The cohomology ring of moduli spaces of 1-dimensional sheaves on $\mathbb{P}^2$

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Gopakumar–Vafa invariants

Let X be the quintic 3-fold.

$$\mathrm{GW}^X_{g=0,d=1} = 2875$$
$$\mathrm{GW}^X_{g=0,d=2} = \frac{4876875}{8} = 609250 + \frac{2875}{2^3}.$$

Problem

How to define the "true curve counts" $n^X_{g=0,d=1} = 2875$, $n^X_{g=0,d=2} = 609250$ intrinsically, in a way that they are obviously integers?

Proposal by Maulik–Toda (ideas of Gopakumar–Vafa, Katz, Kiem–Li, Osono–Saito–Takahash, ...): use moduli spaces of 1-dimensional sheaves.

Moduli stacks of 1-dimensional sheaves

Given a sheaf F on $\mathbb{P}^2$ with 1-dimensional support, its slope is

$$\mu(F) = \frac{\chi(F)}{d(F)} \in \mathbb{Q}$$

where $d(F) = c_1(F) \in H^2(\mathbb{P}^2) \simeq \mathbb{Z}$ is the degree of the curve where F is supported. A sheaf is (semi)stable if

$$\mu(G)(\leqslant)\mu(F) \text{ for every } G \subsetneq F \,.$$

Let

$$\mathfrak{M}_{d,\chi} \longrightarrow M_{d,\chi}$$

be the moduli stack and coarse moduli space of semistable sheaves on $\mathbb{P}^2$ with $d(F) = d$, $\chi(F) = \chi$.

Basic properties

1. The coarse moduli space $M_{d,\chi}$ is projective and irreducible, of dimension
$$\dim M_{d,\chi} = d^2 + 1\,.$$
2. For any d, χ, the stack $\mathfrak{M}_{d,\chi}$ is smooth of dimension
$$\dim \mathfrak{M}_{d,\chi} = d^2\,.$$
3. When $\gcd(d, \chi) = 1$, all the semistable sheaves parametrized by $\mathfrak{M}_{d,\chi}$ or $M_{d,\chi}$ are automatically stable, the coarse moduli space $M_{d,\chi}$ is smooth, and
$$\mathfrak{M}_{d,\chi} \simeq M_{d,\chi} \times B\mathbb{G}_m\,.$$
4. Twisting by line bundles and duality produces isomorphisms
$$\mathfrak{M}_{d,\chi} \xrightarrow{\sim} \mathfrak{M}_{d,\chi+kd} \qquad M_{d,\chi} \xrightarrow{\sim} M_{d,\chi+kd}$$
$$\mathfrak{M}_{d,\chi} \xrightarrow{\sim} \mathfrak{M}_{d,-\chi} \qquad M_{d,\chi} \xrightarrow{\sim} M_{d,-\chi}$$

Hilbert–Chow morphism

There is a morphism

$$h\colon M_{d,\chi} \to |\mathcal{O}_{\mathbb{P}^2}(d)|$$

that sends a 1-dimensional sheaf to its (fitting) support.
A generic point $[C] \in |\mathcal{O}_{\mathbb{P}^2}(d)|$ corresponds to a smooth curve C of genus

$$g = \frac{(d-1)(d-2)}{2}\,,$$

and the fiber over $[C]$ is

$$h^{-1}([C]) \simeq \mathrm{Jac}(C)\,.$$

Note:

$$|\mathcal{O}_{\mathbb{P}^2}(d)| \simeq \mathbb{P}^{\frac{d(d+3)}{2}}\,.$$

Examples

1. For $d = 1, 2$ and any χ, the Hilbert–Chow morphism is an isomorphism

$$M_{1,\chi} \xrightarrow{\sim} |\mathcal{O}_{\mathbb{P}^2}(1)| \simeq \mathbb{P}^2$$
$$M_{2,\chi} \xrightarrow{\sim} |\mathcal{O}_{\mathbb{P}^2}(2)| \simeq \mathbb{P}^5$$

