The Surgery Theoretic Classification of High-Dimensional Smooth and Piecewise Linear Simply Connected Manifolds

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Preface

The aim of this thesis is provide a down-to-earth presentation of the classification of compact simply connected smooth and piecewise linear manifolds in dimensions greater than or equal to five. The primary tool used in this classification is surgery theory.

Surgery theory began with Kervaire and Milnor's work on the classification of exotic spheres. In 1956, Milnor published the surprising result that there exist smooth manifolds that are homeomorphic but not diffeomorphic to S^7 [Mil56]. In their famous 1963 *Annals* paper [KM63], Kervaire and Milnor reduced the classification of smooth manifolds that are homotopy equivalent to S^n, $n > 4$, to the calculation of the homotopy groups of spheres. In doing so, they set forth the techniques that would form the basis for simply connected surgery theory.

Browder [Bro72] and Novikov [Nov64] generalized these techniques to reduce the classification of all smooth manifolds of dimension greater than or equal to five in a given *simply connected* homotopy type to the homotopy theory of a certain classifying space called G/O. Their results easily translated into the piecewise linear category, thereby classifying high-dimensional simply connected PL manifolds in a given homotopy type in terms of the homotopy theory of a different space, G/PL.

In his landmark 1970 book, Wall further generalized surgery theory to manifolds with arbitrary fundamental groups [Wal70]. Roughly concurrently, Kirby and Siebenmann overcame vast technical difficulties to obtain a version of surgery theory that classified topological manifolds [KS77].

Taken as a whole, this corpus of work reduced many of the basic questions about manifolds to the homotopy theory of the classifying spaces G/PL, G/TOP, and G/O. The homotopy groups of G/O involve homotopy groups of spheres, so its homotopy type is complicated and not fully understood. The other two spaces, on the other hand, are much simpler. By applying the localization of homotopy types, Sullivan computed the homotopy type of G/PL. Kirby and Siebenmann then showed that there is a fibration $K(\mathbb{Z}, 3) \to G/PL \to G/TOP$, from which they deduced the homotopy type of G/TOP.

These foundational results enabled the solution of many of the major problems in the theory of manifolds. These included (but were certainly not limited to) Novikov's proof the topological invariance of the rational Pontrjagin classes [Nov65], Siebenmann's disproof of the manifold Hauptvermutung [KS77], Sullivan's proof of the Hauptvermutung for simply connected manifolds with trivial third homology [RCS$^+$96], and Freedman's classification of simply connected topological 4-manifolds [Fre82].

The literature on surgery theory is very difficult to penetrate. Most of what is written about it seems to be aimed at seasoned homotopy theorists and assumes a vast amount of background material. In the first part of this thesis, I shall attempt to provide an elementary treatment of the surgery theoretic classification of simply connected smooth and piecewise linear manifolds. I shall assume no background beyond elementary algebraic topology, as contained for example in Hatcher's *Algebraic Topology* ([Hat02]), basic vector bundle theory, as in Milnor and Stasheff's *Characteristic Classes* ([MS74]), and introductory Morse theory, as in Milnor's *Morse Theory* ([Mil63]). As such, I shall provide brief discussions (primarily without proof) of the major prerequisites to surgery theory, including the Thom transversality theorem, Whitney's embedding and immersion theorems, handle decompositions of manifolds, Smale's h-cobordism theorem and his use of it to prove the generalized Poincaré conjecture in high dimensions, and the existence of exotic spheres. I shall then develop the basics of simply connected surgery theory. This will culminate in the surgery exact sequence, which describes the classification of simply connected manifolds in terms of certain homotopy theoretic objects.

This theory will reduce the classification of smooth and PL manifolds to seemingly intractable homotopy theory. Most expositions of surgery theory content themselves with this, often leaving the reader wondering how to obtain any concrete version of the promised "classification of manifolds." In addition, the calculation of the homotopy type of G/PL does not appear to be discussed at an elementary level anywhere in the literature. In the second part of my thesis, I shall attempt to remedy these issues by reviewing Sullivan's computation of the homotopy type of G/PL, which is the first step in using this theory to perform concrete computations.

A complete discussion of all of the results mentioned above would far exceed the space limitations of a senior thesis. In order to only slightly exceed them, I shall be forced to outsource the proofs of some of these results. I shall provide references to the literature whenever I do this, and I shall make an effort to outsource technical lemmas while retaining the steps that convey the geometric intuition.

I would like to take this opportunity to thank several people whose help has been invaluable in the preparation of this thesis. I would first like to thank my advisor, Peter Kronheimer, for many very helpful discussions about both the substance and the form of this document. I would also like to thank Michael Hopkins for his help with the homotopy theoretic aspects of the theory, and Andrew Ranicki for answering all of my questions and for looking over a draft of this thesis. I owe a great debt of gratitude to all three.

In addition, I would like to thank Nathan Dunfield, Raoul Bott, and Jonathan Rosenberg for looking over drafts of this thesis, Chris Mihelich for his help with TeX and his sage advice about hyphenation, Mike Hill for not laughing too hard at the questions I e-mailed him at 4:30 in the morning, and Anjanette Chan Tack for bringing me sustenance and allowing me to never leave the room for all of spring "break." Finally, last but certainly not least, I would like to thank Starbucks for giving me the strength to go on during the last several days before the deadline.

Jonathan Kelner,
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Contents

Chapter 1

Introduction and Outline of the Argument

The flavor of the classification of closed manifolds depends heavily on the dimension. In one dimension, the classification is trivial—all we have is the circle. In two dimensions, it is easy. Manifolds are determined by their orientability and their genus, and the smooth, piecewise linear, topological, and homotopy categories coincide. In three dimensions, the classification problem is extremely hard, and it is presently unsolved. Even such fundamental questions as whether there is a manifold homotopy equivalent to S^3 but not homeomorphic to it remain unanswered. In greater than or equal to four dimensions, it is provably impossible. As we shall show in Example 2.3.6, for any $n \geq 4$ and any finitely presented group G, there is a closed n-dimensional manifold whose fundamental group is G. Since homotopy equivalent manifolds have isomorphic fundamental groups, and the classification of finitely presented groups is provably undecidable, this renders the classification of n-dimensional manifolds undecidable as well.

Naturally, we shall restrict our attention to the impossible case and consider manifolds of dimension greater than or equal to five. As we shall see, the large number of dimensions allows us to make copious use of general position arguments and renders the classification problem much more tractable than in dimensions three and four, once we compensate for the fact that it is impossible. We do this by fixing a homotopy type. For notational simplicity, we adopt the convention hereafter that *all manifolds will be taken to be closed (i.e., compact and without boundary) unless stated otherwise.*

Our main goal is:

Goal. *Let X be a finite CW complex.*

1. *Determine if X is homotopy equivalent to a smooth manifold.*
2. *If so, classify all such smooth manifolds.*

Before we proceed, a brief remark about the categories of manifolds is in order. There are three categories of manifolds in which all of our questions can be stated: topological, smooth, and piecewise linear (PL). The smooth and PL categories are in many ways the same. Almost all of the methods of differentiable topology commonly used in the smooth category, such as transversality and Morse theory, are available in the PL category as well[1]. As such, most of our results about one category apply to the other, and the modifications to the proofs are essentially linguistic in

[1] If the piecewise linear category is unfamiliar, see [RS72] for a systematic treatment.

nature. To avoid the rampant proliferation of notation, we shall choose one language, namely that of smooth manifolds, and use it for all of the arguments that apply to both categories. We shall explicitly note when there are substantive differences.

The topological category, on the other hand, presents significant technical difficulties that are not encountered in the other two. As such, we shall essentially avoid it in this thesis. For an exposition of how surgery theory extends to topological manifolds, see [KS77].

As stated, our goal seems very difficult. Smooth manifolds and diffeomorphisms do not behave well under deformations, so almost all of our methods from algebraic topology appear inapplicable. The category of manifolds appears very large, and there isn't very much obvious structure on it. Our theory will thus begin by transforming our questions into ones about more manipulable objects. The two main tools that we shall use to do this are the handle decomposition theorem and the h-cobordism theorem.

A handle on a manifold is roughly analogous to a cell in a CW complex. If a manifold can be built up through the successive attachment of handles, continuous results about the manifold can often be proven using discrete inductions over the handle decomposition. We shall use Morse theory to prove the handle presentation theorem, which states that any smooth manifold admits such a decomposition.

The first major application of this result was to proof the Smale's h-cobordism theorem. An *h-cobordism* (W^{k+1}, M^k, N^k) is a cobordism in which the inclusions of M and N into W are homotopy equivalences. The h-cobordism theorem states that if W is simply connected and $k \geq 5$, then W is diffeomorphic to the product $M \times I$, so that M and N are diffeomorphic. Since diffeomorphic manifolds are obviously h-cobordant, it follows that the diffeomorphism and h-cobordism classifications of simply connected high-dimensional manifolds coincide.

For manifolds with nontrivial π_1, the picture is not quite so simple: there exist h-cobordant manifolds that are not diffeomorphic. When this occurs is described by the s-cobordism theorem of Barden, Mazur, and Stallings using the language of simple homotopy. A homotopy equivalence f is a simple homotopy equivalence if a certain invariant known as its *torsion* vanishes. The torsion lives in a group called the *Whitehead group* $Wh(\mathbb{Z}[\pi_1])$. The group $Wh(\mathbb{Z})$ is trivial, so every homotopy equivalence of simply connected manifolds is simple. The s-cobordism theorem states that an h-cobordism (W^{k+1}, M^k, N^k) is diffeomorphic to $M \times I$ if and only if the inclusions $M \hookrightarrow W$ and are $N \hookrightarrow W$ are *simple* homotopy equivalences. This is the first of a series of situations in which the theory is substantively simpler for simply connected manifolds. (See [Kos93] for a proof of the s-cobordism theorem.)

The h- and s-cobordism theorems allow us to rephrase our diffeomorphism classification problem in terms of cobordisms and homotopy equivalences, which we can more easily analyze. This leads us to study the *structure set* $\mathcal{S}(X)$ of a finite CW complex X. Its elements consist of pairs (M, f), where M is a manifold and $f\colon M \to X$ is a homotopy equivalence. Two such pairs (M, f) and (N, g) are equivalent if there is an h-cobordism (W, M, N) and a map $F\colon W \to X$ such that $F|_M = f$, and $F|_N = g$. For simply connected manifolds, the h-cobordism theorem implies that this is the same as calling (M, f) and (N, g) equivalent if there is a diffeomorphism $\phi\colon M \to N$ such that the diagram

$$\begin{tikzcd} M \arrow[rr, "\phi"] \arrow[dr, "f"'] & & N \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

commutes, so the elements of $\mathcal{S}(X)$ are precisely what we set out to classify.

For simply connected X, our goal therefore becomes the computation of $\mathcal{S}(X)$. For nonsimply connected spaces, torsion complicates matters, but the computation of $\mathcal{S}(X)$ remains an important first step. Once $\mathcal{S}(X)$ is understood, there exist further tools to take simple homotopy into account and obtain the full diffeomorphism classification.

Our first step in the computation of $\mathcal{S}(X)$ will be the observation that there are certain homological and homotopy theoretic properties obeyed by every manifold, and thus also by every X with a nonempty structure set. The first such property is Poincaré duality. This will lead us to define a *Poincaré complex* to be a CW complex whose homology obeys the constraints that Poincaré duality imposes, and to restrict our attention to these spaces.

The next requisite property is bundle-theoretic. As we shall show, every manifold can be embedded in a sufficiently high-dimensional Euclidean space, and any two such embeddings are isotopic. A manifold M thus has a well-defined stable normal bundle whose class in $[M, BO]$ (where BO is the doubly infinite real Grassmannian) is independent of the embedding. While this homotopy class of maps depends on your choice of manifold in the homotopy type of M, it turns out that the class of the associated stable spherical fibration in $[M, BG]$ (where BG is the appropriate classifying space) does not. This will allow us to associate to every Poincaré complex X a unique stable spherical fibration ν_X called its *Spivak normal fibration*, such that the stable spherical fibration associated to the stable normal bundle of any manifold homotopy equivalent to X must be isomorphic to ν_X. It will follow that $\mathcal{S}(X)$ can be nonempty only if ν_X admits a bundle reduction to a stable vector bundle

$$\begin{tikzcd} & BO \arrow[d] \\ X \arrow[ur, dashed] \arrow[r, "\nu_X"'] & BG \end{tikzcd},$$

i.e. there exists a stable vector bundle whose associated stable sphere bundle is isomorphic to ν_X.

We shall see that there is a simple construction by which a vector bundle reduction ξ of the Spivak normal fibration ν_X gives rise to a pair (M, f), where M is a smooth manifold, and $f\colon M \to X$ is a degree one map such that $f^*(\xi) = N_M$, where N_M is the stable normal bundle of M. We call such maps *normal maps*, and denote the set of them under a yet-to-be-described equivalence relation called normal bordism by $\mathcal{N}(X)$. This equivalence relation will be such that the elements of $\mathcal{N}(X)$ will be in one-to-one correspondence with isomorphism classes of vector bundle reductions of ν_X. Since an element of the structure set is a normal map by the uniqueness of the Spivak normal fibration, the structure set $\mathcal{S}(X)$ will be nonempty if and only if one of the normal bordism classes of normal maps contains a homotopy equivalence.

The determination of the nonemptiness of the structure set $\mathcal{S}(X)$ of a Poincaré complex will thus proceed in two stages. In the first stage, we shall check if X admits a normal map, which it will if and only if ν_X admits a vector bundle reduction. This will be shown equivalent to the vanishing of a certain homotopy class.

In the second stage, we shall check which of the classes of normal maps in $\mathcal{N}$ contain a homotopy equivalence. To do this, we shall define the homotopy and homology groups of a *map* and show that a map is a homotopy equivalence if and only if they all vanish. Given a normal map, we will then investigate when we can produce a normally bordant map with simpler homotopy groups. It is here that we employ surgery, which can be thought of as a (fully general) method of constructing normal bordisms. The main technical theorem of surgery theory asserts that if $n \geq 5$, we can associate to a normal map (M^n, f) a well-defined obstruction, living in a group $L_n(\mathbb{Z}[\pi_1(X)])$ dependent only on the fundamental group of X, such that we can use surgery to transform (M, f) into a homotopy equivalence if and only if its obstruction vanishes.

These results will fit together to form the *Browder-Novikov-Sullivan-Wall surgery exact sequence* (of pointed sets, not groups):

$$L_{n+1}(\mathbb{Z}[\pi_1(X)]) \to \mathcal{S}(X) \to \mathcal{N}(X) \to L_n(\mathbb{Z}[\pi_1(X)]),$$

where X is an n-dimensional Poincaré complex with nonempty structure set. There is an action of $L_{n+1}(\mathbb{Z}[\pi_1(X)])$ on $\mathcal{S}(X)$ that respects the exactness, so the sequence will be exact in a stronger sense as well.

There is a construction that permits the identification of $\mathcal{N}(X)$ with the set $[X, G/O]$, where G/O is a fixed classifying space, so that the surgery exact sequence takes the form:

$$L_{n+1}(\mathbb{Z}[\pi_1(X)]) \to \mathcal{S}(X) \to [X, G/O] \to L_n(\mathbb{Z}[\pi_1(X)]).$$

The entire theory goes through essentially verbatim in the PL case, and all that changes are the classifying spaces used. As such, there is a similar exact sequence:

$$L_{n+1}(\mathbb{Z}[\pi_1(X)]) \to \mathcal{S}_{\mathrm{PL}}(X) \to [X, G/PL] \to L_n(\mathbb{Z}[\pi_1(X)]).$$

This will reduce the calculation of $\mathcal{S}(X)$ and the analogous PL structure set $\mathcal{S}_{\mathrm{PL}}(X)$ to the computation of the obstruction groups and of $[X, G/O]$ and $[X, \mathrm{PL}]$. There has been a lot written about the calculation of surgery obstruction groups using a theory called *algebraic L-theory*, and they can be computed for most reasonable fundamental groups. In particular, the obstruction groups for trivial π_1 are quite easy to derive.

The calculations of $[X, G/O]$ and have $[X, \mathrm{PL}]$ have very different natures. The former depends on the computation of stable homotopy groups of spheres, so we cannot do it in general. The latter, however, is a good deal more tractable. Sullivan has proved a very nice characterization of the homotopy type of G/PL, using which one can perform a great many computations. Unfortunately, a proof of this characterization is not very accessible in the literature, so we shall provide one.

In this thesis, we shall carry out the program outlined above *for simply connected manifolds*. The general case requires a lot more machinery, and space limitations preclude a proper treatment of it in the present document. The goal will be to provide an elementary, accessible treatment of surgery theory and the homotopy theory it requires, and to develop the necessary machinery to actually do some simple computations.

We shall begin in chapter 2 by briefly reviewing the basic prerequisites for surgery theory. These will include transversality, Whitney's embedding and immersion theorems, handles, the h-cobordism theorem, and the existence of exotic spheres. In chapter 3, we shall formally introduce the Spivak normal fibration, the structure set, and normal maps, and we shall show how the nonemptiness of the structure set $\mathcal{S}(X)$, the existence of degree one normal maps into X, and the existence of bundle reductions of the Spivak normal fibration relate. This will lay the groundwork for surgery, which we shall discuss in chapter 4. Our main result in chapter 4 will be the surgery exact sequence, which reduces the computation of the structure set to certain problems in homotopy theory. In chapter 5, we shall address some of this homotopy theory by computing the homotopy type of the classifying space G/PL. Finally, in chapter 6, we shall look at some simple examples of how one can apply this theory.

