Cohomological Hall algebras and Hecke operators on surfaces

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This talk will be based on the papers [1, 3]. In the proof of $P = W$ in [1], one of the fundamental steps is to construct the action of a large algebra $W(S)$ on homology/cohomology of (the elliptic loci of) moduli spaces of Higgs bundles. This action interacts in a controlled way with tautological classes, and hence with the Chern filtration – which, by a result of Schende presented in a previous talk, matches the W filtration. Using this action one, constructs a $\mathfrak{sl}_2$ triple, which is later used to prove that the perverse filtration matches the Chern filtration (on the elliptic loci; further work is required to reduce the statement to the elliptic loci).

This talk focuses on the construction of this algebra action. I will explain how the algebra $W(S)$ comes from the theory of Cohomological Hall algebras (CoHA), following [3].

1. Cohomological Hall algebra

Cohomological Hall algebras are algebra structures that one can define on the homology of a moduli stack parametrizing objects in abelian conditions satisfying appropriate conditions. The one that we are interested in is the cohomological Hall algebra of 0-dimensional sheaves on a surface S, which for simplicity we will assume is projective. The main example in applications to Higgs bundles is $S = T^*C$, but we can also compactify it to $\mathbb{P}(T^*C \oplus \mathcal{O}_C)$.

Let

$$\mathfrak{Coh}^0(S) \subseteq \mathfrak{Coh}(S)$$

be the derived stacks parametrizing sheaves on S and 0-dimensional sheaves on S. The CoHA of (0-dimensional) sheaves on S is the Borel–More homology of the stack:

$$\mathbb{H}(S) := H_*(\mathfrak{Coh}(S)) \supseteq H_*(\mathfrak{Coh}^0(S)) =: \mathbb{H}^0(S)\,.$$

We define a product on $\mathbb{H}(S)$ and $\mathbb{H}^0(S)$ as follows: let $\mathcal{E}\mathrm{xt}$ be a stack parametrizing short exact sequences of the form

$$0 \to F_1 \to F_2 \to F_3 \to 0\,;$$

this stack admits a map q to $\mathfrak{Coh} \times \mathfrak{Coh}$ remembering F_1, F_3 and another map p to $\mathfrak{Coh}$ remembering F_2. The product on the CoHA is defined by

$$H_*(\mathfrak{Coh}) \otimes H_*(\mathfrak{Coh}) \to H_*(\mathfrak{Coh} \times \mathfrak{Coh}) \xrightarrow{q^!} H_*(\mathcal{E}\mathrm{xt}) \xrightarrow{p_*} H_*(\mathfrak{Coh})\,.$$

where $q^!$ is a "virtual pullback".

The goal of the talk is to understand $\mathbb{H}^0(S)$ and its action on the (co)homology of moduli spaces.

2. Hecke patterns

Let $\mathfrak{M}$ be a substack of $\mathfrak{Coh}(S)$. Under some conditions on $\mathfrak{M}$, it is possible to define a left (and also a right) action of $\mathbb{H}^0(S)$ on $H_*(\mathfrak{M})$. When these appropriate conditions are met, $\mathfrak{M}$ is called a Hecke pattern. The most crucial property of a Hecke pattern is that it should be closed under point modifications. Note that this forces $\mathfrak{M}$ to have many connected components, since point modifications change the topological type of sheaves. The definition of the action resembles the definition of the CoHA product.

Example 2.1. For $S = \mathbb{P}(T^*C \oplus \mathcal{O}_C)$, pure 1-dimensional sheaves on S with integral support away from the ∞ divisor (=the elliptic loci of the moduli of Higgs bundles) form a Hecke pattern,

$$\mathfrak{M}_r^{\mathrm{ell}} = \bigsqcup_{d \in \mathbb{Z}} \mathfrak{M}_{r,d}^{\mathrm{ell}} .$$

On the other hand, the stack of all semistable 1-dimensional sheaves (with support away from the ∞-divisor) is not a Hecke pattern. This is why the proof of $P = W$ in [1] requires a reduction to the elliptic loci.