2. For $d = 3$, the Hilbert–Chow morphism identifies $M_{3,1}$ with the universal cubic:

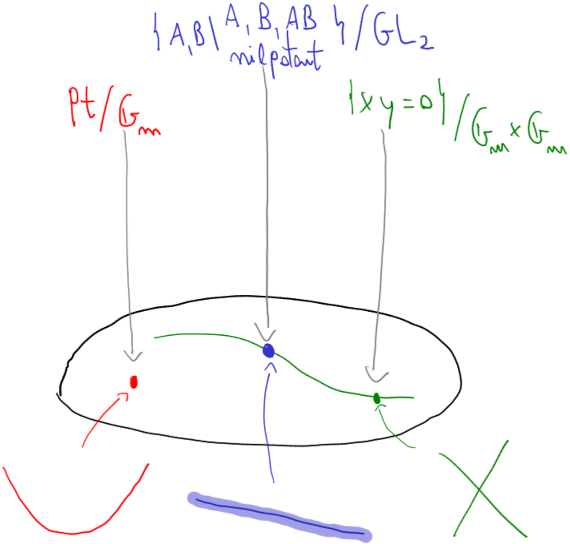
$$|\mathcal{O}_{\mathbb{P}^2}(3)| \times \mathbb{P}^2 \supseteq \mathcal{C}_3 \simeq M_{3,1} \xrightarrow{h} |\mathcal{O}_{\mathbb{P}^2}(3)| \simeq \mathbb{P}^9 .$$

In other words,

$$h^{-1}([E]) \simeq E .$$

Examples

$$\mathcal{M}_{2,0} \longrightarrow M_{2,0} \simeq |\mathcal{O}(2)| \simeq \mathbb{P}^5$$



Gopakumar–Vafa/Gromov–Witten, genus 0

Theorem (Toda+Konishi, conjectured by Katz)

We have $n^{K\mathbb{P}^2}_{g=0,d} = (-1)^{d^2+1} e(M_{d,\chi})$ *for any* χ *coprime with* d, *i.e.*

$$\mathrm{GW}^{K\mathbb{P}^2}_{g=0,d} = \sum_{d=kd'} \frac{(-1)^{(d')^2+1}}{k^3} e(M_{d',\chi})\,.$$

In particular, $e(M_{d,\chi})$ *does not depend on* χ.

- The same holds for an arbitrary χ (not necessarily coprime with d) if we replace the Euler characteristic $e(M_{d,\chi})$ by the intersection Euler characteristic.
- To get higher genus Gopakumar–Vafa/Gromov–Witten invariants we need to introduce the perverse filtration.

Perverse filtration

There is a filtration on the intersection cohomology of $M_{d,\chi}$ associated to the Hilbert–Chow morphism $h\colon M_{d,\chi} \to |\mathcal{O}_{\mathbb{P}^2}(d)|$:

$$P_0 IH^*(M_{d,\chi}) \subseteq P_1 IH^*(M_{d,\chi}) \subseteq \ldots \subseteq P_{2g} IH^*(M_{d,\chi}) = IH^*(M_{d,\chi}) .$$

Theorem (de Cataldo–Migliorini)

Let $f : X \to Y$ be a morphism between smooth and proper varieties with equidimensional fibers. Denote by $L\colon H^*(X) \to H^{*+2}(X)$ the operator of multiplication by $f^*\eta$ where η is an ample class in Y. Then the perverse filtration associated to f is

$$P_k H^m(X) = \sum_{i \geqslant 1} \ker(L^{\dim Y + k + i - m}) \cap \operatorname{im}(L^{i-1}) \cap H^m(X) .$$

χ-independence

Theorem (Maulik–Shen)

Given any χ, χ', there is a natural isomorphism of graded vector spaces

$$IH^*(M_{d,\chi}) \simeq IH^*(M_{d,\chi'})$$

that respects the perverse filtrations.