Chapter 2

Preliminaries

In this chapter we set forth several foundational tools on which our classification of manifolds will rely. We begin in section 2.1 with some basic transversality and general position results. In section 2.2 we define handle decompositions of manifolds and cobordisms and explain how Morse theory guarantees their existence. In section 2.3, we describe the operation of surgery, and we see how it interacts with handles, cobordisms, homotopy, and homology. We discuss the h-cobordism theorem in section 2.4. Finally, we examine the difference between the smooth and piecewise linear categories in section 2.5, where we prove the generalized Poincaré conjecture and construct an exotic sphere.

As the proofs of most of these results are somewhat technical and tangential to our basic goals, we either sketch or omit them and provide references to where they may be found in full.

2.1 A Salute to General Position

One of the basic aims of our theory is to apply the methods of homotopy theory to problems in differential topology. An obvious obstacle to doing this is that homotopy theory concerns itself with arbitrarily badly behaved continuous maps, whereas the questions of differential topology require one to deal with more nicely behaved geometric maps, such as immersions, embeddings, and maps transverse to a given submanifold of the codomain. In this section, we present several results that allow us to bridge this gap between topology and geometry by providing geometric representatives of given homotopy classes.

2.1.1 Transversality

We begin by recalling the basic notions of transversality. For a more respectable treatment of transversality, see [Kos93].

For concreteness, all manifolds in this section are taken to be smooth. All of our results. however, apply equally well in the PL category.

Definition 2.1.1. Let $f\colon M \to N$, $g\colon M' \to N$ be smooth maps of manifolds. We say that f *is transverse to* g if for all $p \in M$, $q \in M'$ such that $f(p) = g(q)$, we have $\mathrm{Span}(Df(T_pM), Dg(T_qM')) = T_{f(p)}N$. If $g\colon M' \hookrightarrow N$ is the inclusion of a submanifold, then we say that f is *transverse to* $M' \subseteq N$. Similarly, if f and g are both inclusions of submanifolds, we say that $M \subseteq N$ *is transverse to* $M' \subseteq N$.

Remark 2.1.2. We note that if M, M' and N are of dimensions m, m', and n respectively, and M and M' intersect transversely in N, then $\dim(M \cap M') = m + m' - n$. It follows that $M \cap M' = \emptyset$ if $m + m' < n$, and $M \cap M'$ is a collection of isolated points if $n = m + m'$.

The following theorem follows easily from the definition and the implicit function theorem:

Theorem 2.1.3.

1. *Let N^n be a smooth manifold of dimension n, and let $S^{n-k} \subseteq N$ be a submanifold of dimension $n-k$. If $f\colon M^m \to S$ is transverse to Y, then $f^{-1}(Y)$ is a smooth manifold of dimension $m-k$.*

2. *If ν^k is the normal bundle of S in N, then the pullback bundle $f^*(\nu)$ is isomorphic to the normal bundle of $f^{-1}(S)$ in M.*

Proof. See [Kos93] or [MS74]. □

The notion of transversality derives much of its usefulness from the following result, which allows us to approximate arbitrary maps with smooth maps that are transverse to a given submanifold of the codomain.

Theorem 2.1.4 (Thom Transversality Theorem). *Let $S \subseteq N$ be a submanifold, and let $f\colon M \to N$ be any continuous map of manifolds. Then f is homotopic by an arbitrarily small homotopy to a smooth map g that is transverse to S. Furthermore, if A is a closed subset of M such that f is already transverse to X in an open neighborhood containing A, then this homotopy can taken rel A.*

Proof. See [Kos93]. □

Remark 2.1.5. The proof of this result never uses the fact that X is smooth at A, nor does it use the fact that N is smooth outside an open neighborhood of $f(M \setminus A)$. As such, the following more technical statement follows immediately from the proof:

Theorem 2.1.6. *Let M be a topological space (not necessarily a manifold), $B \subseteq M$ an open subset, and $A \subseteq \operatorname{Int}(B)$ a closed subset such that $M \setminus A$ is smooth. Furthermore, let $f\colon M \to N$, where N is a space such that an open neighborhood U of $f(M \setminus A)$ is a smooth manifold, and let $S \subseteq U$ be a smooth submanifold. Then f is homotopic rel A, by an arbitrarily small homotopy, to a map f' whose restriction $f'|_{M\setminus B}\colon M \setminus B \to U$ is transverse to S.*

2.1.2 Whitney's Theorems

In this section, we present Whitney's results about when a function can be approximated by an embedding or an immersion. For their proofs, see [Whi36] and [Whi44].

Theorem 2.1.7 (Whitney Immersion Theorem). *Let M^m and N^n be manifolds, and suppose that $n \geq 2m$. Any map $f\colon M \to N$ is homotopic to an immersion by an arbitrarily small homotopy. Furthermore, this immersion can be taken so that all self-intersections of the image are transverse, and no three points in M have the same image.*

We note that if $n \geq 2m + 1$, the transversality of the self intersections of the image actually means that the image does not intersect itself at all, by Remark 2.1.2. We thus obtain:

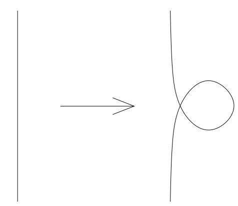

Figure 2.1: Inserting a double point

Theorem 2.1.8 (Weak Whitney Embedding Theorem).

1. *Let $n \geq 2m+1$. Any map $f\colon M^m \to N^n$ is homotopic to an embedding.*

2. *If $n \geq 2m+2$, any two homotopic embeddings of M into N are isotopic.*

The second part of Theorem 2.1.8 follows by applying the first part of it to the map from $M \times I$ to N given by the homotopy between the two embeddings.

Sketch of Proof of Theorem 2.1.7. Let $M_{m,n} \cong \mathbb{R}^{mn}$ be the nm-dimensional manifold of $m \times n$ matrices, and let $R_k^{m,n} \subseteq M_{m,n}$ be the open submanifold comprising matrices of rank k. An easy count shows that $\dim R_k^{m,n} = k(m+n-k)$.

Smooth functions are dense in the space of continuous functions from M to N, so we may perturb f slightly and make it smooth. Let $U \subseteq M$ be a sufficiently small open set that $U \cong \mathbb{R}^m$ and $f(U) \subseteq V \cong R^n$. The differential may be regarded as a map $df|_U \colon \mathbb{R}^m \to M_{m,n}$. By the Thom transversality theorem, df is homotopic by an arbitrarily small homotopy to a map g that is transverse to each $R_k^{m,n}$. We have $\dim R_k^{m,n} = k(m+n-k)$, so the dimension of $g^{-1}(R_k^{m,n})$ for $k < m$ is

$$\dim(g^{-1}(R_k^{m,n})) = m-(mn-k(m+n-k) \leq m-(mn-(m-1)(m+n-(m-1))) = 2m-n-1.$$

However, $2m-n-1 < 0$ by hypothesis, so $g^{-1}(R_k^{m,n}) = \emptyset$ for $k < n$, and the image of g is contained in the set of nondegenerate matrices. The proof now employs simple gluing arguments to show that this small perturbation of the differential can be realized as the differential of a small perturbation of f, and this local perturbation can be done simultaneously everywhere. The result is a map with an everywhere nondegenerate differential, which is an immersion, as desired. Small local perturbations, as made in the proof of the Thom transversality theorem, make the self-intersections of this immersion transverse. □

For a full proof of Theorem 2.1.7, see [Whi36].

We thus see that Theorems 2.1.7 and 2.1.8 follow from general position arguments. When N is simply connected, however, the dimensional restriction in Theorem 2.1.8 can actually be made one dimension better than general position guarantees:

Theorem 2.1.9 (Strong Whitney Embedding Theorem). *Let M^m and N^n be manifolds, and suppose that $\pi_1(N) = 0$. If $n \geq 2m \geq 6$, then any map $f\colon M^m \to N^n$ is homotopic to an embedding.*

Sketch of Proof. By Theorem 2.1.7, f is homotopic to an immersion whose only self-intersections are isolated transverse double points. A simply connected manifold is orientable, and a choice of orientation on N allows us to assign a ± 1 to every self-intersection.

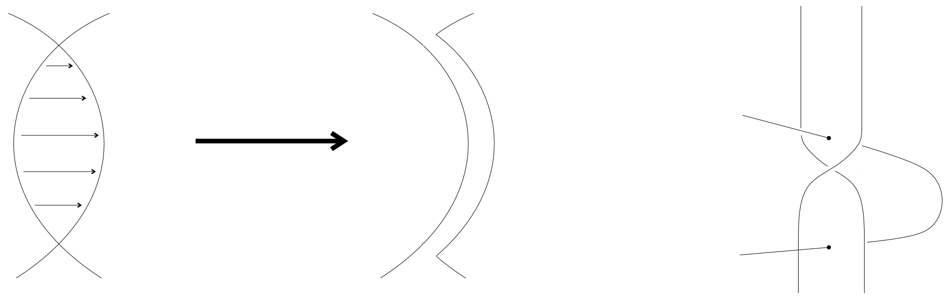

Figure 2.2: On the left is a picture of the use of a Whitney disk to remove a pair of double points of opposite signs. On the right is an attempt at drawing what goes wrong if the double points have the same sign.

It is easy to construct an immersion of $\mathbb{R}^k$ into $\mathbb{R}^{2k}$ that has exactly one double point, and such a double point can be constructed to have either sign. (See Figure 2.1.) Using this map locally allows us to add a double points of either sign to f. Doing this if necessary, we can therefore assume that f has an even number of double points, half of which have sign $+1$, and half of which have sign -1.

Now, the key to the proof is the famous "Whitney trick," which allows one to "cancel" two double points of opposite signs. The precise details of this construction require some work, but the basic idea is as follows. First of all we can locally model the self intersection as the intersection of two distinct manifolds, V and W inside N. Let $p_\pm \in V \cap W$ be the two points. Let $\gamma_1 \colon [0,1] \to V$ and $\gamma_2 \colon [0,1] \to W$ be two distinct arcs that avoid all other intersections and have $\gamma_i(0) = p_-$ and $\gamma_i(1) = p_+$. By the assumption that $\pi_1(N) = 0$, we can map a disk into N so that its boundary is the loop given by $\gamma_1 \cup \gamma_2$. The map of the disk into N may be assumed to be an embedding by the dimensional assumptions and the weak Whitney embedding theorem. One can make the interior of this disk disjoint from $V \cup W$. The idea now is to push γ_1 across this disk, dragging V along with it, to eliminate the two intersections. For this to work, the orientations of the intersections must be opposite. Otherwise, we will just move the intersections, not remove them. See Figure 2.2 for an attempt at drawing this, and see [Whi44] for an actual proof. $\square$

Unlike Thom's transversality theorem, which applies in any dimension, Whitney's results hinge on assumptions about the dimensions of the manifolds involved. We shall apply Whitney's theorems extensively in our classification of manifolds, and it is these dimensional restrictions that will cause our classification to fail for dimensions less than five.

2.2 Getting a Handle on Manifolds

In algebraic topology, putting a CW structure on a topological space allows us to discretize many questions, reducing seemingly intractable questions about uncountable sets to combinatorial questions about the cell structure. The analogous tool in the study of manifolds is handle decomposition.

The motivating example for the study of handles is the classification of compact orientable 2-manifolds. Given a genus g surface S_g, one can construct a genus $g+1$ surface as follows. First, cut out two small disks from S_g, thereby producing a new surface S_g' whose boundary is $S^1 \amalg S^1 = S^0 \times S^1$. Now attach $S^1 \times D^1$ to S_g' along their boundaries. This "handle attachment" produces a genus $g+1$ surface. (See Figure 2.3.) Since a compact orientable surface is determined by its genus, any such surface can be produced by starting with a sphere and successively cutting

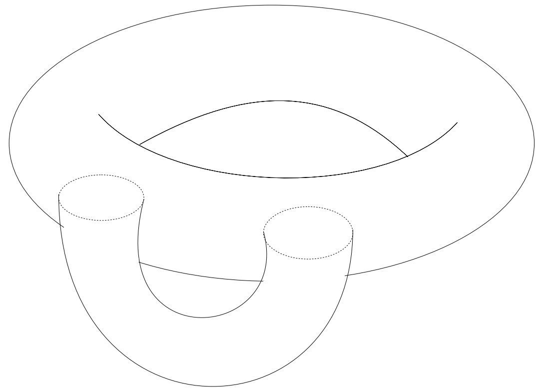

Figure 2.3: Attaching a handle to a genus one surface

out disks and attaching handles. This leads us to define:

Definition 2.2.1. Let $(M, \partial M)$ be an m-dimensional manifold with boundary. Consider the m-disk D^m as the product $D^k \times D^{m-k}$, and suppose we have an embedding $f\colon \partial(D^k) \times D^{m-k} \hookrightarrow M$. Now let

$$M' = M \cup_f ((D^k \times D^{m-k}).$$

We say that M' is obtained by *attaching the k-handle $D^k \times D^{m-k}$ to M*, and that $D^k \times D^{m-k} \subseteq M'$ *is a handle on M'*.

We note that attaching a 0-handle amounts to taking the disjoint union with an m-disk, and attaching an m-handle amounts to identifying the entire boundary of an m-disk with a connected component of ∂M.

Remark 2.2.2. The union in this definition only expresses M' as a topological manifold. It is not *a priori* obvious that there is a unique smooth structure that eliminates the cusps along the region where M and $D^k \times D^{m-k}$ are attached. However, this turns out to be the case, and it is not very difficult to prove. See [Kos93].

Remark 2.2.3. We note that a k-handle $D^k \times D^{m-k}$ on a manifold M deformation retracts onto its "core" $D^k \times \{0\}$, so that attaching a k-handle is, up to homotopy equivalence, the same as attaching a k-cell along its boundary.

Our primary application of handles will be to the study of cobordisms, so the remainder of our treatment of them will be in that setting. Since a manifold with boundary $(M, \partial M)$ can be treated as a cobordism $(M, \emptyset, \partial M)$, this generalizes the case of manifolds with boundary.

Definition 2.2.4. Let (W, M, M') be a cobordism. A handle on (W, M, M') is a handle on W attached along an embedding $f\colon \partial(D^k) \times D^{m-k} \hookrightarrow W$ whose image lies entirely in M'. We note that this results in a new cobordism between M and some new manifold M''.

Definition 2.2.5. An *elementary cobordism of index i* is a cobordism (W, M, M') obtained by attaching an i-handle to the trivial cobordism $(M \times I, M, M)$.

Given cobordisms (W, M, M') and (W', M', M''), one can form a new cobordism $(W \cup_{M'} W', M, M'')$, which we shall call their union and denote by $(W, M, M') \cup (W', M', M'')$. A particularly nice structure on a cobordism, analogous to a CW structure on a topological space, is a *handle decomposition* of (W, M, M') as a union of elementary cobordisms of ascending index.

Definition 2.2.6. Let (W, M, M') be a cobordism. A *handle decomposition* of (W, M, M') is a presentation

$$\begin{aligned}(W, M, M') = (W, M_0, M_k) =&(M_0 \times I, M_0, M_0) \cup (W_1, M, M_1) \\ &\cup (W_2, M_1, M_2) \cup \cdots \cup (W_k, M_{k-1}, M_k) \cup (M_k \times I, M_k, M_k),\end{aligned} \tag{2.1}$$

where (W_i, M_{i-1}, M_i) is an elementary cobordism.

By basic Morse theory, there is a Morse function $f\colon (W, M, M') \to [0,1]$ with $f^{-1}(0) = M$ and $f^{-1}(1) = M'$. Since a collared neighborhood of M is diffeomorphic to $M \times I$, and similarly for M', we may assume that our Morse function has all of its critical values in $(0,1)$. One can arrange for the critical values $v_1, \ldots, v_k$ to be distinct. Let the i^{th} critical point have index ι_i. The main theorem of Morse theory precisely tells us that $f^{-1}([0, v_i + \epsilon]))$ is obtained from $f^{-1}([0, v_i - \epsilon])$ by attaching an ι_i-handle along some subset of $f^{-1}(v_i - \epsilon)$, and that $f^{-1}([0, v_{i+1} - \epsilon])$ is diffeomorphic to $f^{-1}([0, v_i + \epsilon])$. This means that we can build (W, M, M') from $(M \times I, M, M)$ by attaching a sequence of handles. We have thus proven:

Theorem 2.2.7 (Handle Decomposition Theorem). *Every cobordism (W, M, M') admits a handle decomposition.*

2.3 Surgery, Handles, and Cobordism

Attaching a handle to the trivial cobordism $(M \times I, M, M)$ produces a cobordism (W, M, M') between M and some manifold M'. In this section, we will study the operation, called *surgery*, that produces M' from M. Since the handle decomposition theorem guarantees that every cobordism arises as a sequence of elementary cobordisms, we will be able to construct every manifold cobordant to M by performing a sequence of surgeries on M. We begin with some terminology:

Definition 2.3.1. Let M^m be a manifold.

1. A *framed embedding* of a manifold N^n in M is an embedding $f\colon N^n \times D^{m-n} \hookrightarrow M$. Equivalently, (by the tubular neighborhood theorem), it is an embedding of N in M with a fixed trivialization of the normal bundle.

