X)$ is closely related to $\mathbb{D}^X$. Integrating descendents against the virtual fundamental class is identified with pairing between a homology and cohomology class in $\mathcal{M}_X$, i.e.

$$\int_{[M]^{\mathrm{vir}}} \xi_{\mathbb{G}}(D) = \langle [M]^{\mathrm{vir}}_{\mathbb{G}}, D \rangle\,.$$

Wall-crossing

Moduli spaces of sheaves often depend on a choice of a stability parameter μ. Denote by $M_\alpha(\mu)$ the moduli of μ-stable sheaves with Chern character equal to α. Wall-crossing is about trying to understand how $M_\alpha(\mu)$ and the corresponding numerical invariants change when μ changes.

Theorem (Joyce)

For every α there exist classes

$$[M_\alpha(\mu)]^{\mathsf{Jo}} \in \check{V}_\bullet$$

even if $M_\alpha(\mu)$ contains strictly semistable sheaves. When there are no strictly semistables, $[M_\alpha(\mu)]^{\mathsf{Jo}} = [M_\alpha(\mu)]^{\mathsf{vir}}$.

Wall-crossing

Theorem (Joyce)

Let μ, τ be two stability conditions. Then

$$[M_\alpha(\mu)]^{\mathrm{Jo}} = \sum_{\alpha_1+\ldots+\alpha_l=\alpha} U(\alpha_1, \ldots, \alpha_l; \mu, \tau) \times \Big[\ldots \big[[M_{\alpha_1}(\tau)]^{\mathrm{Jo}}, [M_{\alpha_2}(\tau)]^{\mathrm{Jo}} \big], \ldots, [M_{\alpha_l}(\tau)]^{\mathrm{Jo}} \Big]$$

in $\check{V}_\bullet$, where $U(\alpha_1, \ldots, \alpha_l; \mu, \tau)$ are some combinatorial coefficients.

Gross' isomorphism

Theorem (Gross+BLM)

Under some conditions, the vertex algebras $V_\bullet$, $V_\bullet^{\mathrm{pa}}$ are isomorphic to the lattice vertex algebras

$$V_\bullet \cong \mathbb{C}[K^0_{sst}(X)] \otimes SSym\big[H^\bullet(X)[t]\big]$$
$$V_\bullet^{\mathrm{pa}} \cong \mathbb{C}[K^0_{sst}(X)^{\oplus 2}] \otimes SSym\big[H^\bullet(X)^{\oplus 2}[t]\big]$$

constructed from the bilinear forms on $H^\bullet(X)$ and $H^\bullet(X)^{\oplus 2}$ given by

$$\chi_{sym}(\gamma,\delta) = \chi(\gamma,\delta) + \chi(\delta,\gamma)$$
$$\chi^{\mathrm{pa}}_{sym}((\gamma_1,\gamma_2),(\delta_1,\delta_2)) = \chi(\gamma_2-\gamma_1,\delta_2) + \chi(\delta_2-\delta_1,\gamma_2)$$

where

$$\chi(\gamma,\delta) = \int_X (-1)^{\lfloor \frac{\deg\gamma}{2} \rfloor}\gamma\cdot\delta\cdot\mathrm{td}(X).$$

Conformal element

A lattice vertex algebra constructed from a bilinear pairing Q comes naturally equipped with a conformal element provided we have two things:

1. Q is non-degenerate. In our case, $\chi^{\mathrm{pa}}_{\mathsf{sym}}$ is non-degenerate but χ_{sym} is not in general.
2. We are given an isotropic decomposition of the fermionic part, which in our case is $H^{\mathrm{odd}}(X)$.

Assumption (†)

$$H^{p,q}(X) = 0 \text{ if } |p-q| > 1\,.$$

Holds for curves and surfaces with $h^{0,2} = 0$.

In this case we have an isotropic decomposition

$$H^{\mathrm{odd}}(X) = H^{\bullet,\bullet+1}(X) \oplus H^{\bullet+1,\bullet}(X)\,.$$

Virasoro fields

Under (†), there is a natural conformal element ω on $V_\bullet^{\mathrm{pa}}$ and corresponding Virasoro fields $\mathrm{L}_n \colon V_\bullet^{\mathrm{pa}} \to V_\bullet^{\mathrm{pa}}$.

Theorem (Bojko-Lim-M)

Assume (†). Under the duality between $V_\bullet^{\mathrm{pa}}$ and $\mathbb{D}^{X,\mathrm{pa}}$, the Virasoro fields L_n induced by ω are dual to the pair Virasoro operators $\mathsf{L}_n^{\mathrm{pa}}$ defined in the algebra of descendents $\mathbb{D}^{X,\mathrm{pa}}$.

Virasoro fields

Corollary (Bojko-Lim-M)

1. *A moduli of sheaves M satisfies the sheaf Virasoro constraints if and only if*

$$[M]^{\mathrm{vir}} \in \check{P}_0$$

is a physical state.

2. *A moduli of pairs P with universal pair $q^*V \to \mathbb{F}$ satisfies the pair Virasoro constraints if and only if*

$$[P]^{\mathrm{vir}}_{q^*V\to\mathbb{F}} \in P_0^{\mathrm{pa}}$$

is a physical state.

Corollary

The Virasoro constraints are compatible with wall-crossing.

Results

Theorem (Bojko-Lim-M)

The Virasoro constraints hold for the following moduli spaces:

1. *Moduli spaces of stable bundles on curves* $M_C(r,d)$*;*
2. *Moduli spaces of stable torsion-free sheaves* $M_S^H(r,\beta,n)$ *on surfaces* S *with* $h^{0,1}=h^{0,2}=0$ *and a polarization* H*;*
3. *Moduli spaces of stable 1 dimensional sheaves* $M_S^H(\beta,n)$ *on surfaces* S *with* $h^{0,1}=h^{0,2}=0$ *and a polarization* H *(assuming a technical condition necessary for the proof of the relevant wall-crossing formula).*

Sketch of proof

I will focus on the torsion-free cases (1 and 2).

1. Let P^t_α be the moduli spaces of Bradlow pairs $\mathcal{O}_X \to F$, depending on a parameter $0 < t < \infty$; we will prove by induction on $\mathrm{rk}(\alpha)$ that M_α satisfies the sheaf Virasoro constraints and P^t_α satisfies the pair Virasoro constraints.
2. The base case is when $\mathrm{rk}(\alpha) = 1$. In the case of curves, it amounts to proving Virasoro for the symmetric powers of a curve, which can be done directly. For surfaces, it reduces to proving Virasoro constraints for Hilbert scheme of points on S, which was done previously.
3. For $s \gg 1$ and $\mathrm{rk}(\alpha) > 1$ the moduli space P^s_α becomes empty. The wall-crossing formula between s-stability and t-stability writes $[P^t_\alpha]^{\mathrm{vir}}_{\mathcal{O}\to\mathbb{F}}$ in terms of iterated brackets involving lower rank classes.

Sketch of proof

4. By induction and the compatibility between wall-crossing and Virasoro, P^t_α satisfies the pair Virasoro constraints.
5. If M_α doesn't have strictly semistables and $0 < t \ll 1$ then $P^t_\alpha \to M_\alpha$ is a projective bundle.
6. The projective bundle structure can be used to prove that if P^t_α satisfies the pair Virasoro then M_α satisfies the sheaf Virasoro.

Thanks!