Let

$$\Omega_d(q,t) = (-1)^{d^2+1} q^{-g} t^{-b} \sum_{i,j \geqslant 0} \dim \mathrm{Gr}_i^P IH^{i+j}(M_{d,\chi}) q^i t^j$$

This encodes (refined) Gopakumar–Vafa invariants of $K\mathbb{P}^2$, as defined by Maulik–Toda. It is a Laurent polynomial symmetric under $q \leftrightarrow q^{-1}$ and $t \leftrightarrow t^{-1}$ by Hard Lefschetz symmetries. Notation: $\mathrm{Gr}_i^P = P_i/P_{i-1}$.

Gopakumar–Vafa/Gromov–Witten correspondence

Conjecture (GV/GW)

We have

$$\exp\left(\sum_{g,d} \mathrm{GW}_{g,d}^{K\mathbb{P}^2} u^{2g-2} Q^d\right) = \mathrm{PE}\left(-\frac{q}{(1-qt)(1-q/t)} \sum_{d\geqslant 1} \Omega_d(q,t) Q^d\right)$$

after setting $t = 1$, $q = e^{iu}$.

Above, PE is the plethystic exponential:

$$\mathrm{PE}(f(q,t,Q)) = \exp\left(\sum_{k\geqslant 1} \frac{1}{k} f(q^k, t^k, Q^k)\right)$$

Tautological classes

Goal: describe the rings in terms of generators and relations.

Definition

Let $\mathcal{F}$ be the universal sheaf on $\mathfrak{M}_{d,\chi} \times \mathbb{P}^2$. Let p, q be the projections of $\mathfrak{M}_{d,\chi} \times \mathbb{P}^2$ onto $\mathfrak{M}_{d,\chi}$ and $\mathbb{P}^2$, respectively. We define for $k \geqslant 0, j = 0, 1, 2$

$$c_k(j) = p_*\big(\mathrm{ch}_{k+1}(\mathcal{F}) q^* H^j\big) \in H^{2k+2j-2}(\mathfrak{M}_{d,\chi})\,.$$

If $\gcd(d,\chi) = 1$, the coarse moduli space $M_{d,\chi}$ also has a universal sheaf, but it is not unique! We choose a normalized universal sheaf by requiring that $c_1(1) = 0$ in $H^2(M_{d,\chi})$.

Remark

The class $c_2(0) \in H^2(M_{d,\chi})$ is relatively ample with respect to the Hilbert–Chow map. The class $c_0(2) \in H^2(M_{d,\chi})$ is the pullback of an ample class from $|\mathcal{O}_{\mathbb{P}^2}(d)|$.

Tautological generation

We have algebra homomorphisms

$$\mathbb{D} := \mathbb{Q}[c_0(2), c_1(1), c_2(0), c_1(2), c_2(1), c_3(0), \ldots] \to H^*(\mathfrak{M}_{d,\chi})$$

$$\widetilde{\mathbb{D}} := \mathbb{D}/\langle c_1(1)\rangle \to H^*(M_{d,\chi})$$

Theorem (Pi–Shen, KLMP)

The homomorphisms above are surjective, i.e. $H^*(\mathfrak{M}_{d,\chi})$ and $H^*(M_{d,\chi})$ are generated as algebras by tautological classes. More precisely, $H^*(\mathfrak{M}_{d,\chi})$ is generated by the tautological classes of (algebraic) degree $\leqslant d$ and $H^*(M_{d,\chi})$ is generated by tautological classes of degree $\leqslant d-2$.

Problem

Describe the ideal of relations, i.e.

$$\ker\left(\mathbb{D} \to H^*(\mathfrak{M}_{d,\chi})\right), \qquad \ker\left(\widetilde{\mathbb{D}} \to H^*(M_{d,\chi})\right).$$

(Generalized) Mumford relations

Proposition

Let F, F' be semistable sheaves of type (d, χ) and (d', χ'), respectively. If

$$\frac{\chi'}{d'} < \frac{\chi}{d} < \frac{\chi'}{d'} + 3\,,$$

then

$$\mathrm{Hom}(F, F') = \mathrm{Ext}^2(F, F') = 0\,.$$

Proof.

