Virasoro constraints for moduli spaces of sheaves

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Enumerative geometry

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The problem has an interesting story and a complete (recursive) answer was given by Kontsevich (1994).

$$N_1 = 1\,, \quad N_2 = 1\,, \quad N_3 = 12 \text{ (Steiner, 1848)}$$
$$N_4 = 620 \text{ (Zeuthen, 1873)}\,, \quad N_5 = 87304 \text{ (Ran, 1989)}$$
$$N_6 = 6312976, N_7 = 14616808192, \ldots \text{ (Kontsevich, 1994)}$$

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Enumerative geometry and moduli spaces

- To get finitely many objects we need to impose the correct amount of conditions on the objects, which amounts to choose a naturally defined cohomology class $D \in H^{2\text{vdim}}(M)$. The enumerative invariants are given by

$$\boxed{\int_{[M]^{\text{vir}}} D := \langle D, [M]^{\text{vir}} \rangle \in \mathbb{Q}\,.}$$

E.g. let D_i be the locus where the curves pass through a fixed point q_i in the plane and

$$D = \text{PD}([D_1]) \dots \text{PD}([D_{3d-1}]).$$

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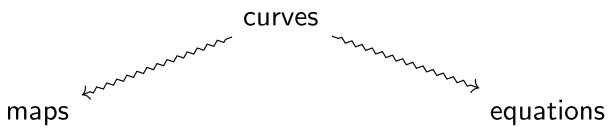
Question

How to compute the enumerative invariants of a moduli space? What structural properties do they have?

Counting curves

Let X be a smooth compact variety (e.g. $X = \mathbb{CP}^2$ the projective plane). How to define a moduli space to count curves on X?

Counting curves



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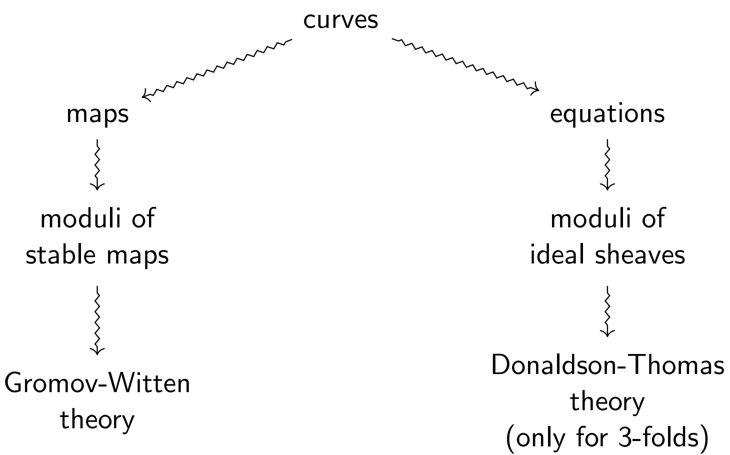

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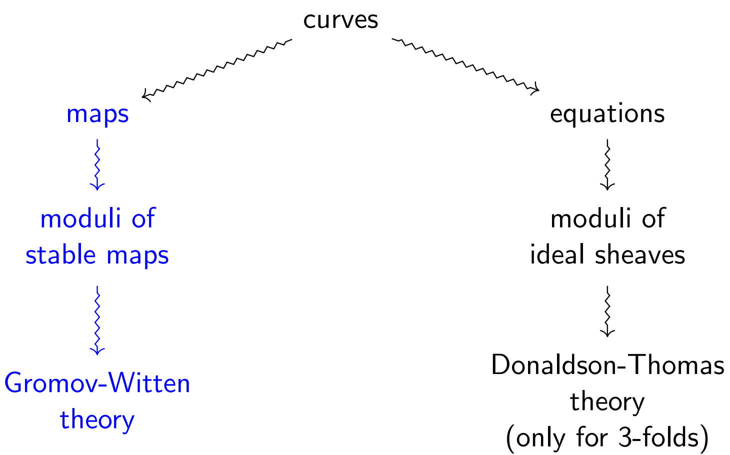

Stable maps and Gromov-Witten theory

The moduli space of stable maps is

$$\overline{M}_{g,m}(X,\beta) = \Big\{ f\colon C \to X \mid C \text{ nodal curve of genus } g \\ p_1,\dots,p_m \in C^{\text{smooth}}, \\ \beta = f_*[C],\ \#\mathrm{Aut}(f) < \infty \Big\}$$

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Gromov-Witten invariants are defined by integrating certain cohomology classes in $\overline{M}_{g,m}(X,\beta)$ called descendents:

$$\boxed{\int_{[\overline{M}_{g,m}(X,\beta)]^{\text{vir}}} \psi_1^{k_1}\mathrm{ev}_1^*(\gamma_1)\ldots\psi_m^{k_m}\mathrm{ev}_m^*(\gamma_m) \in \mathbb{Q}}$$

E.g. $\quad N_d = \displaystyle\int_{[\overline{M}_{0,3d-1}(\mathbb{CP}^2,d)]^{\text{vir}}} \mathrm{ev}_1^*(\mathrm{pt})\ldots\mathrm{ev}_{3d-1}^*(\mathrm{pt}).$

Gromov-Witten theory of the point

Even the Gromov-Witten theory of a point is highly non-trivial as it amounts to study

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We can compute these integrals thanks to a striking prediction due to Witten (90) and proven by Kontsevich (92).