2. An *n-embedding* in M is an embedding $f\colon S^n \hookrightarrow M$.

3. A *framed n-embedding* is a framed embedding $f\colon S^n \times D^{m-n} \hookrightarrow M$.

We can now define this section's primary object of study:

Definition 2.3.2. Let M^m be a manifold (possibly with boundary), let $f\colon S^n \times D^{m-n} \hookrightarrow M$ be a framed n-embedding, and let B be such that

$$\partial(\mathrm{cl}(M^m \setminus f(S^n \times D^{m-n}))) = \partial M \amalg (S^n \times S^{m-n-1}) = \partial M \cup B.$$

The operation of cutting out $S^n \times D^{m-n}$ and gluing in $D^{n+1} \times S^{m-n-1}$ along B is called *n-surgery* (or just *surgery* when no ambiguity results) on $f(S^n \times D^{m-n}) \subseteq M$. The new manifold

$$\mathrm{cl}(M \setminus f(S^n \times D^{m-n})) \cup_{S^n \times S^{m-n-1}} D^{n+1} \times S^{m-n-1}$$

that results is called the *effect* of the surgery, and $\overline{f} = f|_{S^n \times \{0\}} \in \pi_n(M)$ is said to be *killed* by the surgery. We note that the effect of the surgery has the same boundary as M.

Now consider the elementary cobordism (W^{m+1}, M, M') obtained by attaching an $(n+1)$-handle to the trivial cobordism $(M^m \times I, M, M)$. We have

$$W = M \times I \cup_{(S^n \times D^{m-n}) \times \{1\}} D^{n+1} \times D^{m-n},$$

so that

$$\partial W = M \amalg (\mathrm{cl}(M \setminus S^n \times D^{m-n}) \cup_{S^n \times S^{m-n-1}} D^{n+1} \times S^{m-n-1}),$$

i.e., M' is the effect of an n-surgery on M, and M is cobordant to M'.

Definition 2.3.3. With the notation above, we say that W is the *trace* of the n-surgery that produces M' from M.

An elementary cobordism from M to M',

$$(W, M, M') \cong (M \times I, M, M) \cup (W, M, M') \cup (M' \times I, M', M'),$$

can equally well be viewed as an elementary cobordism

$$(W, M', M) \cong (M' \times I, M', M') \cup (W, M', M) \cup (M \times I, M, M)$$

from M' to M. In this case, we regard the handle $D^{n+1} \times D^{m-n}$ as an $(m-n)$-handle attached to $M' \times I$, so that (W, M', M) is an elementary cobordism of index $m-n$. We call this the *dual cobordism* to (W, M, M').

Since the dual cobordism is an elementary cobordism, it is the trace of a surgery on M' with effect M. We call this the *dual surgery* to the one that turns M into M'. It is an $(m-n-1)$-surgery, obtained by performing surgery along the framed embedding $D^{n+1} \times \partial(D^{m-n}) \hookrightarrow M'$.

We now consider the effects of surgery on the homotopy and homology of a manifold. By Remark 2.2.3, W is obtained up to homotopy from $M \times I$ by attaching an n-cell. The following proposition now follows easily from the long exact sequence on homology and the relative Hurewicz theorem:

Proposition 2.3.4. *Let $f\colon S^n \times D^{m-n} \to M$ be a framed n embedding, let $\overline{f} \in \pi_n(M)$ be the restriction $\overline{f} = f|_{S^n \times \{0\}}$, and let (W, M, M') be the trace of the surgery along $f(S^n \times D^{m-n})$. In addition, let $\langle \overline{f} \rangle$ denote the normal subgroup of $\pi_n(M)$ generated by $\overline{f}$. We have:*

$$\pi_i(W) = \begin{cases} \pi_i(M) & \text{if } i < n, \\ \pi_n(M)/\langle \overline{f} \rangle & \text{if } i = n \end{cases}$$

and

$$H_i(W, M) = \begin{cases} 0 & \text{if } i \neq n+1, \\ \mathbb{Z} & \text{if } i = n+1. \end{cases}$$

Applying these equations to the dual cobordism, we also obtain:

$$\pi_i(W) = \begin{cases} \pi_i(M') & \text{if } i < m-n-1, \\ \pi_n(M')/\langle \overline{f'} \rangle & \text{if } i = m-n-1 \end{cases}$$

and

$$H_i(W, M') = \begin{cases} 0 & \textit{if } i \neq m-n, \\ \mathbb{Z} & \textit{if } i = m-n, \end{cases}$$

where $f' \colon S^{m-n-1} \times D^{n+1}$ *is the dual framed* n*-embedding to* f*, and* $\overline{f'} \in \pi_{m-n-1}(M')$ *equals* $f'|_{S^{m-n-1}\times\{0\}}$*. In particular, if* $m > 2n+1$*,*

$$\pi_i(M') = \begin{cases} \pi_i(M) & \textit{if } i < n, \\ \pi_n(M)/\langle\overline{f}\rangle & \textit{if } i = n. \end{cases}$$

As such, if $n \leq (m-1)/2$, n-surgery simplifies $\pi_n(M)$. One can use this to systematically kill off all elements of the homotopy groups below the middle dimension that can be represented by framed embeddings, thereby producing a highly connected manifold in the same cobordism class. If $m = 2n$ or $m = 2n+1$, however, n-surgery need not simplify π_n. In the case $m = 2n+1$, for example, all we get is

$$\pi_n(M)/\langle\overline{f}\rangle = \pi_n(M')/\langle\overline{f'}\rangle.$$

The question of when the middle homotopy group can be simplified by surgery is a complicated one, and its solution will be discussed in section 4.4, although there we shall discuss it in the slightly different context of surgery on normal maps, and the dimensions will be shifted by one.

Remark 2.3.5. Proposition 2.3.4 suggests a nice interpretation of surgery. In the category of CW complexes, one can simplify the n^{th} homology and homotopy groups of a space by attaching an $(n+1)$-cell along its boundary to a nontrivial element of π_n. If we do this to a manifold M^m, $2n+1 < m$, this will change $H_n(X)$ and $H_{n+1}(X)$, but it will not affect $H_{m-n}(X)$ or $H_{m-n-1}(X)$. As such, the new space will not obey Poincaré duality, so it won't be a closed manifold. Accordingly, when we attach an n-cell, we need to remove an $(m-n)$-cell to preserve Poincaré duality if we would like to have any chance of obtaining a new manifold as the result. This is precisely the homotopy theoretic effect of n-surgery.

Example 2.3.6. As a fun application of surgery, we now use it show that for every finitely presented group G and integer $n \geq 4$, there is an n-dimensional manifold M with $\pi_1(M) = G$. To this end, suppose that G is generated by $g_1, \ldots, g_k$, subject to the relations $r_1, \ldots, r_\ell$, and let $N = (S^1 \times S^{n-1})\# \ldots \#(S^1 \times S^{n-1})$ be the connected sum of k copies of $(S^1 \times S^{n-1})$. The fundamental group of N is the free group on k generators, generated by a copy of S^1 from each of the connected summands. Now, represent r_1 by some loop in N, which we may take to be an embedding of S^1 into M by the weak Whitney embedding theorem (Theorem 2.1.8. N and S^1 are orientable, so the normal bundle of S^1 in N is orientable as well. The only orientable bundles over S^1 are the trivial bundles, so we can extend our embedding to a framed 1-embedding and kill r_1 by surgery. By the dimensional hypothesis and Theorem 2.3.4, this produces a new manifold N' with $\pi_1(N') = \pi_1(N)/\langle r_1\rangle$, and N' is clearly still orientable. We can thus repeat this process for each of the relations $r_2, \ldots, r_\ell$, obtaining a manifold with fundamental group $\mathbb{Z} * \mathbb{Z} \cdots * \mathbb{Z}/\langle r_1, \ldots, r_\ell\rangle = G$, as desired.

2.4 The H-Cobordism Theorem

Our primary goal is to classify simply connected manifolds of dimension greater than or equal to five up to diffeomorphism. Our general program is to reduce this classification to homotopy theory and then solve the homotopy theory. The first step in the reduction to homotopy theory is to reduce statements about diffeomorphisms to statements about cobordisms. This is accomplished by Smale's h-cobordism theorem.

Definition 2.4.1. An *h-cobordism* is a cobordism (W, M, M') such that the inclusions $M \hookrightarrow W$ and $M' \hookrightarrow W$ are homotopy equivalences. We say that (W, M, M') is is *trivial* if there exists a diffeomorphism $(W, M, M') \cong (M \times I, \{0\}, \{1\})$, so that $M \cong M'$.

Theorem 2.4.2 (Smale's H-Cobordism Theorem). *Let (W, M, M') be an h-cobordism, and suppose that $W \simeq M \simeq M'$ is simply connected and $\dim(M) \geq 5$. Then (W, M, M') is a trivial h-cobordism, and M is diffeomorphic to M'.*

Proof. The proof is an induction on the handle decomposition of an h-cobordism. Smale shows that the handle decomposition of an h-cobordism is highly nonunique, and there are certain operations one can perform on a handle decomposition to obtain a new one. He uses the Whitney trick to show that one can make the geometric intersections of handles the same as what their homological intersections predict, and then he systematically simplifies the homological intersections of the handles in the decomposition using the aforementioned operations.

The main step is the famous "handle cancellation lemma," which says that if two adjacent handles of complementary dimension intersect in a point, attaching both of them to a manifold doesn't change its diffeomorphism type, so they can be "cancelled." This eventually permits him to remove all of the handles on an h-cobordism.

The dimensional hypothesis comes from the fact that the handle cancellation lemma requires two handles to intersect geometrically in a point, whereas the operations one performs affect the algebraic (homological) intersection of the handles. The use of the Whitney trick to make the algebraic and geometric intersections coincide requires the dimension to be greater than or equal to five. For a proof, see [Kos93], [Mil65], [Sma61], or [RS72] (for the PL version). ☐

Since two diffeomorphic manifolds are obviously h-cobordant, this implies that the classification of simply connected manifolds of dimension greater than or equal to five up to diffeomorphism is identical to their classification up to h-cobordism, as desired.

Remark 2.4.3. The hypothesis that the manifolds be simply connected is, in fact, essential. There is an invariant of a homotopy equivalence called its *Whitehead torsion* that that lives in a group called the *Whitehead group* $Wh(\mathbb{Z}[\pi_1(M)])$ of the group algebra of the fundamental group. The s-cobordism theorem of Barden, Mazur, and Stallings states that an h-cobordism (W, M, M') is trivial if and only if the homotopy equivalences $W \simeq M$ and $W \simeq M'$ have vanishing Whitehead torsion. In general, this torsion need not vanish, so it is possible to have h-cobordisms that are not trivial. However, the group $Wh(\mathbb{Z})$ is trivial, so homotopy equivalences of simply connected manifolds never have torsion. As such, the s-cobordism theorem reduces to the h-cobordism theorem in the simply connected case. For a discussion and proof of the s-cobordism theorem, see [Kos93] or [RS72].

2.5 Exotic Spheres and the Generalized Poincaré Conjecture

Thus far, everything that we have done carries over to the piecewise linear category without incident. Thom's transversality theorem, Whitney's theorem, the theory of handlebodies, and the h-cobordism theorem all remain true, and their proofs are the same in nature if different in language. In fact, nothing that that we have done so far has given us any reason to believe that the two categories differ in any substantive way. However, in this section, we shall see that there is a marked difference between the two. We shall show that there exist high-dimensional smooth manifolds homotopy equivalent to but not diffeomorphic to S^n. On the other hand, we shall also prove the generalized Poincaré conjecture, which states that for $n \geq 5$, any piecewise linear manifold homotopy equivalent to S^n is in fact PL-isomorphic to it, and any smooth manifold homotopy equivalent to S^n is homeomorphic (but not necessarily diffeomorphic) to it. This shows that the smooth category is, in a sense, much more complicated than the PL one.

The root cause of his difference, as we shall see, is that every PL-isomorphism $\partial D^n \to \partial D^n$ extends to a PL-isomorphism $D^n \to D^n$, but the same is not true of diffeomorphisms. As such, there is, in essence, only one PL-isomorphism from $S^n \to S^n$, but there can be many diffeomorphisms of the sphere to itself. This means that there are more different ways to attach two disks to each other along their boundaries in the smooth case than in the PL one.

2.5.1 The Generalized Poincaré Conjecture

Theorem 2.5.1 (Generalized Poincaré Conjecture [Sma61]).

1. *If M is a smooth manifold that is homotopy equivalent to S^n, $n \geq 5$, then M is homeomorphic to S^n.*

2. *If M is a PL manifold that is homotopy equivalent to S^n, $n \geq 5$, then M is PL-isomorphic to S^n.*

Proof for $n \geq 6$. We shall only prove this theorem for $n \geq 6$. The case $n = 5$ is slightly harder; see [Kos93] or [Sma61] for a proof. The generalized Poincaré conjecture actually holds in dimension 4 as well, but the proof is much more difficult and uses different methods than those discussed here. See [Fre82]. The conjecture that it holds in dimension 3 is the (ungeneralized) Poincaré conjecture. It is presently unresolved, and it remains one of the major open problems in mathematics.

For $n \geq 6$, the theorem follows relatively easily from the h-cobordism theorem. Let M be an n-dimensional smooth (respectively PL) manifold that is homotopy equivalent to S^n. Remove two small disjoint n-disks D_1 and D_2 from M to produce $N \cong cl(M \setminus (D_1 \cup D_2))$. N is now a manifold with boundary $\partial D_1 \cup \partial D_2 \cong S^{n-1} \cup S^{n-1}$. A simple application of excision and Lefschetz duality shows that the relative homology groups of $(N, \partial D_i)$ vanish, so the inclusions $\partial D_i \hookrightarrow N$ induce homotopy equivalences by Whitehead's theorem. N is therefore an h-cobordism (N, S^{n-1}, S^{n-1}). Since $n \geq 6$, the h-cobordism theorem tells us the N is diffeomorphic (respectively PL-isomorphic) to the product $S^{n-1} \times I$.

Gluing D_1 and D_2 back onto N, we see that

$$M \cong N \cup D_1 \cup D_2 \cong (S^{n-1} \times I) \cup_f D_1 \cup_g D_2,$$

where $f\colon S^{n-1} \times \{0\} \to \partial D_1$ and $g\colon S^{n-1} \times \{1\} \to \partial D_2$ are diffeomorphisms (respectively PL-isomorphisms). We may choose coordinates on $S^{n-1} \times I$ such that f is the identity, so we obtain

$$M \cong (S^{n-1} \times I) \cup_{Id} D_1 \cup_g D_2 \cong D_3 \cup_g D_2.$$

M is therefore a union of two disks identified along their boundaries by a diffeomorphism (respectively PL-isomorphism) from S^{n-1} to itself.

We now note:

Lemma 2.5.2. *If D and D' are PL n-disks and $f\colon \partial D \to \partial D'$ is a PL-isomorphism, then f extends to a PL-isomorphism from D to D'. If f is a homeomorphism between smooth n-disks, then f extends to a homeomorphism from D to D'.*

This follows immediately by using polar coordinates and extending the map radially in the smooth case, and by doing the analogous construction with cones and conical extensions in the PL case.

Remark 2.5.3. This lemma fails if we replace "PL-isomorphism" with "diffeomorphism," as we do not know that the extension is a diffeomorphism in a neighborhood of the center of the disks. This is what prevents our proof from showing that M is diffeomorphic to S^n.

Now, map D_3 into S^n as the "southern hemisphere." The map g gives us a diffeomorphism (respectively PL-isomorphism) between ∂D_2 and the boundary of the northern hemisphere of S^n. By the lemma, this extends to a homeomorphism (respectively PL-isomorphism) between D_2 and the northern hemisphere of S^n, which completes the proof. □

2.5.2 An Exotic Sphere

In this section, we shall sketch a counterexample to the generalized Poincaré conjecture for smooth manifolds. In particular, we shall construct a seven-dimensional manifold M^7 that is homotopy equivalent to S^7 but not diffeomorphic to it. The counterexample that we shall give, due to Milnor ([Mil56] or [Mil00]), was historically the first one to be found, and it is also the lowest-dimensional. However, there exist so-called "exotic spheres" in most dimensions greater than or equal to seven. These were classified, using essentially the first application of surgery theory, by Kervaire and Milnor in [KM63]. We shall not cover in depth the surgery theoretic classification of exotic spheres, but we shall prove in Theorem 6.1.1 that there are finitely many in each dimension greater than or equal to five.

We shall only sketch the proof that our counterexample has the desired properties. For the details, see [Mil56]. The manifold M^7 will be constructed as the sphere bundle associated to a 4-disk bundle over S^4. Four-dimensional disk bundles over S^4 are classified by $\pi_3(SO(4)) = \mathbb{Z} \oplus \mathbb{Z}$. Let E be the four-dimensional disk bundle corresponding to $(i, j) \in \mathbb{Z} \oplus \mathbb{Z}$. E is an eight-dimensional manifold with boundary $\partial E = M^7$. It is not difficult to show that M^7 has the homotopy type of S^7 if and only if $i + j = \pm 1$. For concreteness, set $j = 1 - i$. We claim that, for appropriate values of i and j, M^7 provides our desired counterexample.

Assume to the contrary that M^7 is diffeomorphic to S^7. We claim that we can attach an 8-cell to E along $\partial E \cong S^7$ to obtain a smooth closed manifold. For this, it suffices to construct any 8-manifold with boundary S^7 to which we can attach a single 8-cell to form a smooth manifold.

To this end, we note that the quaternionic projective plane $\mathbb{HP}^8$ is an 8-dimensional manifold of the form $S^4 \cup e^8$, where e^8 is an 8-cell. The S^4 may be taken to be embedded in $\mathbb{HP}^8$ by the Whitney embedding theorem (Theorem 2.1.9), and a regular neighborhood of it is a disk bundle over S^4. Since $\mathbb{HP}^8$ is a closed manifold obtained by attaching a single 8-cell to the boundary of this regular neighborhood, the boundary of this disk bundle must be diffeomorphic to S^7.