A map from a semistable object to another semistable object with smaller slope is necessarily trivial, so $\mathrm{Hom}(F, F') = 0$. By Serre duality,

$$\mathrm{Ext}^2(F, F') = \mathrm{Hom}(F', F(-3))^\vee = 0.$$

□

(Generalized) Mumford relations

1. This means that

$$\mathcal{V} = Rp_* \mathcal{R}Hom(\mathcal{F}, \mathcal{F}')[1]$$

   is a vector bundle on $\mathfrak{M}_{d,\chi} \times \mathfrak{M}_{d',\chi'}$ of rank dd', with fibers

$$\mathcal{V}_{|(F,F')} = \mathrm{Ext}^1(F, F') \,.$$

2. Hence

$$c_j(\mathcal{V}) = 0 \,, \quad \text{for } j > dd' \,.$$

3. Using Grothendieck–Riemann–Roch and Newton's identities, we express $c_j(\mathcal{V})$ in terms of tautological classes on $\mathfrak{M}_{d,\chi}$ and $\mathfrak{M}_{d',\chi'}$.
4. If we already understand $H^*(\mathfrak{M}_{d',\chi'})$, we obtain relations on $H^*(\mathfrak{M}_{d,\chi})$ by taking Kunneth components of $c_j(\mathcal{V})$.

BPS integrality

Problem

How do we know when we found all the relations?

- If $\gcd(d, \chi) = 1$, $M_{d,\chi}$ satisfies Poincaré duality, so we can obtain all the relations once we cut the dimension of the top degree cohomology to 1.
- Alternatively: Betti numbers of $M_{d,\chi}$ for small d are known (Choi–Chung, Bousseau,...).
- Betti numbers for the stacks $\mathfrak{M}_{d,\chi}$ can be obtained using BPS integrality + χ-independence.

BPS integrality

Let

$$\Omega_{M_{d,\chi}}(q) = (-q)^{-d^2-1} \sum_{i \geqslant 0} \dim IH^i(M_{d,\chi}) q^i = \Omega_d(q,q)$$

$$\Omega_{\mathfrak{M}_{d,\chi}}(q) = (-q)^{-d^2} \sum_{i \geqslant 0} \dim H^i(\mathfrak{M}_{d,\chi}) q^i \,.$$

Theorem (Mozgovoy–Reineke)

Let $\mu \in \mathbb{Q}$ be a fixed slope. We have

$$\sum_{\substack{d \geqslant 0 \\ \chi = d\mu}} \Omega_{\mathfrak{M}_{d,\chi}}(q) Q^d = \mathrm{PE}\left(\frac{q}{q^2-1} \sum_{\substack{d \geqslant 1 \\ \chi = d\mu}} \Omega_{M_{d,\chi}}(q) Q^d \right).$$

BPS integrality: $\mathfrak{M}_{2,0}$

Using $\mu = 0$ and taking the Q^2 coefficient on both sides:

$$\sum_{i \geqslant 0} \dim H^i(\mathfrak{M}_{2,0}) q^i = (-q)^4 \left(\frac{q}{q^2-1} \Omega_2(q) + \frac{q^2}{2(q^4-1)} \Omega_1(q^2) + \frac{q^2}{2(q^2-1)^2} \Omega_1(q)^2 \right)$$

$$= \frac{1 + q^2 + 2q^4 + 2q^6 + 3q^8 + q^{10} - q^{14}}{(1-q^2)(1-q^4)}$$

$$= 1 + 2q^2 + 5q^4 + 8q^6 + 14q^8 + 18q^{10} + \cdots$$

Virasoro representation

Definition

Let $\mathrm{R}_n \colon \mathbb{D} \to \mathbb{D}$, $n \geqslant -1$, be a derivation of the algebra

$$\mathbb{D} = \mathbb{Q}[c_0(2), c_1(1), c_2(0), c_1(2), \ldots]$$

defined on generators by

$$\mathrm{R}_n(c_k(j)) = \frac{(k+j+n-1)!}{(k+j-2)!} c_{k+n}(j)\,.$$

These operators define a representation of $\mathrm{Vir}_{\geqslant -1}$ on $\mathbb{D}$, i.e.