2d gravity

$\rightsquigarrow$

integration over

(∞-dimensional)

space of metrics on surfaces

2d gravity

↓

integration over
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space of metrics on surfaces

↙ ↘

holomorphic curves

↓

GW of the point

triangulations of surfaces

↓

matrix models

2d gravity

⇝

integration over
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holomorphic curves ⇝ GW of the point

triangulations of surfaces ⇝ matrix models

KdV/Virasoro constraints? $\Longleftarrow$ KdV/Virasoro constraints

Witten's conjecture

Define the generating function

$$F(t_0, t_1, t_2, \ldots) = \sum_{g,m \geqslant 0} u^{2g-2} \sum_{k_1,\ldots,k_m} \frac{t_{k_1} \ldots t_{k_m}}{m!} \int_{\overline{M}_{g,m}} \psi_1^{k_1} \ldots, \psi_m^{k_m}$$

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and the differential operators L_n for $n \geqslant -1$ in the variables $T_{2i+1} = t_i/(2i+1)!!$.

$$L_n = \frac{1}{4} \sum_{k+l=2n} \frac{\partial^2}{\partial T_k \partial T_l} + \frac{1}{2} \sum_{k \geqslant 0} (2k+1) T_{2k+1} \frac{\partial}{\partial T_{2k+2n+1}} - \frac{1}{2u^2} \frac{\partial}{\partial T_{2n+3}} + \frac{\delta_{n,-1} T_1^2}{4} + \frac{\delta_{n,0}}{16}$$

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Theorem (Conjecture by Witten (90), proof by Kontsevich (92))

$$L_n \exp(F) = 0 \quad \text{for every } n \geqslant -1 \,.$$

Virasoro constraints in Gromov-Witten theory

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Known in two large families:

- When X is a curve, by work of Okounkov-Pandharipande (03).
- When X is toric, by work of Givental (01) or more generally when X is semisimple by Teleman (07) classification theorem .

curves

maps

moduli of
stable maps

Gromov-Witten
theory

equations

moduli of
ideal sheaves

Donaldson-Thomas
theory
(only for 3-folds)

Ideal sheaves and DT invariants

Let X be a 3-fold and

$$I_n(X,\beta) = \Big\{ I_Z \colon Z \subseteq X \text{ 1 dimensional subscheme}, \\ [Z] = \beta, \chi(\mathcal{O}_Z) = n \Big\}.$$

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There is a universal subscheme $\mathcal{Z} \subseteq I_n(X,\beta) \times X$. For $k \geqslant 0, \gamma \in H^\bullet(X)$ define sheaf theoretical descendents:

$$\boxed{\mathrm{ch}_k(\gamma) = p_*\big(\mathrm{ch}_k(I_{\mathcal{Z}})q^*\gamma\big) \in H^\bullet(I_n(X,\beta))\,.}$$

Define the Donaldson-Thomas invariants of X by

$$\boxed{\int_{[I_n(X,\beta)]^{\mathrm{vir}}} \mathrm{ch}_{k_1}(\gamma_1)\ldots\mathrm{ch}_{k_m}(\gamma_m) \in \mathbb{Q}\,.}$$

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Conjecture (Maulik-Nekrasov-Okounkov-Pandharipande, 06)

There are universal formulas expressing the DT invariants of a 3-fold X in terms of its GW invariants and vice-versa.

World of sheaves

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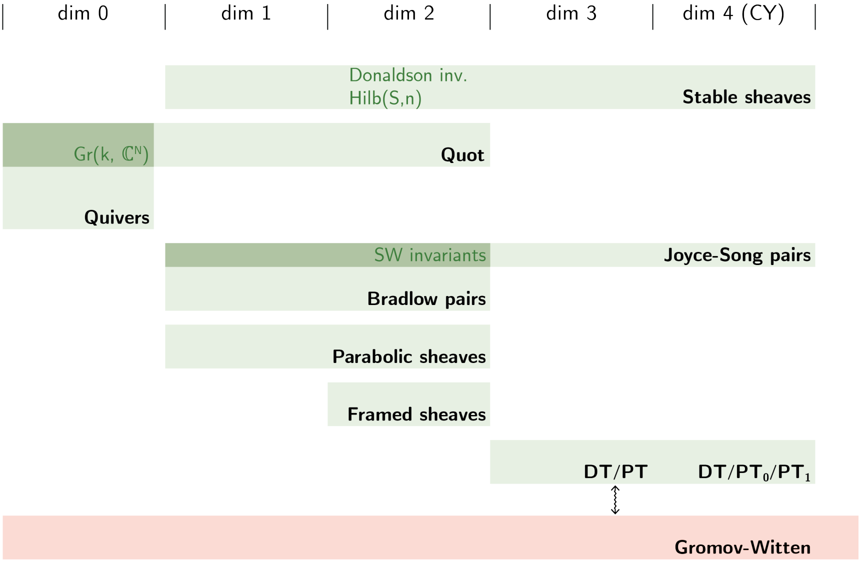