If $\partial E \cong S^7$, is therefore possible to attach an eight-dimensional cell to E and produce a smooth manifold. Accordingly, form a new manifold $N^8 = E \cup e^8$. By cellular cohomology, N has nonzero cohomology groups $H^0(N) = H^4(N) = H^8(N) = \mathbb{Z}$, so, by Poincaré duality, its intersection form must have signature ± 1, and we may choose orientations so that the signature $I(N) = 1$.

We note now that the first Pontrjagin class of the tangent bundle of N is equal to $2(i - j) = 2(2i - 1)$ times the generator of $H^4(N)$. To see this, we observe that $p_1(\tau_N)$ is clearly linear in i and j, and it is invariant under a reversal of the orientation of the fiber. Reversing the orientation of the fiber replaces (i, j) with $(-i, -j)$. This implies that $p_1(\tau_N) = c(i - j)$ for some c. An explicit examination of the quaternionic projective plane shows that $c = 2$.

However, the Hirtzebruch signature theorem tells us that

$$I(N) = \frac{1}{45}(7p_2 - (p_1)^2)[N],$$

where p_i is the i^{th} Pontrjagin class of tangent bundle of N. Plugging in the values that we have for I and p_1 and solving for p_2, we obtain

$$p_2[N] = \frac{4(2i-1)^2 + 45}{7}.$$

If we set $i = 2$, this yields $p_2[N] = 81/7$. Since p_2 belongs to $H^8(N, \mathbb{Z})$, but $81/7$ is not an integer, this is a contradiction. We thus have that M^7 is a smooth manifold that is homotopy equivalent, but not diffeomorphic, to S^7. We note that M^7 is homeomorphic to S^7 by the generalized Poincaré conjecture, so what we have constructed is an exotic smooth structure on the topological seven-sphere.

Chapter 3

The Structure Set and Normal Maps

We now have all of the pieces in place to begin our diffeomorphism classification of manifolds of dimension ≥ 5 in a given simply connected homotopy type.

3.1 Manifold Structures and the Structure Set

Definition 3.1.1. Let X be a simply connected CW complex. A *manifold structure* on X is a pair (M, f), where M is a (closed) smooth manifold and $f\colon M \to X$ is a homotopy equivalence. Two manifold structures (M, f) and (M', f') are said to be equivalent (or *h-bordant*) if there exists an h-cobordism (W, M, M') and a map $F\colon W \to X$ such that $F|_M = f$ and $F|_{M'} = f'$.

Definition 3.1.2. The *(smooth) structure set* $\mathcal{S}(X)$ of a CW complex X is the set of equivalence classes of manifold structures on X.

Remark 3.1.3. The natural equivalence relation to put on manifold structures is to take (M, f) to be equivalent to (M', f') if there exists a diffeomorphism $\phi\colon M \to M'$ such that the diagram

$$\begin{tikzcd} M \arrow[d, "\phi"'] \arrow[dr, "f"] & \\ M' \arrow[r, "f'"'] & X \end{tikzcd}$$

commutes up to homotopy. However, these two equivalence relations coincide in dimensions ≥ 5 by the h-cobordism theorem, and h-cobordisms will prove much easier to analyze than diffeomorphisms. This reduction is the first place in our theory where we make essential use of the hypothesis that our manifolds are of dimension greater than or equal to 5.

3.2 Poincaré Complexes

From here on in, we shall focus on two main questions about a simply connected CW complex X:

1. Is $\mathcal{S}(X)$ nonempty?

2. If so, how can we compute it?

The first step in answering these questions is to note that there are certain homological and homotopy theoretic conditions obeyed by all manifolds, and thus by anything homotopy equivalent to a manifold. As such, these must be satisfied by X if $\mathcal{S}(X)$ is to stand a chance at being nonempty. The most obvious such restriction is Poincaré duality, which leads us to define a *Poincaré complex* to be a finite CW complex whose homology obeys Poincaré duality:

Definition 3.2.1. A *n-dimensional simply connected Poincaré complex* is a finite simply connected CW complex X with an element $[X] \in H^n(X)$ such that $[X] \cap -\colon H^k(X) \to H_{n-k}(X)$ is an isomorphism for all k. We call $[X]$ the *fundamental class* of X.

Remark 3.2.2. Here we make weak use of our hypothesis of simple connectivity. For nonsimply connected Poincaré complexes, we would require Poincaré duality with twisted coefficients.

Since every manifold is manifestly a Poincaré complex, any CW complex that is not a Poincaré complex must have an empty structure set. As such, we have reduced our problem to the computation of the structure sets of Poincaré complexes.

3.3 Bundle Theoretic Properties of Manifolds

The second property that all manifolds obey is bundle theoretic in nature. It roughly corresponds to the statement that a manifold has a well-defined stable normal bundle, and the stable sphere bundle associated to it depends only on the homotopy type of the manifold.

In order to state this property, we shall need the language of spherical fibrations, which slightly generalizes that of sphere bundles.

3.3.1 Spherical Fibrations

Definition 3.3.1. A k-dimensional spherical fibration over X is a pair (ξ, π), $\pi\colon \xi \to X$, that fits into a fibration

$$S^{k-1} \longrightarrow \xi \xrightarrow{\pi} X\ .$$

As with vector bundles, we shall often abuse notation and use the total space ξ to denote the spherical fibration (ξ, π).

Definition 3.3.2. Two spherical fibrations (ξ, π) and (ξ', π') are said to be *isomorphic* or *fiber homotopy equivalent* if there exists a homotopy equivalence $f: \xi \to \xi'$ such that

$$\begin{tikzcd} \xi \arrow[rr, "f"] \arrow[dr, "\pi"'] & & \xi' \arrow[dl, "\pi'"] \\ & X & \end{tikzcd}$$

commutes up to homotopy. In this case, we write $\xi \simeq \xi'$.

Definition 3.3.3. If $E \to X$ is a vector bundle with associated disk bundle $D \to X$, the restriction $\partial D \to X$ is a spherical fibration. We call it the spherical fibration *associated to E* and denote it $J(E)$.

As in the case of vector bundles, we have a trivial spherical fibration:

Definition 3.3.4. The *trivial spherical fibration* ϵ^k is the fiber homotopy equivalence class of the fibration $\pi_2\colon S^{k-1} \times X \to X$.

Note the shifted dimensions in the previous definitions. This is so that the sphere bundle associated to a k-dimensional vector bundle is k-dimensional. In particular, the sphere bundle associated to the trivial k-dimensional vector bundle is ϵ^k.

To a spherical fibration (ξ, π), we can associate the mapping cylinder of π, which we denote $D(\xi)$. This gives us a fibration of pairs

$$(D^{k+1}, S^k) \longrightarrow (D(\xi), \xi) \longrightarrow X \ ,$$

so $D(\xi)$ can be thought of as analogous to the disk bundle associated to a sphere bundle. We note that we have *not* shown that there is a vector bundle whose associated sphere bundle is isomorphic to ξ. This does not occur in general. We shall discuss when this fails in section 5.1.

Just as for vector bundles, we can define pullbacks and Whitney sums of spherical fibrations, and our definitions are entirely analogous. In fact, note in that which follows that $f^*(J(E)) = J(f^*(E))$, and $J(E \oplus E') = J(E) \oplus J(E')$.

Definition 3.3.5. Let (ξ, π) be a spherical fibration over Y and let $f\colon X \to Y$. We define the *pullback* $f^*(\xi)$ to be the spherical fibration with total space

$$f^*(\xi) = \{(x, s) \in X \times \xi | f(x) = \pi(s)\}$$

and projection map $(x, s) \to x$, so that we have the diagram

$$\begin{tikzcd} f^*(\xi) \arrow[r, "(x,s)\mapsto s"] \arrow[d, "(x,s)\mapsto x"'] & \xi \arrow[d, "\pi"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \ .$$

As with vector bundles, a spherical fibration over X and a spherical fibration over Y naturally give rise to a spherical fibration over $X \times Y$. We define its total space so that, in the case of sphere bundles over manifolds, it will just be the boundary of the product of the associated disk bundles.

Definition 3.3.6. Let $\xi \to X$ and $\xi' \to Y$ be spherical fibrations of dimensions k and l respectively. We define the $(k+l)$-dimensional spherical fibration $\xi \times \xi'$ to have total space $(D(\xi) \times \xi') \cup (\xi \times D(\xi'))$ and the obvious projection map.

This allows us to define the Whitney sum of two spherical fibrations:

Definition 3.3.7. Let ξ and ξ' be spherical fibrations of dimensions k and l, respectively, over X, and let $\Delta\colon X \to X \times X$ be the diagonal map. We define the *Whitney sum* $\xi \oplus \xi'$ to be the $(k + l)$-dimensional spherical fibration $\Delta^*(\alpha \times \alpha)$.

There is also an analogue of stable vector bundles:

Definition 3.3.8. We say that ξ and ξ are *stably fiber homotopy equivalent* if $\xi \oplus \epsilon^m \simeq \xi' \oplus \epsilon^n$ for some m and n. The equivalence classes of spherical fibrations up to stable fiber homotopy equivalence are called *stable spherical fibrations.*

We note that all of the operations described above for spherical fibrations are well-defined as operations on stable spherical fibrations.

Finally, we shall need the notion of a vector bundle reduction of a spherical fibration, along with an equivalence relation on these objects. Intuitively, a vector bundle reduction of a spherical fibration ξ is a vector bundle along with a fiber homotopy equivalence between its associated spherical fibration and ξ, and two such reductions are equivalent if there is a homotopy equivalence of one to the other through vector bundles whose associated spherical fibrations are equal to ξ. Rigorously, we have:

Definition 3.3.9. Let $\pi\colon \xi \to X$ be a spherical fibration. A *vector bundle reduction* of ξ is a vector bundle E along with a fiber homotopy equivalence $\Phi\colon J(E) \simeq \xi$ between its associated spherical fibration $J(E)$ and ξ. (By a fiber homotopy equivalence, we mean the maps from $J(E) \to \xi$ and $\xi \to J(E)$ along with the homotopies of their compositions to the identity.) Two such pairs (E, Φ) and (F, Ψ) are isomorphic if there is a homotopy of pairs between them.

We define stable vector bundle reductions of stable spherical fibrations and isomorphisms thereof in the obvious analogous way.

Remark 3.3.10. For a description of vector bundle reductions in terms of classifying spaces, see section 5.1.

3.3.2 The Spivak Normal Fibration

Let M^m be a smooth manifold. By the Whitney embedding theorem (Theorem 2.1.8), M embeds into a high-dimensional sphere S^n, and any two such embeddings are isotopic for $n \gg m$. For sufficiently large n, M thus has a well-defined normal bundle N_M^{n-m} in S^n. Furthermore, $N_M^{n-m+1} \cong N_M^{n-m} + \epsilon^1$, where ϵ^1 is the trivial line bundle, so this endows M with a well-defined stable vector bundle N_M called its *stable normal bundle.*

The stable normal bundle depends on the diffeomorphism class of M, not just its homotopy type. (It is difficult to provide homotopy equivalent manifolds with different stable normal bundles at present, but the homotopy theoretic tools we develop in chapter 5 will render this quite simple.) However, it turns out that the stable spherical fibration associated to the stable normal bundle of M depends solely on the homotopy type of M. In fact, a more general theorem holds:

Theorem-Definition 3.3.11 (Spivak). *Let $X^k \subseteq S^n$ be a finite simply connected CW complex embedded in a high-dimensional sphere, and let N be a tubular neighborhood of X. The map $\partial N \hookrightarrow N \simeq X$ has homotopy fiber S^{n-k-1} if and only if X is a Poincaré complex. Furthermore, if X is a Poincaré complex and we treat this spherical fibration as a stable spherical fibration, its fiber homotopy equivalence class is independent of the embedding and thus depends only on the homotopy type of X. We call this fibration the* Spivak normal fibration of X, *and we denote it ν_X.*

Proof. See [Bro72]. □

By the tubular neighborhood theorem, the Spivak normal fibration of a $M \subseteq S^n$ is the stable spherical fibration associated to the stable normal bundle of M. If X is a Poincaré complex homotopy equivalent to X via a map $f\colon X \to M$, it follows from Theorem-Definition 3.3.11 that $f^*\nu_M \cong \nu_X$, and thus f^*N_M is a stable vector bundle reduction of ν_X. This gives us a bundle theoretic property that any Poincaré complex with nonempty structure set must obey:

Theorem 3.3.12. *Let X be a Poincaré complex. If $\mathcal{S}(X)$ is nonempty then ν_X admits a bundle reduction to a stable vector bundle.*

3.4 Normal Maps

The discussion above provides us with a necessary condition for the nonemptiness of the structure set of a Poincaré complex. It does not, however, provide us with any obvious clues about how close to sufficient this condition is, nor does it suggest how one might compute the structure set of the complex, or even construct a single element of it. In this section, we shall begin to answer these questions by showing how a vector bundle reduction of the Spivak normal fibration of a Poincaré complex gives us a candidate for membership in its structure set. The language that we shall use to do this is that of *degree one normal maps* and *normal bordisms.*

We begin with the notion of degree. The standard definition of the degree of a map of manifolds uses only that the manifold has a fundamental class. As such, it immediately carries over to the realm of Poincaré complexes:

Definition 3.4.1. Let $f\colon X \to Y$ be a map of n-dimensional connected Poincaré complexes, so that $H_n(X) = H_n(Y) = \mathbb{Z}$ are generated by $[X]$ and $[Y]$ respectively, and $f_*[X] = k[Y]$ for some $k \in \mathbb{Z}$. We call k the *degree* of f.

In particular, any homotopy equivalence has degree one if we correctly choose our signs. We now define this section's primary objects of study:

Definition 3.4.2. Let X be a Poincaré complex, and let M be a manifold. A *normal map* is a map $f\colon M \to X$ such that $f^*(\nu(X)) = \nu_M(= J(N_M))$. We shall sometimes denote such a map by (f, b), where $b\colon \nu_M \to \nu_X$ is the induced map of Spivak normal fibrations that covers f.

Definition 3.4.3. We say that two normal maps (f, b), $f\colon M \to X$, and (f', b'), $f'\colon M' \to X$, are *normally bordant* if there exists a cobordism (W, M, M') and a normal map $(F, B)\colon (W, M, M') \to (X \times I, X \times \{0\}, X \times \{1\})$ such that $(F, B)|_M = (f, b)$, and $(F, B)|_{M'} = (f', b')$.

As such, a normal bordism is, in a sense, simply a bordism that preserves the normal structure.

Definition 3.4.4. The set of normal bordism classes of degree one normal maps into a Poincaré complex X is called the *normal structure set of* X, and we denote it by $\mathcal{N}(X)$.

Theorem 3.4.5. *An element (M, f) of the structure set $\mathcal{S}(X)$ is a degree one normal map to (X, E), where E is a stable vector bundle reduction of the Spivak normal fibration ν_X.*

Proof. The map $f\colon M \to X$ is a homotopy equivalence, so it has degree one. Let g be a homotopy inverse to f, and take $E = g^*(N_M)$. The Spivak normal fibration ν_M is the stable spherical fibration associated to N_M, so $g^*\nu_M$ is the stable spherical fibration associated to E. By Theorem-Definition 3.3.11, $g^*\nu_M \cong \nu_X$, so E is a stable vector bundle reduction of ν_X. Since $f^*E = f^*(g^*N_M) = N_M$, our desired result follows. $\square$

Remark 3.4.6. It follows that the structure set $\mathcal{S}(X)$ is nonempty if and only if some degree one normal map to a stable vector bundle reduction of ν_X is a homotopy equivalence.

Theorem 3.4.7. *There is a one-to-one correspondence between isomorphism classes of stable vector bundle reductions of ν_X and normal bordism classes of degree one normal maps. Precisely, for every stable vector bundle reduction E of ν_X, there exists a degree one normal map to (X, E), and all such maps are normally bordant.*

This reduces the question of whether there exists $(M, f) \in \mathcal{S}(X)$ with $N_M = E$ to the question of whether the normal bordism class corresponding to E contains a homotopy equivalence. We shall show how to answer this question using surgery theory in the next section.

Before we can prove Theorem 3.4.7, we shall need to make a brief digression into the theory of Thom spaces of stable spherical fibrations. Let $\pi\colon \xi^k \to X^n$ be a spherical fibration with fiber S^{k-1}. Proceeding analogously to the vector bundle case, we define the Thom space of ξ to be the mapping cone of π. We note that the Thom space of a vector bundle is homotopy equivalent to that of its associated sphere bundle.

As in the vector bundle case, there exists a *Thom class* $U(\xi) \in H^k(T(\xi))$ such that the maps

$$U(\xi)\cup\colon H^*(X) \to H^{*+k}(T(\xi))$$

and

$$U(\xi)\cap\colon H_*(T(\xi)) \to H_{*-k}(X)$$

are isomorphisms. We note that if ϵ^j is a trivial S^{j-1} bundle, $T(\xi + \epsilon^j) = \Sigma^j T(\xi)$. We thus have a spectrum $T(\eta)$ associated to a stable spherical fibration η, and we define the stable homotopy groups $\pi_m^S(T(\eta))$ of a stable spherical fibration η realized by a $(k-1)$-sphere bundle ξ^k to be

$$\pi_m^S(T(\eta)) = \pi_{m+k}^S(T(\xi)) = \varinjlim_j \pi_{m+j+k}\Sigma^j T(\xi).$$

We note that there is a Hurewicz map $h^S\colon \pi_{n+k}^S T(\xi) \to H_{n+k}(T(\xi))$ induced by the usual Hurewicz map $h\colon \pi_{n+k}T(\xi) \to H_{n+k}T(\xi)$.