$$[\mathrm{R}_n, \mathrm{R}_m] = (m-n)\mathrm{R}_{n+m}\,.$$

Virasoro geometricity

Theorem (Lim–M)

The action of R_n, $n \geqslant -1$, *descends to* $H^*(\mathfrak{M}_{d,\chi})$ *via the realization morphism* $\mathbb{D} \to H^*(\mathfrak{M}_{d,\chi})$.
In other words, R_n *preserves the ideal of tautological relations.*

Theorem (KLPM)

The derivations R_n *preserve each of the ideals of (generalized) Mumford relations obtained from* $\mathfrak{M}_{d',\chi'}$ *for each fixed* d', χ'.

Virasoro representation

Theoretically we do not get new relations, but very useful in practice for implementation on the computer:

Theorem (KLPM)

The ideal of Mumford relations is the smallest ideal containing the relations obtained from the vanishing

$$c_{dd'+1}(\mathcal{V}) = c_{dd'+2}(\mathcal{V}) = 0$$

which is closed under R_n.

Calculations

Theorem (KLMP)

The procedure explained determines completely the cohomology rings $H^*(M_{d,\chi})$ for $d \leqslant 5$ and $\gcd(d,\chi) = 1$ and $H^*(\mathfrak{M}_{d,\chi})$ for $d \leqslant 4$.

All the rings are available online.

Corollary

The GV/GW (and refined GV/PT) correspondence holds up to degree 5.

Remark

It turns out that the relations proven for $H^*(M_{5,1})$ are not all of them (but for $H^*(M_{5,2})$ they are). However, they cut down the top degree of the ring to dimension 1, so we can recover the missing relations by Poincaré duality.

Example: $H^*(M_{5,1}), H^*(M_{5,2})$

Both are generated by

$$c_0(2), c_2(0), c_1(2), c_2(1), c_3(0), c_2(2), c_3(1), c_4(0)$$

Relations for $M_{5,1}$:

degrees	H^{10}	H^{12}	H^{14}	H^{30}	H^{34}	H^{36}	H^{38}	H^{40}
# of rel.	3	12	13	1	1	$1^\times$	$2^\times$	1

Relations for $M_{5,2}$:

degrees	H^{10}	H^{12}	H^{14}	H^{16}	H^{18}	H^{28}	H^{30}	H^{32}
# of rel.	3	12	13	2	1	1	1	1

	H^{34}	H^{36}	H^{38}	H^{40}
	2	3	1	1

Example: $H^*(\mathfrak{M}_{2,0})$

$$H^*(\mathfrak{M}_{2,0}) \simeq \mathbb{D}/\big(U(\mathrm{Vir}_{\geqslant -1}) \cdot I_1\big)$$

where

$$\begin{aligned} I_1 = \Big\langle & 8c_2(0) - c_0(2), \\ & c_2(2) + 2c_1(1)^2 c_2(0) + \frac{32}{3} c_2(0)^3 - 4c_1(1)c_3(0) - 4c_2(0)c_2(1), \\ & c_3(1) + \frac{1}{12} c_1(1)^3 + 4c_1(1)c_2(0)^2 - 8c_2(0)c_3(0) - \frac{1}{2} c_1(1)c_2(1) \Big\rangle \end{aligned}$$

These 3 relations are obtained by using the (generalized) Mumford relations with $d' = 1, \chi' = -1, -2$.