Virasoro constraints

Definition (Descendent algebra)

Let $\mathbb{D}^X$ be the free (super)commutative $\mathbb{C}$-algebra generated by symbols

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When we have a moduli space of sheaves M on X with a universal sheaf $\mathbb{F}$ on the product $M \times X$ we can realize $\mathrm{ch}_k^{\mathrm{H}}(\gamma)$ as

$$\boxed{\mathrm{ch}_k^{\mathrm{H}}(\gamma) \mapsto p_*\left(\mathrm{ch}_{k+\dim(X)-s}(\mathbb{F})q^*\gamma\right) \in H^\bullet(M)}$$

for $\gamma \in H^{s,t}(X)$.

We get numerical invariants

$$\boxed{\int_{[M]^{\mathrm{vir}}} \mathrm{ch}_{k_1}^{\mathrm{H}}(\gamma_1)\ldots\mathrm{ch}_{k_m}^{\mathrm{H}}(\gamma_m) \in \mathbb{Q}\,.}$$

Virasoro operators

Definition

For $n \geqslant -1$ define the operators $\mathsf{L}_n : \mathbb{D}^X \to \mathbb{D}^X$ by

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1. The operator $\mathsf{R}_n \colon \mathbb{D}^X \to \mathbb{D}^X$ is a derivation defined on generators by

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2. The operator $\mathsf{T}_n \colon \mathbb{D}^X \to \mathbb{D}^X$ is multiplication by

$$\boxed{\mathsf{T}_n = \sum_{i+j=n} i!j!\sum_s(-1)^{\dim X - p_s^L}\mathrm{ch}_i^{\mathrm{H}}(\gamma_s^L)\mathrm{ch}_j^{\mathrm{H}}(\gamma_s^R)\,.}$$

Virasoro constraints

The operators define a representation of half of the Virasoro Lie algebra:

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Conjecture (Bojko-Lim-M, 22)

Let M be a moduli space of sheaves. For any $D \in \mathbb{D}^X$ we have

$$\int_{[M]^{\mathrm{vir}}} \mathsf{L}_{\mathrm{wt}_0}(D) = 0$$

where

$$\mathsf{L}_{\mathrm{wt}_0} = \sum_{n \geqslant -1} \frac{(-1)^n}{(n+1)!} \mathsf{L}_n \mathsf{R}_{-1}^{n+1}\,.$$

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- *For the Hilbert scheme of points on a simply-connected surface.* [M, 20]
- *For the moduli spaces of stable torsion-free sheaves on curves and surfaces with* $h^{0,1}(S) = h^{0,2}(S) = 0$. [Bojko-Lim-M, 22]
- *For the moduli spaces of Bradlow pairs on curves and surfaces with* $h^{0,1}(S) = h^{0,2}(S) = 0$. [BLM]
- *For the moduli spaces of 1-dimensional sheaves on surfaces with* $h^{0,1}(S) = h^{0,2}(S) = 0$, *assuming a conjectural wall-crossing formula.* [BLM]

Toric 3-folds and Hilbert scheme

- For toric 3-folds the proof uses (new but based on previous work by many people) very explicit formulas for the DT/GW correspondence in the stationary regime to transport the Virasoro constraints in GW theory to DT theory.

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- For Hilbert schemes of points on a surface S the proof uses the 3-fold result applied to $\mathbb{P}^1 \times S$ when S is toric and a bootstrapping argument based on ideas of Ellingsrud-Göttsche-Lehn (99).
- Afterwards, van Bree (21) suggested a generalization to moduli spaces of stable sheaves on surfaces and gave strong numerical evidence.

The vertex algebra story

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- Moduli spaces of sheaves define a class $[M]^{\text{vir}} \in \check{V}_\bullet$.
- Joyce (21) shows that wall-crossing formulas can be expressed using the Lie bracket on $\check{V}_\bullet$ (proved in some cases, conjectural in others).

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- Virasoro constraints $\Leftrightarrow$ $[M]^{\mathrm{vir}}$ is a physical state in $\check{V}_\bullet$.
- The Virasoro constraints are compatible with wall-crossing.
- With the wall-crossing compatibility we prove Virasoro constraints in new cases using an inductive rank reduction argument.

Thank you for listening