We now return to the Spivak normal fibration. Let X^k be a Poincaré complex embedded in S^n with tubular neighborhood N, so that the map $\partial N \hookrightarrow N \simeq X$ has homotopy fiber S^{n-k-1} by Theorem-Definition 3.3.11, and let ξ be this spherical fibration, which provides a finite realization of ν_X. The composition

$$S^n \to S^n/(S^n \setminus N) \simeq N/\partial N \simeq T(\xi)$$

gives rise to an element α of $\pi_n(T(\xi))$, and thus of $\pi_n^S(T(\xi))$. We note that the composition sends the fundamental class $[S^n]$ to the fundamental class of $(N, \partial N)$, so that $U(\xi) \cap h^S(\alpha) = [X]$.

Theorem 3.4.8 (Spivak). *Let ξ be a stable spherical fibration over a simply connected Poincaré complex X, and let $\alpha \in \pi_n^S(T(\xi))$ be such that $U(\xi) \cap h^S(\alpha) = [X]$. In addition, let ν_X be the Spivak normal fibration, and let $\beta \in \pi_n^S(T(\nu_X))$ obey $U(\nu_X) \cap h^S(\beta) = [X]$. Then ξ is fiber homotopy equivalent to the Spivak normal fibration ν_X by a fiber homotopy equivalence whose induced map of Thom spectra sends α to β.*

Proof. See [Bro72] ☐

This theorem can be thought of as a stronger version of the uniqueness of the Spivak normal fibration. With it, we may now prove Theorem 3.4.7.

Proof of Theorem 3.4.7. By Theorem 3.4.8, stable vector bundle reductions of the Spivak normal fibration of a Poincaré complex X^k are in one-to-one correspondence with triples $(\xi, \alpha, [E, \Phi])$, where ξ is a stable spherical fibration, $\alpha \in \pi_{m+k}^S(T(\xi))$ obeys $U(\xi) \cap h^S(\alpha) = [X]$, and two such triples $(\xi, \alpha, [E, \Phi])$ and $(\xi', \alpha', [E', \Phi'])$ are taken to be equivalent if there is an isomorphism of bundle reductions $[E, \Phi] \simeq [E', \Phi']$ whose induced map of Thom spectra takes $\alpha \in \pi_m^S(T(\xi))$ to $\alpha' \in \pi_m^S(T(\xi'))$.

Given a triple $(\xi, \alpha, [E, \Phi])$, we construct a normal map as follows. Let F^m be a high-dimensional bundle that realizes E, and use the lift Φ to treat α as an element of $\pi_{m+k}^S(T(F)) = \pi_{m+k}(T(F)))$. By our stronger version of the Thom transversality theorem (Theorem 2.1.6), there exists a map $\beta\colon S^{m+k} \to T(F)$ that is homotopic to α and transverse to $X \subseteq T(F)$. It follows from transversality and the tubular neighborhood theorem that $M = \beta^{-1}(X)$ is a manifold, and the restriction of β to a tubular neighborhood of $\beta^{-1}(X)$ gives a bundle map b from the stable normal bundle N_M of M to E. If $T(b)\colon T(N_M) \to T(E)$ is the induced map of Thom spaces and γ is the fundamental class of $T(N_M)$, it follows from the naturality of the Thom isomorphism that $\beta_*([M]) = \beta_*(U(M) \cap \gamma) = T(b)_*(U(N_M)) \cap T(b)_*(\gamma) = U(E) \cap h^S(\alpha) = [X]$, so we have constructed a degree one normal map.

If we choose a different representative $F \oplus \epsilon^\ell$ of E and choose the transverse representative of the homotopy class compatibly, this construction clearly yields the same normal map.

We claim that homotopy equivalent β give normally bordant normal maps. Indeed, let $f\colon S^{m+k} \times I \to T(E)$ be a homotopy between two maps β and β' that are both transverse to $X \in T(E)$. By the stronger version of Thom transversality (Theorem 2.1.6) we can make f transverse to X without changing $f|_{S^{m+k}\times\{0,1\}}$. The preimage of X under this map yields the desired normal bordism.

In the other direction, we wish to construct a triple $(\xi, \alpha, [E, \Phi])$ from a normal map (M, f, b). Let ξ be the spherical fibration over X given by $b(J(N_M))$, and let $[E, \Phi]$ be the canonical lift of ξ given by $b(N_M)$. Embed $M \subseteq S^n$ for $n \gg 0$ with regular neighborhood N. The composition

$$S^n \longrightarrow S^n/(S^n \setminus N) \simeq T(N_M) \xrightarrow{T(b)} T(E)$$

gives us a map α from S^n to $T(E)$, which is easily seen to have the required properties, so we have constructed our desired pair. Furthermore, suppose (M, f) is normally bordant to (M', f') by a normal bordism (W, F, B). Embed W in $S^n \times I$ with tubular neighborhood Q so that $W \cap S^n \times \{0\} = M$, $W \cap S^n \times \{1\} = M'$, and B sends the normal bundle of $W \cap S^n \times \{t\}$ to E. The composition

$$S^n \times I \longrightarrow S^n \times I/(S^n \times I \setminus Q) \simeq T(N_W) \xrightarrow{T(B)} T(E)$$

gives us a homotopy between the α induced by the two normal maps. $\square$

Chapter 4

Surgery Theory

Theorems 3.4.7 and 3.4.5 of the last section told us that there is a unique normal bordism class of degree one normal maps corresponding to every stable vector bundle reduction of the Spivak normal fibration ν_X of a simply connected Poincaré complex X, and every element of the structure set is in one of these normal bordism classes. In chapter 5, we shall show how to enumerate all of the stable vector bundle reductions of ν_X. This reduces the calculation of $\mathcal{S}(X)$ to the determination of when a normal bordism class contains a homotopy equivalence, and how many distinct homotopy equivalences it contains. We shall answer both of these questions in this section using surgery theory.

The basic procedure is to start with a degree one normal map and manufacture normal bordisms that transform it into a homotopy equivalence. Before we can do this, we need a precise measure of how much a map deviates from being a homotopy equivalence. We provide this by defining the homotopy and homology groups *of a map.*

4.1 Homotopy, Homology, and Cohomology Groups of a Map

The homotopy, homology, and cohomology groups of a space measure how close the space is to being contractible; the homotopy, homology, and cohomology groups of a map measure how close the map is to being a homotopy equivalence. If $f\colon X \to Y$ is the inclusion of a subspace, we already know how to do this—we consider the relative homotopy, homology, and cohomology groups $\pi_k(Y, X)$, $H_k(Y, X)$, and $H^k(Y, X)$.

However, in the homotopy category, every map can be treated as an inclusion. X includes into the mapping cylinder $M_f = ((X \times I) \amalg Y)/(x \times \{1\} \sim f(x))$, which deformation retracts onto Y. We thus define:

Definition 4.1.1. The homotopy, homology, and cohomology groups of a map $f\colon X \to Y$ are the relative groups $\pi_k(M_f, X)$, $H_k(M_f, X)$, and $H^k(M_f, X)$. We say f is *simply connected* if $\pi_0(f) = \pi_1(f) = 0$ and *n-connected* if $\pi_k(f) = 0$ for all $k \leq n$.

Theorem 4.1.2. *These groups fit into long exact sequences*

$$\cdots \longrightarrow \pi_{k+1}(f) \longrightarrow \pi_k(X) \xrightarrow{f_*} \pi_k(Y) \longrightarrow \pi_k(f) \longrightarrow \pi_{k-1}(X) \longrightarrow \cdots,$$

$$\cdots \longrightarrow H_{k+1}(f) \longrightarrow H_k(X) \xrightarrow{f_*} H_k(Y) \longrightarrow H_k(f) \longrightarrow H_{k-1}(X) \longrightarrow \cdots,$$

and

$$\cdots \longrightarrow H^{k-1}(X) \longrightarrow H^k(f) \longrightarrow H^k(Y) \xrightarrow{f^*} H^k(X) \longrightarrow H^{k+1}(f) \longrightarrow \cdots.$$

Proof. This follows immediately from the long exact sequences for the pair (M_f, X). □

Remark 4.1.3. It will occasionally be useful to use an alternative representation of the homotopy groups of a pointed map, in which we represent an element of $\pi_k(f)$ as a pointed homotopy class of pairs (α, β), where $\alpha\colon S^{k-1} \to X$ is a pointed map and β is a pointed nullhomotopy of the composition $f \circ \alpha$. These pairs are homotopy classes of diagrams

$$\begin{tikzcd} S^{k-1} \arrow[r, "\alpha"] \arrow[d, hook] & X \arrow[d, "f"] \\ D^k \arrow[r, "\beta"] & Y \end{tikzcd},$$

in which all maps are pointed. The equivalence of the two representations follows immediately from Theorem 4.1.2.

Now, from Theorem 4.1.2 and the standard Hurewicz and Whitehead theorems, we get:

Theorem 4.1.4.

1. *A map f is a homotopy equivalence if and only if $\pi_k(f) = 0$ for all k.*
2. *If X^m, Y^n, and f are simply connected, and f is n-connected, then $H_i(f) = 0$ for all $i \leq n$, and the Hurewicz map $\pi_{n+1}(f) \to H_{n+1}(f)$ is an isomorphism.*

The homology and cohomology groups of f behave quite nicely with respect to the Poincaré duality map if f is a degree one normal map of simply connected Poincaré complexes.

Theorem 4.1.5. *Let $K_*(f) = H_{*+1}(f)$ and $K^* = H^{*+1}(f)$, and assume $f\colon X \to Y$ to be a degree one normal map of simply connected Poincaré complexes. Then*

$$H_*(X) \cong K_*(f) \oplus H_*(Y)$$

and

$$H^*(X) \cong K^*(f) \oplus H^*(Y).$$

Furthermore, this decomposition respects the Poincaré duality map on X, so that

$$[X] \cap -\colon K^k(f) \to K_{m-k}(f),$$

and this map is an isomorphism.

Sketch of Proof. The point is that the Poincaré duality maps give rise to splittings of f on homology and cohomology. Consider the diagram

$$\begin{tikzcd} H_*(X) \arrow[r, "f_*"] \arrow[d, shift left, "p"] & H_*(Y) \arrow[d, shift left, "p"] \\ H^{m-*}(X) \arrow[u, shift left, "q"] & H^{m-*}(Y) \arrow[l, "f^*"'] \arrow[u, shift left, "q"] \end{tikzcd},$$

where p and q are the Poincaré duality isomorphisms, and define

$$f^! = q \circ f^* \circ p$$

and

$$f_! = p \circ f_* \circ q.$$

We have $f^! \circ f_* = Id$ and $f^* \circ f_! = Id$. This and an easy diagram chase give us $K_k(f) \cong \ker(f_*\colon H_k(X) \to H_k(Y))$, $K_k(f) \cong \ker(f_!\colon H^k(X) \to H^k(Y))$, and the asserted direct sum decompositions. Furthermore, if $\alpha \in \ker(f_!\colon H^k(X) \to H^kY)$, then $f_!(\alpha) = p(f_*(\alpha \cap [X])) = 0$, so $f_*(\alpha \cap [X]) = 0$, and $\alpha \cap [X] \in K_{m-k}$. A similar argument shows the other case of the assertion that the Poincaré duality map respects the direct sum decomposition. The fact that it restricts to an isomorphism on K^k then follows immediately from the fact that the original map was an isomorphism. □

It follows easily from the above and standard exact sequence arguments that if $(f, \partial f)\colon (M, \partial M) \to (X, \partial X)$ is a map of pairs, then the relative groups $K_*(f, \partial f) = H_{*+1}(f, \partial f)$ and $K^*(f, \partial f) = H^{*+1}(f, \partial f)$ fit into exact sequences

$$K_{n+1}(M, \partial M) \to K_n(\partial M) \to K_n(M) \to K_n(M, \partial M) \to K_{n-1}(\partial M).$$

4.2 Surgery on Normal Maps

We recall that surgery is a general method for producing cobordisms of manifolds. In section 2.3, we mentioned that surgery can be used to simplify the first $\lfloor n/2 \rfloor - 1$ homotopy groups of a manifold. We aim here to apply analogous ideas to find normal bordisms that kill off the homotopy groups of a normal map. This requires us to generalize our previous methods in two ways: we need to be able to perform surgery on a *map* instead of a manifold, and we need to make these surgeries respect the normal structure of our map. Once we do this, we shall actually get an even stronger result than we had for manifolds—we shall show that every degree one normal map is normally bordant to an $(\lfloor \frac{n}{2} \rfloor - 1)$-connected one, and there is a well-defined obstruction that vanishes if and only if the map is normally bordant to a homotopy equivalence. We begin by describing the operation of surgery on a map.

Definition 4.2.1. Let $f\colon M^m \to X$ be a map from a manifold M to a Poincaré complex X.

1. An *n-embedding* into f is an element of $\pi_{n+1}(f)$

$$\begin{tikzcd} S^n \arrow[r, "\alpha"] \arrow[d, hook] & M \arrow[d, "f"] \\ D^{n+1} \arrow[r, "\beta"] & X \end{tikzcd}$$

in which α is an embedding.

2. A *framed n-embedding* into f is commutative diagram

$$\begin{tikzcd} S^n \times D^{m-n} \arrow[r, "\widetilde{\alpha}"] \arrow[d, hook] & M \arrow[d, "f"] \\ D^{n+1} \times D^{m-n} \arrow[r, "\widetilde{\beta}"] & X \end{tikzcd}$$

in which $\widetilde{\alpha}$ is an embedding.

As in the case of framed n-embeddings into manifolds, we would like to be able to perform surgery on a framed n-embedding $(\widetilde{\alpha}, \widetilde{\beta})$ into a map. The result of this should be a bordism between f and some new map $f' \colon M' \to X$. Since $\widetilde{\alpha}$ gives us a framed n-embedding into M, we can construct a cobordism (W, M, M') by performing surgery on M along $\widetilde{\alpha}$, with $W = (M \times I) \cup_{\widetilde{\alpha} \times \{1\}} (D^{n+1} \times D^{m-n})$. We now need a map $F \colon W \to X \times I$. We can use $f \times Id$ on $M \times I \subseteq W$, and we need to extend this map to $D^{n+1} \times D^{m-n}$. How to perform this extension is precisely the information contained in the map $\widetilde{\beta}$. We thus define:

Definition 4.2.2. Using the notation above, let M' be the effect of performing surgery on M along $\widetilde{\alpha}$, and let $W = M \times I \cup_{\widetilde{\alpha} \times \{1\}} (D^{n+1} \times D^{m-n})$ be the trace of this surgery. The bordism $F \colon (W, M, M') \to (X \times I, \{0\}, \{1\})$ given by $F|_{M \times I} = f \times Id$ and $F|_{D^{n+1} \times D^{m-n}} = (\widetilde{\beta}, \{1\})$ is called the *trace of the surgery on f along $(\widetilde{\alpha}, \widetilde{\beta})$*. The map $f' \colon M' \to X$ given by $f' = F|_{M'}$ is called the *effect* of this surgery, and the element $(\alpha, \beta) \in \pi_{n+1}(f)$ is said to be *killed* by this surgery.

If f is a normal map, we would like to be able to perform surgery in such a way that we produce a *normal* bordism. However, there is no guarantee that surgery along an n-embedding $(\widetilde{\alpha}, \widetilde{\beta})$ produces a normal bordism. As such, we amend our definitions to include this additional information.

Definition 4.2.3. A *normal framed n-embedding* into a normal map (f, b) is a framed n-embedding $(\widetilde{\alpha}, \widetilde{\beta})$ into (f, b) along with an extension B of b to the trace of the surgery on f along $(\widetilde{\alpha}, \widetilde{\beta})$. We call the normal bordism $(F, B) \colon (W, M, M') \to X \times I$ the *trace of the normal surgery* along this n-embedding, and we call the restriction $(f', b') = (F, B)|_{M'}$ the *effect* of this surgery.

We note that a normal n-surgery has a dual $(m - n - 1)$-surgery, defined in the obvious way by analogy to the dual of an n-surgery on a manifold.

These definitions give rise to the obvious questions of when an n-embedding into a map can be extended to a framed n-embedding, and when a framed n-embedding into a normal map can be extended to a normal framed n-embedding. We shall show that each of these questions is answered by a well-defined obstruction class.

The first one is easy. We are given an embedding $\alpha \colon S^n \to M$, and we want to know when it can be extended to a map from $S^n \times D^{m-n} \to M$. By the tubular neighborhood theorem, such maps are in one-to-one correspondence with framings (trivializations) of the normal bundle N_α^{m-n} of α. We may represent N_α as a map $N_\alpha \colon S^n \to BO(m - n)$ to the classifying space of $(m - n)$-bundles, and our condition is that N_α is nullhomotopic, i.e., it is trivial as an element of $\pi_n(BO(m - n))$.

Definition 4.2.4. We call the element $N_\alpha \in \pi_n(BO(m - n))$ the *framing obstruction $\mathcal{O}(\alpha, \beta)$* of (α, β).

The second question will require a bit more work. We have a normal map $(f, b) \colon (M, N_M) \to (X, E)$, where E is a stable vector bundle reduction of the Spivak normal fibration ν_X. Given a framed n-embedding $(\widetilde{\alpha}, \widetilde{\beta})$ into M, we want to know if we can extend b to a map from the normal bundle N_W of the trace W of the surgery along $(\widetilde{\alpha}, \widetilde{\beta})$. We note that W is homotopy equivalent to $M \cup_{\widetilde{\alpha}} D^{n+1}$.