Example: $H^*(\mathfrak{M}_{2,0})$

Eliminating the redundant variables:

$$H^*(\mathfrak{M}_{2,0}) \simeq \mathbb{Q}[c_1(1), c_2(0), c_2(1), c_3(0)]/I_2$$

where

$$\begin{aligned} I_2 = \Big\langle & c_1(1)c_2(0)^4 - 2c_2(0)^3c_3(0), \\ & 16c_2(0)^5 + c_1(1)^2c_2(0)^3 - 6c_1(1)c_2(0)^2c_3(0) \\ & \quad + 6c_2(0)c_3(0)^2 + 2c_2(0)^3c_2(1), \\ & c_1(1)^3c_2(0)^3 - 6c_1(1)c_2(0)c_3(0)^2 + 4c_3(0)^3 \\ & \quad - 6c_1(1)c_2(0)^3c_2(1) + 12c_2(0)^2c_3(0)c_2(1), \\ & 2c_1(1)c_2(0)^3c_3(0) - 3c_2(0)^2c_3(0)^2 - c_2(0)^4c_2(1) \Big\rangle \end{aligned}$$

The $P = C$ conjecture

- Our moduli $M_{d,\chi}$ are analogues of moduli of Higgs bundles (replace $\mathbb{P}^2$ by T^*C).
- The P filtration for Higgs bundles has been identified with a weight filtration on a character variety ($P = W$ conjecture, proofs by Maulik–Shen, Hausel–Mellit–Minets–Schiffmann, Maulik–Shen–Yin).
- We do not have a character variety side, but there is an intermediate filtration C that is used in the proof ($P = C = W$) which makes sense for $M_{d,\chi}$.
- Non Calabi–Yau/compact setting changes many things, but $P = C$ seems to be true.

The $P = C$ conjecture

Definition

Let $C_\bullet H^*(M_{d,\chi})$ be the filtration defined by

$$C_k H^*(M_{d,\chi}) = \mathrm{span}\{c_{k_1}(j_1)\dots c_{k_l}(j_l) : k_1 + \dots + k_l \leqslant k\}.$$

Conjecture ($P = C$)

$$P_\bullet H^*(M_{d,\chi}) = C_\bullet H^*(M_{d,\chi}).$$

Corollary (KLMP)

$P = C$ holds up to $d = 5$.

Theorem (Maulik–Shen–Yin)

$$P_\bullet H^{\leqslant 2d-2}(M_{d,\chi}) \supseteq C_\bullet H^{\leqslant 2d-2}(M_{d,\chi}).$$

Consequences of $P = C$

1. The P filtration is multiplicative (but, unlike for Higgs bundles, it does not admit a multiplicative splitting).
2. The C filtration is χ-independent (for other del Pezzo surfaces, also polarization independent).
3. The C filtration has "curious hard Lefschetz" symmetries.
4. $C_{2g}H^*(M_{d,\chi}) = H^*(M_{d,\chi})$.
5. Vanishing of integrals:

$$\int_{M_{d,\chi}} c_{k_1}(j_1)\ldots c_{k_l}(j_l) = 0$$

for $k_1 + \ldots + k_l < 2g = (d-1)(d-2)$.

Stacky $P = C$

There is a P filtration on $H^*(\mathfrak{M}_{d,\chi})$ defined by Ben Davison, which is compatible with the BPS integrality formula. The C filtration can be defined on the stack easily.

Conjecture (Stacky $P = C$)

$$P_\bullet H^*(\mathfrak{M}_{d,\chi}) = C_\bullet H^*(\mathfrak{M}_{d,\chi}).$$

Corollary (KLMP)

The numerical stacky $P = C$ holds up to $d = 4$, i.e.

$$\dim \mathrm{Gr}_i^P H^{i+j}(\mathfrak{M}_{d,\chi}) = \dim \mathrm{Gr}_i^C H^{i+j}(\mathfrak{M}_{d,\chi})$$

for every $i, j \geqslant 0$ and $d \leqslant 4$.

Thanks you!