3. Length 1 Hecke patterns and Negut's lemma

The first step to understand $\mathbb{H}^0(S)$ and its action on $H_*(\mathfrak{M})$ more concretely is to start with the action of elements in

$$H_*(\mathfrak{Coh}_\delta) \subseteq \mathbb{H}^0(S) ,$$

where $\mathfrak{Coh}_\delta$ is the stack of 0-dimensional sheaves of length 1. Note that $\mathfrak{Coh}_\delta \simeq S \times B\mathbb{G}_m$, so

$$H_*(\mathfrak{Coh}_\delta) \simeq H^*(S)[u]$$

by Poincaré duality. The (left) action of $u^n\lambda \in \mathbb{H}^0(S)$, for $n \geqslant 0$ and $\lambda \in H^*(S)$, produces an operator

$$T_n^+(\lambda)\colon H_*(\mathfrak{M}) \to H_*(\mathfrak{M}) .$$

Similarly, there is another operator $T_n^-(\lambda)$ coming from the right action.

Negut's lemma [2, Proposition 2.19] gives a very concrete understanding of these operators by identifying some of the maps that come up in the definition of the action with (virtual) projective bundles. In particular, it gives

(1) A formula for the image of the fundamental class $[\mathfrak{M}]$ under $T_n^{\pm}(\lambda)$;
(2) A formula for the commutator between $T_n^{\pm}(\lambda)$ and the operators of capping with tautological classes.

4. (Deformed) W-algebras

The main result of [3], and of this talk, is an isomorphism

$$\mathbb{H}^0(S) \simeq W^+(S)$$

with an explicitly defined algebra.

Let $W(S)$ be the algebra generated by

$$\psi_n(\lambda), T_n^+(\lambda), T_n^-(\lambda) \quad n \geqslant 0, \lambda \in H^*(S)$$

and a central element c, modulo certain explicit relations which have the following shape:

(1) ψ commute.
(2) $[T^\pm, \psi] = T^\pm$.
(3) Quadratic relation on $T^\pm$.
(4) Cubic relation on $T^\pm$.
(5) $[T^+, T^-] = \psi$.

Let $W^0, W^\pm$ be the algebras generated by $\psi_n(\lambda)$ and $T_n^\pm(\lambda)$, respectively. Let $W^\geqslant$ be the algebra generated by W_0 and W^+, and $W^\leqslant$ similarly defined.

The algebra $W(S)$ acts on the (tautological part of the) homology $H_*(\mathfrak{M})$ of a Hecke pattern; the actions of $W^+(S) \simeq \mathbb{H}^0(S)$ and $W^-(S) \simeq \mathbb{H}^0(S)^{\mathrm{op}}$ are identified with the left and right CoHA actions previosuly mentioned, while the action of $W^0(S)$ is essentially capping with tautological classes.

There are a few important consequences that we can extract from this description of the algebra $W(S)$: the CoHA is generated as an algebra by $H_*(\mathfrak{Coh}_\delta)$; $W(S)$ contains copies of the Heisenberg and Virasoro Lie algebras; when $c_1 = 0$, $W^\geqslant(S)$ is the universal envelopping algebra of a certain Lie algebra of differentials;

References

[1] Hausel, T., Mellit, A., Minets, A., Schiffmann, O. (2022). $P = W$ via H_2. ArXiv preprint 2209.05429.

[2] Negut, A. (2022). Hecke correspondences for smooth moduli spaces of sheaves. Publ. Math. Inst. Hautes Études Sci.135, 337–418.

[3] Mellit, A., Minets, A., Schiffmann, O., Vasserot, E. (2023). Coherent sheaves on surfaces, COHAs and deformed $W_{1+\infty}$-algebras. ArXiv preprint 2311.13415.