We start by making all of our stable bundles finite by embedding M into a large dimensional sphere S^{n+k}. We thus have the sequence of embeddings

$$S^n \xhookrightarrow{\alpha} M^m \xhookrightarrow{\iota} S^{m+k} .$$

Let N_α, N_ι, and $N_{\iota\alpha}$ be the normal bundles of S^n in M, M in S^{m+k}, and S^n in S^{m+k} respectively, and let ξ be a finite bundle reduction of the Spivak normal fibration ν_X that pulls back to N_ι

under f. $N_{\iota\alpha} \simeq \epsilon^{m-n+k}$, and all such trivializations are homotopic; the latter assertion follows from the fact that all embeddings of S^n in S^{m+k} are isotopic by the Whitney embedding theorem (Theorem 2.1.8).

The normal map gives us the equivalence $N_\iota \simeq f^*\xi$, so we have $\alpha^* N_\iota \simeq \alpha^* f^* \xi$. But the map β in our diagram

$$\begin{tikzcd} S^n \arrow[r, "\alpha"] \arrow[d, hook] & M \arrow[d, "f"] \\ D^{n+1} \arrow[r, "\beta"] & X \end{tikzcd}$$

provides a nullhomotopy of $f \circ \alpha$, so $\alpha^* f^* \xi \simeq \epsilon^k$. Putting this all together, we get

$$N_\alpha \oplus \epsilon^k \simeq N_\alpha \oplus \alpha^* f^* \xi \simeq \alpha^* N_\iota \oplus N_\alpha \simeq N_{\iota\alpha} \simeq \epsilon^{m-n+k}.$$

The normal map (f, b) therefore endows N_α with a stable trivialization. We can treat N_α as a map $N_\alpha \colon S^n \to BO$, so that the normal structure gives us a nullhomotopy of this map, which we can think of as a map $\gamma \colon D^{n+1} \to BO$ that fits into the diagram

$$\begin{tikzcd} S^n \arrow[r, "N_\alpha"] \arrow[d, hook] & BO(m-n) \arrow[d, hook] \\ D^{n+1} \arrow[r, "\gamma"] & BO \end{tikzcd} . \tag{4.1}$$

We already know N_α to be nullhomotopic, since we assumed (α, β) could be framed, but the normal structure gives us more. The trace W of the surgery along $(\widetilde{\alpha}, \widetilde{\beta})$ is homotopy equivalent to $M \cup_\alpha D^{n+1}$. As such, α is nullhomotopic in W, say by a homotopy $\alpha_t \colon S^n \to W$, $t \in [0,1]$. An extension of the normal map to W along with this nullhomotopy gives a diagram as in equation (4.1) for each t. As such, an extension of the normal map to W provides a nullhomotopy of the diagram in equation (4.1). This is equivalent to saying that the element $(\gamma, N_\alpha) \in \pi_{n+1}(BO, BO(m-n))$ is trivial. By running these arguments in reverse, it is not difficult to see that vanishing of $(\gamma, N_\alpha) \in \pi_{n+1}(BO, BO(m-n))$ is sufficient to guarantee that this extension exists. We thus define:

Definition 4.2.5. We call this element $(\gamma, N_\alpha) \in \pi_{n+1}(BO, BO(m-n))$ the *b-framing obstruction* $\mathcal{O}_b(\alpha, \beta)$ of (α, β).

Collecting the results of the above discussion, we have therefore shown:

Theorem 4.2.6. *Let $f \colon M^m \to X$ be a map from a manifold to a Poincaré complex, and let $(\alpha, \beta) \in \pi_{n+1}(f)$ be an n-embedding into f.*

1. *The n-embedding (α, β) extends to a framed n-embedding if and only if the framing obstruction $\mathcal{O}(\alpha, \beta) \in \pi_n(BO(m-n))$ vanishes.*

2. *If f is actually a normal map, (α, β) extends to a normal n-embedding if and only if the b-framing obstruction $\mathcal{O}_b(\alpha, \beta) \in \pi_{n+1}(BO, BO(m-n))$ vanishes.*

For $m \leq 2n$, standard fibration and exact sequence arguments allow us to relatively easily compute the groups in which the b-framing obstruction lives:

Theorem 4.2.7. *We have*

$$\pi_{n+1}(BO, BO(m-n)) = \begin{cases} 0 & \text{if } 2n+1 \leq m \\ \mathbb{Z} & \text{if } m = 2n,\ n \text{ even}, \\ \mathbb{Z}/2\mathbb{Z} & \text{if } m = 2n,\ n \text{ odd}. \end{cases}$$

Our eventual goal is to perform normal surgeries to simplify the homotopy and homology groups of a map as much as possible in order to see if the map is normally bordant to a homotopy equivalence. To do this, we shall need to understand the effects of normal surgery on the homotopy and homology groups of the map. When the map is between simply connected spaces, these effects are described by the following theorem. This theorem is analogous to, and proved by essentially the same methods as, Theorem 2.3.4, which described the effects of surgery on the homotopy and homology of a manifold.

Theorem 4.2.8. *Let $(\alpha, \beta) \in \pi_{n+1}(f)$ be an n-embedding into a map of* simply connected *spaces $f\colon M^m \to X$ that extends to a framed n-embedding $(\widetilde{\alpha}, \widetilde{\beta})$, let $\langle(\alpha, \beta)\rangle$ be the normal subgroup of $\pi_{n+1}(f)$ generated by (α, β), and let $F\colon (W, M, M') \to (X \times I, \{0\}, \{1\})$ be the trace of the surgery along $(\widetilde{\alpha}, \widetilde{\beta})$. If $(\gamma, \delta) \in \pi_{m-n}(f')$ is the element killed by the dual $(m-n-1)$-surgery, and $\langle(\gamma, \delta)\rangle$ is the normal subgroup of $\pi_{m-n}(f')$ generated by it, we have:*

$$\begin{aligned}
\pi_i(F) &= \begin{cases} \pi_i(f) & \text{if } i < n+1, \\ \pi_{n+1}(f)/\langle(\alpha, \beta)\rangle & \text{if } i = n+1 \end{cases} \\
&= \begin{cases} \pi_i(f') & \text{if } i < m-n, \\ \pi_{m-n}(f')/\langle(\gamma, \delta)\rangle & \text{if } i = m-n, \end{cases}
\end{aligned}$$

$$K_i(F, f) = \begin{cases} 0 & \text{if } i \neq n+1, \\ \mathbb{Z} & \text{if } i = n+1, \end{cases}$$

and

$$K_i(F, f') = \begin{cases} 0 & \text{if } i \neq m-n, \\ \mathbb{Z} & \text{if } i = m-n. \end{cases}$$

In particular, if $m > 2n+1$,

$$\pi_i(f') = \begin{cases} \pi_i(f) & \text{if } i < n+1, \\ \pi_{n+1}(f)/\langle(\alpha, \beta)\rangle & \text{if } i = n+1, \end{cases}$$

and if $m = 2n+1$,

$$\pi_i(f) = \pi_i(f')$$

for $i < n+1$, and

$$\pi_{n+1}(f)/\langle(\alpha, \beta)\rangle = \pi_{n+1}(f')/\langle(\gamma, \delta)\rangle.$$

Relatively straightforward topological arguments also yield:

Theorem 4.2.9. *There is a commutative braid of exact sequences:*

$$\begin{tikzcd}
K_{i+1}(F, f) \arrow[rr, bend left] \arrow[dr] & & K_i(f) \arrow[rr, bend left, "\phi"] \arrow[dr] & & K_i(F, f') \,, \\
& K_{i+1}(F, f \cup f') \arrow[ur] \arrow[dr] & & K_i(F) \arrow[ur] \arrow[dr] & \\
K_{i+1})(F, f') \arrow[ur] \arrow[rr, bend right] & & K_i(f') \arrow[ur] \arrow[rr, bend right, "\psi"'] & & K_i(F, f)
\end{tikzcd}$$

where the maps ϕ and ψ are given by applying Poincaré duality to the cohomology maps $K^i(F, f') \to K^i(f)$ and $K^i(F, f) \to K^i(f')$ respectively.

Remark 4.2.10. In a commutative braid of exact sequences, there are four exact sequences that are intertwined, corresponding to the four "strings" being braided. For example, one exact sequence in the braid above is

$$K_{i+1}(F, f) \to K_i(f) \to K_i(F) \to K_i(F, f).$$

Proof. For the proofs of Theorems 4.2.8 and 4.2.9, see [Ran02]. □

4.3 Surgery on Normal Maps Below the Middle Dimension

We can now begin in earnest the process of performing surgery to attempt to find a homotopy equivalence normally bordant to a given degree one normal map $f: M^m \to X$. The theorems from the last section will make it quite easy to produce a map f' normally bordant to f with $\pi_n(f) = 0$ for all $n \leq m/2$. By the Hurewicz theorem (Theorem 4.1.4) and the universal coefficient theorem, this will imply that $K^{n-1}(f) = K_{n-1}(f) = \pi_n(f) = 0$ for $n \leq m/2$. Our Poincaré duality theorem for the K_j (Theorem 4.1.5) will then imply that $K_{m-n+1}(f) = K^{m-n+1}(f) = 0$ for such n as well. This will leave us with possibly nonzero $K_{m/2}(f)$ if m is even and $K_{(m\pm1)/2}(f)$ if m is odd. In both cases, we shall have to kill one more homotopy group to obtain a homotopy equivalence. (In the odd case, killing $K_{(m-1)/2)}(f)$ will kill $K_{(m+1)/2}(f)$ as well by Poincaré duality.

While the groups below the middle dimension can always be killed, there are times when this last group cannot be. Understanding when this occurs is somewhat subtle, and we shall examine this question in section 4.4 . In this section, we deal with the dimensions below the middle. Our main result is:

Theorem 4.3.1. *Let $f: M^m \to X$ be a normal map, $m \geq 5$, and assume X to be connected. (Or, alternatively, assume that $f_*: H_0(M) \to H_0(X)$ is a surjection.) Further assume $\pi_1(X) = \pi_1(M) = 0$. Then f is normally bordant to an $\lfloor \frac{m}{2} \rfloor$-connected map $f': M' \to X$.*

Remark 4.3.2. We note that $\deg(f) = 1$ implies that f induces a surjection on H_0, so the connectivity hypothesis of the theorem is automatically satisfied if f is a degree one normal map. The simple connectivity hypothesis is unnecessary, and we make it solely for simplicity.

Lemma 4.3.3. *Let f be as above, and suppose $\pi_i(f) = 0$ for all $i < n$. Then $\pi_n(f)$ is finitely generated.*

Proof. M and X are finite CW complexes. Replacing X by the mapping cylinder of f if necessary, we may assume that f is an inclusion, so that $\pi_i(f) = \pi_i(X, M)$. Since $\pi_i(X, M) = 0$ for $i < n$, we may assume, by the cellular approximation theorem, that X is obtained by attaching (finitely many) cells of dimension greater than or equal to n. The desired result now follows from cellular homology and the relative Hurewicz theorem. □

Lemma 4.3.4. *If $f: M^m \to X$ and $2n + 1 \leq m$, $n \geq 5$, every element (α, β) of $\pi_{n+1}(f)$ can be represented by an element that can be extended to normal framed n-embedding, and thus it can be killed by surgery.*

Proof. By the dimensional hypotheses and the weak Whitney embedding theorem (Theorem 2.1.8), we can homotope α by an arbitrarily small homotopy so that it is an embedding. This will extend to a homotopy of the nullhomotopy β of $f \circ \alpha$, so we can assume (α, β) to be an n-embedding.

The b-framing obstruction $\mathcal{O}_b(\alpha,\beta)$ lives in $\pi_{n+1}(BO, BO(m-n))$, $2n+1 \leq m$, which is the zero group by Theorem 4.2.7, so (α,β) is homotopic to an n-embedding that can be normally framed, as desired. □

Proof of Theorem 4.3.1. By our assumptions, f is 0- and 1-connected. (If it weren't, we actually could make it so, but our assumptions allow us to omit this step.) Our proof proceeds by inductively killing off the homotopy groups of f. Suppose we have already modified f so that it is n-connected, $2n+1 < m$, and that $\pi_{n+1}(f)$ is generated by elements $g_1, \ldots, g_r$, where r is known to be finite by Lemma 4.3.3. By Lemma 4.3.4, we can represent g_1 by an n-embedding that can be normally framed. Perform surgery on this framed normal n-embedding, and let the effect of this surgery be f'. Since $2n+1 < m$, this surgery does not affect the lower homotopy groups, so $\pi_i(f') = \pi_i(f)$ for $i < n+1$. Furthermore, $\pi_{n+1}(f') = \pi_{n+1}(f)/\langle g_1 \rangle$. We have thus reduced by one the number of generators of $\pi_{n+1}(f)$. The theorem now follows by induction. □

4.4 Surgery in the Middle Dimension

We have shown that we can perform a sequence of normal surgeries on a normal map $f\colon M^m \to X$, $m \geq 5$, X simply connected, to produce an $\lfloor \frac{m}{2} \rfloor$-connected map that is normally bordant to f. In this section, we examine the question of when we can perform normal surgery to kill off one more homotopy group of f. This would produce a map f' whose homotopy groups are all zero, by the Hurewicz theorem (Theorem 4.1.4) and the Poincaré duality of the groups $K_n(f)$ and $K^n(f)$ (Theorem 4.1.5), so f' would be a homotopy equivalence by Theorem 4.1.4. Unlike the case of surgery below the middle dimension, this final homotopy group cannot always be killed by surgery. There is sometimes an obstruction that must vanish for this to occur.

To kill off the homotopy groups of f below the middle dimension, we used that the b-framing obstruction lived in a trivial group, and that n-surgery always simplified $\pi_{n+1}(f)$. We lose these advantageous conditions when dealing with the middle dimension, which makes our task a good deal subtler.

We shall actually have two cases. When the dimension m is odd and $2n+1 = m$, the b-framing obstruction will still belong to a trivial group. However, if we perform n-surgery along a framed n-embedding $g \in \pi_{n+1}(f)$ with dual embedding $g' \in \pi_{n+1}(f')$, we shall only have

$$\pi_{n+1}(f)/\langle g \rangle = \pi_{n+1}(f')/\langle g' \rangle,$$

which will not in general imply that $\pi_{n+1}(f')$ is simpler than $\pi_{n+1}(f)$. It turns out, however, that in the simply connected case we can overcome this algebraic difficulty and find a collection of elements along which we can perform surgery to completely kill $\pi_{n+1}(f)$. We shall thus always be able to find a homotopy equivalence in the normal bordism class of f.

When m is even, the b-framing obstruction will no longer belong to a trivial group, and it will sometimes be nonzero. This will give rise to a well-defined obstruction that will have to vanish for us to be able to find a homotopy equivalence in the normal bordism class of f. Our main result will therefore be:

Theorem 4.4.1 (Fundamental Theorem of Surgery Theory). *Let $f\colon M^m \to X$ be a degree one normal map. There is a well-defined obstruction $\sigma(f)$, called the* surgery obstruction *of f that vanishes if and only if f is normally bordant to a homotopy equivalence. If m is odd, $\sigma(f)$ will always be zero, so that f will always be normally bordant to a homotopy equivalence.*

We shall refine this theorem by defining the surgery obstruction of a map later in this section.

Remark 4.4.2. In this section, we shall make heavy use of our hypothesis of simple connectivity. In general, the algebraic difficulties arising from the possibility that surgery doesn't simplify $\pi_{n+1}(f)$ for m odd will not be surmountable. It will thus be possible, in general, for the surgery obstruction of a map to be nonzero when m is odd. The surgery obstruction will live in the so-called *surgery obstruction groups* $L_m(\mathbb{Z}[\pi_1(X)])$, which will depend only on the group algebra of the fundamental group of X. The odd-dimensional groups will measure when one can find a collection of elements of $\pi_{n+1}(f)$ such that the algebraic effect of performing surgery along them suffices to completely annihilate $\pi_{n+1}(f)$. The even-dimensional groups will measure when there exists a sufficiently large class of elements of $\pi_n(f)$ that can be represented by framed normal embeddings. See [Ran02] or [Wal70].

Due to space constraints, we shall only sketch the proof of the fundamental theorem of surgery theory, and we really shall not do it justice. For a full treatment of it, see [Bro72], or see [Ran02] for a proof of the multiply connected version that is probably more readable than any proof in the literature of just the simply connected case. We shall begin by analyzing the vanishing of the b-framing obstruction in the even-dimensional case.

4.4.1 Quadratic Forms and the Vanishing of the b-Framing Obstruction

Throughout this section, let $f\colon M^m \to X$ be a degree one normal map, with $m \geq 5$ and $m = 2n$. Furthermore, assume that f is n-connected, so that $\pi_{n+1}(f) = H_{n+1}(f) = H^{n+1}(f) = K_n(f) = K^n(f)$. The crux of simply connected surgery theory in the middle dimension of an even dimensional manifold is a relationship between the geometric condition that the b-framing obstruction of a given element of $K_n(f)$ vanishes and the algebraic condition that a certain quadratic form evaluates to zero on that element. The b-framing obstruction belongs to $\pi_{n+1}(BO, BO(n))$, which equals $\mathbb{Z}$ if $m = 4k$ and $\mathbb{Z}/2\mathbb{Z}$ if $m = 4k + 2$. As such, we shall have two different cases. When $m = 4k$, the quadratic form that we use will be defined over $\mathbb{Z}$, whereas when $m = 4k + 2$, it will be defined over $\mathbb{Z}/2\mathbb{Z}$. We shall provide a relatively full proof of the $m = 4k$ case. The $m = 4k + 2$ case is a bit more technically difficult, and shall not cover it in detail.

Before we begin the geometric aspects of the theory, we provide a brief review of certain aspects of the theory of quadratic forms over $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$.

Quadratic Forms Over $\mathbb{Z}$

Definition 4.4.3. Let V be a finitely generated free $\mathbb{Z}$-module. A *quadratic form* on V is a map $q\colon V \to \mathbb{Z}$ such that $q(x) = (x, x)$ for some symmetric bilinear form $(\,,)\colon V \times V \to \mathbb{Z}$. In this case, we call $(\,,)$ the symmetric bilinear form *associated* to q.

Definition 4.4.4. A quadratic form q over $\mathbb{Z}$ is said to be *nonsingular* if its associated bilinear form has determinant ± 1.

We note that symmetric bilinear forms and quadratic forms over $\mathbb{Z}$ are in a one-to-one correspondence. A bilinear form $(\,,)$ determines a quadratic form q by $q(x) = (x, x)$, and a quadratic form determines a bilinear form by

$$(x, y) = \frac{1}{2}(q(x + y) - q(x) - q(y)).$$

These operations are easily seen to be mutually inversive.

A quadratic form $q\colon V \to \mathbb{Z}$ naturally gives rise to a quadratic form $q_{\mathbb{Q}}\colon V \otimes \mathbb{Q} \to \mathbb{Q}$. Quadratic forms over $\mathbb{Q}$ diagonalize by Sylvester's theorem, so we may define the *signature* $I(q)$ to be the signature of $q_{\mathbb{Q}}$ over $\mathbb{Q}$. We have the following (nontrivial) theorems about quadratic forms over $\mathbb{Z}$, which we present without proof.

Theorem 4.4.5. *Let $q\colon V \to \mathbb{Z}$ be a nonsingular quadratic form that is neither positive definite nor negative definite. There exists an $x \in V$ such that $q(x) = 0$.*

Proof. See [Mil61]. □

Definition 4.4.6. A quadratic form $q\colon V \to \mathbb{Z}$ is said to be *even* if $q(x)$ is even for all $x \in V$.

Theorem 4.4.7. *The signature of a nonsingular even quadratic form $q\colon V \to \mathbb{Z}$ is divisible by 8.*

Proof. See [Mil58]. □

Quadratic Forms Over $\mathbb{Z}/2\mathbb{Z}$

We make similar definitions for quadratic forms over $\mathbb{Z}/2\mathbb{Z}$, although we modify them slightly to take into account the fact that multiplication by two is the zero map in $\mathbb{Z}/2\mathbb{Z}$.

Definition 4.4.8. Let V be a vector space over $\mathbb{Z}/2\mathbb{Z}$. A quadratic form on V is a map $q\colon V \to \mathbb{Z}/2\mathbb{Z}$ such that $q(x) = 0$ for all $x \in V$, and the map $(\,,)\colon V \times V \to \mathbb{Z}/2\mathbb{Z}$ by $(x, y) = q(x + y) - q(x) - q(y)$ is bilinear. We call $(\,,)$ the bilinear form *associated* to q. We say that q is nonsingular if its associated bilinear form has determinant one.

We note that the correspondence between quadratic forms and bilinear forms breaks down over $\mathbb{Z}/2\mathbb{Z}$, essentially because of our inability to put a $\frac{1}{2}$ in front of $q(x+y)-q(x)-q(y)$ in the definition. In fact, we note that if $(\,,)$ is the bilinear form associated to a quadratic form q, then

$$(x, x) = q(2x) - 2q(x) = 0$$

for all x. This means that, if q is nonsingular, there exists a symplectic basis $a_1, \ldots, a_n, b_1, \ldots, b_n$ such that $(a_i, a_j) = 0$, $(b_i, b_j) = 0$, and $(a_i, b_j) = \delta_{ij}$ for all $i, j \in \{1, \ldots, n\}$. (We note that this shows that only even-dimensional $\mathbb{Z}/2\mathbb{Z}$-vector spaces admit nonsingular quadratic forms.)

Definition 4.4.9. Let q, $(\,,)$, and $\{a_i, b_i\}$ be as above. The *Arf invariant* of q is defined to be the sum

$$c(q) = \sum_{i=1}^{n} q(a_i)q(b_i) \in \mathbb{Z}/2\mathbb{Z}.$$

It is not *a priori* obvious that this definition is independent of our choice of symplectic basis, but it can be shown that this is indeed the case. The importance of the Arf invariant is given by the following theorem:

Theorem 4.4.10. *There are exactly two isomorphism classes of quadratic forms $q\colon V \to \mathbb{Z}/2\mathbb{Z}$ on an even-dimensional $\mathbb{Z}/2\mathbb{Z}$-vector space V, and they are classified by their Arf invariants. Furthermore, a quadratic form with Arf invariant zero evaluates to zero on more than half of the elements of V.*

Proof. For a proof that the Arf invariant is well-defined and classifies nonsingular quadratic forms over $\mathbb{Z}/2\mathbb{Z}$, see [Bro72]. □

The Kernel Form

We now return to surgery in the middle dimension. As before, let $f\colon M^m \to X$ be an n-connected degree one normal map, with $m = 2n \geq 5$. Let $(\,,)$ denote the intersection pairing on $H_*(X)$, given by

$$(x, y) = (\mathrm{PD(x)} \cup \mathrm{PD(y)})[X],$$

where PD(x) denotes the Poincaré dual of x. We note that, by the Poincaré duality on $K_*(f)$, if $m = 4k$ then $K_i(f) = 0$ for $i \neq n$, and $K_n(f)$ is a free $\mathbb{Z}$-module.

By Theorem 4.1.5, $(\,,)$ restricts to a nondegenerate bilinear form on $K_n(f)$, which we continue to denote $(\,,)$. If $m = 4k$, this bilinear form is symmetric; if $m = 4k+2$, it is skew-symmetric. In the case $m = 4k+2$, we pass to $K_n(f, \mathbb{Z}/2\mathbb{Z}) = H_{n+1}(f, \mathbb{Z}/2\mathbb{Z})$, on which $(\,,)$ now induces a symmetric bilinear form $(\,,)$. We now define quadratic forms on $K_{4k}(f)$ and $K_{4k+2}(f, \mathbb{Z}/2\mathbb{Z})$ that have these as their associated bilinear forms.

Defining the quadratic form on $K_{4k}(f)$ is easy. There is a one-to-one correspondence between symmetric bilinear forms and quadratic forms, and we simply define $q_{2k}\colon K_{2k}(f) \to \mathbb{Z}$ by $q_{2k}(x) = (x, x)$.

When $m = 4k+2$, our task is a bit more difficult, and we shall not fully address it here. There is no longer a simple formula for q_n in terms of $(\,,)$. Instead, one uses a construction due to Browder involving functional Steenrod squares. See [Bro72] for the details. For our purposes, we shall just assume such a quadratic form exists, and we shall assert its properties when necessary.

Relating the b-Framing Obstruction to the Kernel Form

We now relate the b-framing obstruction to the kernel form, which will provide the basis for our proof of the fundamental theorem of surgery theory. We shall only prove this relation when $m = 4k$. See [Bro72] for the case when $m = 4k+2$.

Theorem 4.4.11. *Let $x = (\alpha, \beta) \in K_n(f)$. If $m = 2n = 4k$, the b-framing obstruction $\mathcal{O}_b(x) = 0$ if and only if $q_n(x) = (x, x) = 0$. If $m = 2n = 4k+2$, and $x^{(2)}$ is the image of x in $K_n(f, \mathbb{Z}/2\mathbb{Z})$, $\mathcal{O}_b(x) = 0$ if and only if $q_n(x^{(2)}) = 0$.*

Proof for $m = 4k$. Using the Whitney embedding theorem (Theorem 2.1.9), we can represent (α, β) by an n-embedding. (Here we again use our hypothesis that $m \geq 5$.) We recall that $\mathcal{O}_b(\alpha, \beta) = (\gamma, N_\alpha) \in \pi_{n+1}(BO, BO(n))$, where N_α is the stable normal bundle of α. We now apply the following standard fact, which follows easily from the appropriate exact sequences. (See [Bro72] for the proof.)

Lemma 4.4.12. *The map $\pi_{n+1}(BO, BO(n)) \to \pi_n(BO(n))$ is injective for all even n. (In fact, it is injective whenever $n \neq 1$, 3, or 7.)*

This implies that $\mathcal{O}_b(\alpha, \beta) = (\gamma, N_\alpha)$ vanishes if and only if its image $N_\alpha \in \pi_n(BO)$ vanishes. (Here we are using our assumption that $m = 4k = 2n$.) Now, in the exact sequence

$$\pi_{n+1}(BO, BO(n)) \xrightarrow{\partial} \pi_n(BO(n)) \xrightarrow{\iota_*} \pi_n(BO)\ ,$$

$N_\alpha \in \operatorname{im} \partial$, so $N_\alpha \in \ker \iota_*$. From another exact sequence argument, we see that $\ker \iota_* \cong \mathbb{Z}$, and it is generated by the class of the tangent bundle $\tau_{S^n}\colon S^n \to BO(n)$. As such, there is some $\lambda \in \mathbb{Z}$ such that $N_\alpha = \lambda \tau_{S^n}$.

Claim 4.4.13. $(x, x) = 2\lambda$.

Proof of Claim 4.4.13. We compute both sides using the Euler classes of various vector bundles. Let E be the total space of the normal bundle of α (which is a finite representative of N_α), and let E_0 be E minus the zero section. We have a collapse $\eta\colon M \to E/E_0$, and we let $[E] = \eta^*[M]$. By the definition of the Euler class, $\chi(E) \cup U(E) = U(E)^2$. We also note that $[M] \cap \eta^* U(E) = x$. We compute

$$\chi(E)[S^n] = (\chi(E) \cup U(E))[E] = U(E)^2[E] = (\eta^* U(E))^2[M] = (\mathrm{PD}(x) \cup \mathrm{PD})(x)[M] = (x, x). \tag{4.2}$$

However, we also have

$$\chi(E)[S^n] = \chi(\lambda \tau_{S^n})[S^n] = \lambda \chi(\tau_{S^n})[S^n] = \lambda \chi(S^n) = 2\lambda. \tag{4.3}$$

Claim 4.4.13 therefore follows by combining equations (4.2) and (4.3). □

Now, $\mathcal{O}_b(x) = 0$ if and only if $\lambda = 0$, which we see occurs if and only if $(x, x) = 0$. This establishes Theorem 4.4.11 when $m = 4k$.

The proof for $m = 4k + 2$ is a little bit more difficult, and it requires us to have more specific knowledge of q_{4k+2}. See [Bro72] □

4.4.2 The Surgery Obstruction

We can now define the surgery obstruction of f. We note that, by Claim 4.4.13, q_{2k} is an even form on $K_{4k}(f)$, so its signature is divisible by 8.

Definition 4.4.14. If $m = 4k$, we define the *surgery obstruction* $\sigma(f) = \sigma_m(f)$ to be $\frac{1}{8} I(q_n) \in \mathbb{Z}$, where $I(q_n)$ is the signature. If $m = 4k + 2$, we define $\sigma(f) = c(q_n) \in \mathbb{Z}/2\mathbb{Z}$, where $c(q_n)$ is the Arf invariant. If m is odd, we declare $\sigma(f)$ to be zero.

To unify notation, we define the *simply connected surgery obstruction groups* to be

$$L_m(\mathbb{Z}[\{1\}]) = L_m(\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n \equiv 0 \pmod 4, \\ \mathbb{Z}/2\mathbb{Z} & \text{if } n \equiv 2 \pmod 4, \\ 0 & \text{if } n \text{ is odd}, \end{cases}$$

so that $\sigma_m(f) \in L_m(\mathbb{Z})$.

Remark 4.4.15. We note that the surgery obstruction can be defined in an analogous way for manifolds with boundary. One defines a *Poincaré pair* $(X, \partial X)$ to be a finite CW pair that obeys Lefschetz duality. One can then define normal maps, the kernel form, and the surgery obstructions in the obvious analogous way, with all of our constructions done rel the boundaries. The only difference is that one must stipulate that f induce homology isomorphisms on the boundaries for the surgery obstruction theory results to carry over.

We now note several properties of σ. They follow from standard arguments for $m = 4k$, and they are not difficult to show for $m = 4k + 2$. See [Bro72] for their proofs.

Theorem 4.4.16. *The following properties of the surgery obstruction hold, where f can be a map of closed manifolds or a map of manifolds with boundary.*

1. *If (f, b) and (f', b') are normally bordant degree one normal maps, then $\sigma(f) = \sigma(f')$.*

2. *If (f, b) is a homotopy equivalence, $\sigma(f) = 0$.*

3. *Let* $(f_1, b_1): (M_1, \partial M_1) \to (X_1, \partial X_1)$, *and* $(f_2, b_2): (M_2, \partial M_2) \to (X_2, \partial X_2)$ *be degree one normal maps of Poincaré pairs. Now suppose* $(M, \partial M)$ *is obtained by attaching* M_1 *and* M_2 *along subsets of their boundaries, and let* $(X, \partial X)$ *be obtained by attaching* X_1 *and* X_2 *along subsets of* ∂X_1 *and* ∂X_2. *Further suppose that there is a degree one normal map* $f: (M, \partial M) \to (X, \partial X)$ *with* $f|_{M_1} = f_1$ *and* $f|_{M_2} = f_2$, *and that all maps induce homology isomorphisms on the homologies of the boundaries. Then*
$$\sigma(f, b) = \sigma(f_1, b_1) + \sigma(f_2, b_2).$$

4. *Let* $(f, b): (M, \partial M) \to (X, \partial X)$ *be a degree one normal map of pairs. Restriction to the boundary gives rise to a degree one normal map* $(\overline{f}, \overline{b}): \partial M \to \partial X$. *This map has surgery obstruction* $\sigma(\overline{f}) = 0$.

4.4.3 The Fundamental Theorem of Surgery Theory

We can now more precisely state the fundamental theorem of surgery theory, along with its rel ∂ analogue.

Theorem 4.4.17 (Fundamental Theorem of Surgery Theory). *Let* $f: M^m \to X$ *be a degree one normal map, where* $m \geq 5$ *and* M *and* X *are simply connected. Then* f *is normally cobordant to a homotopy equivalence if and only if* $\sigma_m(f) \in L_m(\mathbb{Z})$ *is zero.*

Theorem 4.4.18 (Rel ∂ Fundamental Theorem of Surgery Theory). *Let* $f: (M^m, \partial M) \to (X, \partial X)$ *be a degree on normal map of pairs, where* $m \geq 5$ *and* M *and* X *are simply connected. Further suppose that* $f|_{\partial M}$ *induces an isomorphism in homology. Then* f *is normally cobordant* rel ∂ *to a homotopy equivalence.*

Partial Proof. We shall now provide a sketch of the proof of the fundamental theorem of surgery for closed manifolds when m is even. The proof for m is odd is based on a somewhat complicated algebraic argument using the commutative braid of exact sequences from Theorem 4.2.9 that specifies the homology effects of normal surgery, and we omit it. For its proof, see [Bro72]. The proof of the rel ∂ version is essentially identical. See [Bro72] for its details.

We note that the "only if" portion of the theorem is follows immediately from Theorem 4.4.16, so we need only prove the "if" direction. It suffices by the Hurewicz theorem and the Poincaré duality of $K_*(f)$ to show that we can make f $(n+1)$-connected. Our proof for m even uses the following lemma, which follows by a relatively straightforward diagram chase using the exact braid in Theorem 4.2.9 . We assume throughout this proof that $m = 2n$.

Lemma 4.4.19. *Let* $g \in \pi_{n+1}(f) = K_n(f)$ *generate in an infinite cyclic direct summand. Suppose that* g *can be normally framed, and let* f' *be the effect of normal surgery killing* g. *Then*
$$\operatorname{rank} K_n(f') < \operatorname{rank} K_n(f).$$

Similarly, if the image of $g \in \pi_{n+1}(f)$ *in* $K_n(f, \mathbb{Z}/2\mathbb{Z})$ *is nonzero, then*
$$\operatorname{rank} K_n(f', \mathbb{Z}/2\mathbb{Z}) < \operatorname{rank} K_n(f).$$

We remark that this lemma holds when m is odd as well, and it is used in the first step in the proof of the odd part of the fundamental theorem. By choosing elements meeting its hypotheses and performing surgery on them, one reduces $K_n(f)$ to its torsion. Killing this torsion is the subtle part of the proof, which we omit.

Now, first suppose $m = 4k$ and that $\sigma(f) = 0$. By Theorem 4.3.1, we can take f to be n-connected, so that $K_i(f) = 0$ for all $i \neq n$ by Poincaré duality. We note that Poincaré duality also implies $K_n(f)$ to be a free $\mathbb{Z}$-module, and the module is of finite rank by Lemma 4.3.3.

By assumption, the signature of $(,)$ on $K_q(f)$ is zero, so the bilinear form is indefinite. By Theorem 4.4.5, there is therefore some element $x \in K_n(f)$ such that $(x, x) = 0$, and we can, without loss of generality take this element to be indivisible, so that it generates a direct summand of $K_n(f)$, and this direct summand is infinite since $K_n(f)$ is free. By Theorem 4.4.11, $\mathcal{O}_b(x) = 0$, so we can perform normal along x. By Lemma 4.4.19, the effect f' of this surgery has $\operatorname{rank} K_n(f') < \operatorname{rank} K_n(f)$. Since f is normally cobordant to f', $\sigma(f') = \sigma(f) = 0$ by Theorem 4.4.16. We can therefore repeat this process, and our desired result follows by induction.

Now suppose $m = 4k + 2$ and that $\sigma(f) = 0$, and we may again take f to have $K_i(f) = 0$ for $i \neq n$. $K_n(f, \mathbb{Z}/2\mathbb{Z})$ is a finite dimensional $\mathbb{Z}/2\mathbb{Z}$ vector space. The Arf invariant $c(q_n) = 0$, which implies that there exists an element $x \in K_n(f, \mathbb{Z}/2\mathbb{Z})$ with $q_n(x) = 0$ by Theorem 4.4.10. Performing surgery along x reduces the dimension of $K_n(f, \mathbb{Z}/2\mathbb{Z})$ while keeping the surgery obstruction equal to zero. The desired result thus again follows by induction. □

4.5 The Manifold Existence Theorem and the Surgery Exact Sequence

We are now in a position to put all of our work together into the two main theorems of this part of the thesis: the Browder-Novikov-Sullivan-Wall manifold existence theorem, and the the Browder-Novikov-Sullivan-Wall surgery exact sequence. They address, respectively, when the structure set of a Poincaré complex is nonempty, and, if it is nonempty, how to compute it.

Theorem 4.5.1 (Browder-Novikov-Sullivan-Wall Manifold Existence Theorem). *Let X be a simply connected Poincaré complex of dimension greater than or equal to five. The structure set $\mathcal{S}(X)$ is nonempty if and only if there exists a degree one normal map (f, b) into X with surgery obstruction $\sigma(f, b) = 0$ in $L_m(\mathbb{Z})$.*

By our equivalence between bundle reductions of the Spivak normal fibration and degree one normal maps (Theorem 3.4.7), we can assign a surgery obstruction to a bundle reduction of the Spivak normal fibration, and we can restate this theorem as saying that $\mathcal{S}(X)$ is nonempty if and only if there is a bundle reduction of the Spivak normal fibration with surgery obstruction zero.

Proof of Theorem 4.5.1. By Remark 3.4.6, $\mathcal{S}$(X) is nonempty if and only if some degree one normal map is a homotopy equivalence. By Theorem 4.4.17, a normal bordism class of degree one normal maps contains a homotopy equivalence if and only if its surgery obstruction vanishes. □

If the structure set of a Poincaré complex is nonempty, we would like some way to compute it. For this, we shall need one last result, which tells us that all values of the surgery obstruction are achieved:

Theorem 4.5.2 (Milnor's Plumbing Theorem). *Let $m = 2k > 4$, and let $x \in L_m(\mathbb{Z})$. There exists a manifold with boundary $(M^m, \partial M)$ and a degree one normal map of pairs $(f, b)\colon (M, \partial M) \to (D^m, S^{m-1})$ with $f|_{\partial M}\colon \partial M \to S^{m-1}$ a homotopy equivalence such that $\sigma(f, b) = x$.*

Of course, this theorem holds (vacuously) for m odd as well.

Proof. Such manifolds can be constructed explicitly for each m and x using a technique called plumbing. See [Bro72]. □

With this, we can define an action of $L_{m+1}(\mathbb{Z})$ on the structure set $\mathcal{S}(X)$ of a Poincaré complex X. Let $(M^m, f) \in \mathcal{S}(X)$, and let $x \in L_{m+1}(\mathbb{Z})$. By the plumbing theorem, there exists a degree one normal map of pairs $(f, b)\colon (N^{m+1}, \partial N) \to (D^{m+1}, S^m)$, where $f|_{\partial N}\colon \partial N \to S^{m-1}$ is a homotopy equivalence and $\sigma(f, b) = x$. Let W^{m+1} be the manifold obtained by taking the connected sum of $M \times [0, 1]$ and N along their boundaries. We have $\partial W = M \amalg (M \# \partial N)$, and we define $M' = M \# \partial N$. By assumption, ∂N is a homotopy sphere, so we have a homotopy equivalence

$$f'\colon\ M\#\partial N \xrightarrow{\simeq} M \xrightarrow{\simeq} X$$

naturally induced by (M, f). We define $x(M, f)$ to be (M', f').

To see that this is well defined, we have to show that different choices of $(f, b)\colon (N, \partial N) \to X$ with the same surgery obstruction give rise to the same (M', f') in the structure set. To this end, let $(f, b)\colon (N, \partial N) \to X$ and $(f', b')\colon (N', \partial N') \to X$ be normal maps with the same surgery obstruction, and let W and W' be the connected sums along the boundary of $M \times I$ with N and N' respectively. We note that from the identity map on $M \times I$ and f, we can construct a degree one normal map $F\colon W \to (M \times I)\#_b D^{n+1} \cong M \times I \to X \times I$ that induces a homotopy equivalence on the boundary and has surgery obstruction x, and we can construct the analogous map $F'\colon W' \to X \times I$ with surgery obstruction x. If we compose F' with the map $X \times [0, 1] \to X \times [-1, 0]$ obtained by sending (x, t) to $(x, -t)$, we obtain a new map $G\colon W' \to X \times I$ with surgery obstruction $-x$.

Both ∂W and $\partial W'$ have M as a connected component, $F'|_M = F_M\colon M \to X \times \{0\}$, and the bundle maps agree, so we can glue W and W' together along M and obtain a normal bordism $G\colon W \cup_M W' \to X \times [-1, 1]$ with surgery obstruction 0. By our rel ∂ version of the fundamental theorem of surgery theory (Theorem 4.4.18), this implies that G is normally bordant to an h-bordism, since $X \times [-1, 1] \simeq X$. As such, (f, b) and (f', b') represent the same element of the structure set, as desired.

We can now state and prove our main result:

Theorem 4.5.3 (Browder-Novikov-Sullivan-Wall Surgery Exact Sequence). *Let M^m be a simply connected manifold with $m \geq 5$. The sequence*

$$L_{m+1}(\mathbb{Z}) \xrightarrow{\omega} \mathcal{S}(M) \xrightarrow{\eta} \mathcal{N}(X) \xrightarrow{\sigma} L_m(\mathbb{Z})\ ,$$

where:

1. *ω takes $x \in L_{m+1} \mapsto x(M)$ using the action described above,*

2. *η sends an element of $\mathcal{S}(M)$ to its normal bordism class, and*

3. *σ is the surgery obstruction*

is exact as a sequence of pointed sets. Furthermore, the exactness respects the action of $L_{m+1}(\mathbb{Z})$ on $\mathcal{S}(M)$, so that if $\alpha, \beta \in \mathcal{S}(M)$ have $\eta(\alpha) = \eta(\beta)$, then there exists $x \in L_{m+1}(\mathbb{Z})$ such that $\alpha = x(\beta)$.

Proof. By the fundamental theorem of surgery theory (Theorem 4.4.17), a normal bordism class in $\mathcal{N}(X)$ has surgery obstruction zero if and only if it contains a homotopy equivalence, i.e. it is in the image of $\mathcal{S}(M)$. The sequence is therefore exact at $\mathcal{N}(X)$.

The construction used to show that the action of $L_{m+1}(\mathbb{Z})$ on $\mathcal{S}(M)$ is well-defined constructs a normal bordism between $x(\alpha)$ and $y(\alpha)$ for any $x, y \in L_{m+1}(\mathbb{Z})$ and $\alpha \in \mathcal{S}(M)$. Since $0 \in L_{m+1}(\mathbb{Z})$ has $0(\alpha) = \alpha$, this implies that $\eta(x(\alpha)) = \eta(\alpha)$. Furthermore, suppose $\eta(\alpha) = \eta(\beta)$, $\alpha, \beta \in \mathcal{S}(M)$, i.e., that α and β are normally bordant, and let $x \in L_{m+1}(\mathbb{Z})$ be the surgery obstruction of the normal bordism between them, when you interpret it as a normal map into $M \times I$. Attaching to this the normal bordism between α and $-x(\alpha)$ produces a normal bordism F between β and $-x(\alpha)$ with surgery obstruction zero. By the rel ∂ version of the fundamental theorem of surgery theory (Theorem 4.4.18), this implies that F is normally bordant to an h-bordism between β and $-x(\alpha)$, which means that β and $-x(\alpha)$ represent the same element of $\mathcal{S}(X)$. This shows exactness of the sequence at $\mathcal{S}(M)$ and shows that the exactness respects the action of $L_{m+1}(\mathbb{Z})$ on $\mathcal{S}(M)$, which therefore completes the proof. $\square$

Warning 4.5.4. We ought to stress that this is an exact sequence of sets, not groups. There is no obvious group structure on $\mathcal{S}(X)$, and σ is *not* necessarily a homomorphism of groups! Calculations that assume to the contrary will sometimes be wrong.

Remark 4.5.5. The proof of the the exactness at $\mathcal{S}(M)$ of the surgery exact sequence exemplifies a general paradigm that one can use to leverage our theorems about the existence of manifold structures into theorems about their uniqueness. Suppose $f\colon N \to M$ is a homotopy equivalence. Our goal is to determine if f is h-bordant to a diffeomorphism.

To do this, we first try to determine if f is normally bordant to $\mathrm{Id}\colon M \to M$. By Theorem 3.4.7, this occurs if and only if f pulls back the normal bundle of M to that of N.

If f isn't normally bordant to Id, it obviously is not h-bordant to it. If it is, we want to check if the normal bordism $F\colon (W, N, M) \to (M \times I, \{0\}, \{1\})$ between f and Id is bordant to an h-bordism. This occurs if and only if F is normally bordant to a homotopy equivalence of pairs. But we know how to check if this is the case—we simply check if the rel ∂ surgery obstruction $\sigma(F) \in L_{m+1}(\mathbb{Z})$ is zero. As such, our surgery obstruction theory can provide uniqueness results as easily as it can provide existence ones.

Chapter 5

Surgery and Homotopy Theory

5.1 The Transition to Homotopy Theory

Given a simply connected Poincaré complex X of dimension $m \geq 5$, our original stated goal is to:

1. Determine if the structure set $\mathcal{S}(X)$ is nonempty, and
2. If $\mathcal{S}(X)$ is nonempty, compute it.

Our cumulative progress toward this goal can be summed up by:

The Manifold Existence Theorem The structure set $\mathcal{S}(X)$ is nonempty if and only if there exists a degree one normal map (f, b) to X with surgery obstruction $\sigma(f, b) = 0$ in $L_m(\mathbb{Z})$.

The Surgery Exact Sequence Suppose $\mathcal{S}(X)$ is nonempty. The sequence

$$L_{m+1}(\mathbb{Z}) \to \mathcal{S}(X) \to \mathcal{N}(X) \to L_m(\mathbb{Z})$$

is exact as pointed sets, and this exactness respects the action of $L_{m+1}(\mathbb{Z})$ on $\mathcal{S}(X)$.

In one sense, we seem to have answered our questions. The manifold existence theorem tells us when $\mathcal{S}(X)$ is nonempty, and, when it is, the surgery exact sequence describes it. However, in another sense, we haven't really answered anything, since we have no obvious way of computing the normal structure set. In this section, we shall put elements of the normal structure set into one-to-one correspondence with homotopy classes of maps between certain spaces, thereby reducing the computation of the normal structure set to homotopy theory. In the next several sections of this chapter, we shall solve some of this homotopy theory in the PL case, thereby providing something closer to a full answer to our original questions.

Theorem 3.4.7 put elements of the normal structure set into one-to-one correspondence with equivalence classes of stable vector bundle reductions of the Spivak normal fibration. This will be the form in which the normal structure set will be easiest to calculate. To do this, we shall need some basic results about the classifying spaces of stable vector bundles and stable spherical fibrations. These results are somewhat standard, and we shall state them without proof. For their proofs, see [Sta63].

Definition 5.1.1.

1. Let $O(n)$ be the group of orthogonal $n \times n$ matrices. There is an inclusion $O(n) \hookrightarrow O(n+1)$ given by
$$M \mapsto \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix}.$$
We denote the direct limit of $O(1) \hookrightarrow O(2) \hookrightarrow O(3) \hookrightarrow \cdots$ by O.

2. Let $G(n)$ be the monoid of homotopy equivalences $S^{n-1} \to S^{n-1}$, endowed with the compact open topology. Suspension of maps induces an inclusion $G(n) \hookrightarrow G(n+1)$, and we denote the direct limit $G(1) \hookrightarrow G(2) \hookrightarrow G(3) \hookrightarrow \cdots$ by G.

3. Let G/O be the direct limit of the quotients $G(n)/O(n)$, where the map $O(n) \hookrightarrow G(n)$ is given by treating S^{n-1} as the unit sphere is $\mathbb{R}^n$ and treating an $n \times n$ orthogonal matrix as a linear map $\mathbb{R}^n \to \mathbb{R}^n$.

Theorem 5.1.2.

1. *There exist classifying spaces $BO(k)$ and BO such that the isomorphism classes of k-dimensional vector bundles on a finite CW complex X are in a natural one-to-one correspondence with homotopy classes of maps $X \to BO(k)$, and isomorphism classes of stable vector bundles on X are in a natural one-to-one correspondence with homotopy classes of maps $X \to BO$.*

2. *There exist classifying spaces $BG(k)$ and BG such that the isomorphism classes of k-dimensional spherical fibrations on a finite CW complex X are in a natural one-to-one correspondence with homotopy classes of maps $X \to BG(k)$, and isomorphism classes of stable spherical fibrations on X are in a natural one-to-one correspondence with homotopy classes of maps $X \to BG$.*

3. *There is a map $J\colon BO \to BG$ such that if $\alpha\colon X \to BO$ is a vector bundle, $J \circ \alpha\colon X \to BG$ is its associated sphere bundle. We call this map the* J-homomorphism.

4. *G/O is the homotopy fiber of $J\colon BO \to BG$, so we have a fibration*
$$G/O \xrightarrow{\tau} BO \xrightarrow{J} BG\,.$$

5. *The fibration from 4. extends to the right, giving a fibration sequence*
$$G/O \xrightarrow{\tau} BO \xrightarrow{J} BG \xrightarrow{\eta} B(G/O)\,.$$

Proof. See [Sta63]. □

It follows from Theorem 5.1.2, part 5, that we have an exact sequence of pointed sets
$$[X, G/O] \xrightarrow{\tau_*} [X, BO] \xrightarrow{J_*} [X, BG] \xrightarrow{\eta_*} [X, B(G/O)]\,. \tag{5.1}$$

The Whitney sum construction makes BO and BG into H-spaces, and it is clear that J commutes with the H-space structures. This therefore endows the fiber G/O with an H-space structure, and it induces multiplications on $[X, G/O]$, $[X, BG]$, and $[X, BO]$, which correspond to taking Whitney sums of bundles. These multiplications commute with all of the maps in equation 5.1, and both vector bundles and spherical fibrations have stable inverses, so our sequence is actually exact as a sequence of groups.

Now, let $\alpha \in [X, BG]$ be a stable spherical fibration. We would like to know if α lifts to a stable vector bundle

$$\begin{tikzcd} & BO \arrow[d, "J"] \\ X \arrow[r, "\alpha"] \arrow[ur, dashed] & BG \end{tikzcd},$$

and, if so, we would like to classify such lifts up to isomorphism.

Existence is straightforward. Equation 5.1 implies that α lifts if and only $\eta_*\alpha$ is nullhomotopic. We claim that isomorphism classes of lifts are in an unnatural one-to-one correspondence with elements of $[X, G/O]$.

It follows easily from the fact that G/O is the homotopy fiber of J that elements of $[X, G/O]$ are in a one-to-one correspondence with homotopy classes of pairs (E, h), where $E\colon X \to BO$ is a vector bundle with trivial $J(E)$, and h is a nullhomotopy of $J(E)$. (The argument is exactly analogous to the one that gives the long exact sequence of a fiber bundle.)

Now, choose a fixed lift η_0 of α. (This choice is the source of the unnaturality.) Isomorphism classes of lifts are the same as homotopy classes of pairs (η, h), where $\eta\colon X \to BO$ is a vector bundle, and h is a homotopy of $J(\eta) \simeq \alpha\colon X \to BG$. Subtracting η_0 from η and $J(\eta_0) = \alpha$ from each stage in the homotopy of $J(\eta)$ to α puts these objects into a one-to-one correspondence with pairs $(\eta - \eta_0, \text{nullhomotopy of } J(\eta) - \alpha)$, which is precisely the realization of $[X, G/O]$ given in the last paragraph.

Applying this to the Spivak normal fibration, we obtain:

Theorem 5.1.3. *Let X be a Poincaré complex.*

1. *The normal structure set $\mathcal{N}(X)$ is nonempty if and only if the image $\eta_*\nu_X$ of the Spivak normal fibration in $[X, B(G/O)]$ is nullhomotopic.*
2. *If $\mathcal{N}(X)$ is nonempty, then a choice of an element $E \in X$ puts the structure set into an (unnatural) one-to-one correspondence with $[X, G/O]$.*

We note that for a manifold M, the correspondence between $\mathcal{N}(M)$ and $[M, G/O]$ is actually natural, since there is a distinguished vector bundle reduction of the Spivak normal fibration—the stable normal bundle N_M.

We can now rewrite the surgery exact sequence:

Theorem 5.1.4. *Let M be a manifold of dimension greater than or equal to five. The sequence*

$$L_{m+1}(\mathbb{Z}) \to \mathcal{S}(M) \to [M, G/O] \to L_m(\mathbb{Z})$$

is an exact sequence of pointed sets, and the exactness respects the action of $L_{m+1}(\mathbb{Z})$ on $\mathcal{S}(M)$.

5.2 The PL Version

Everything that we have done thus far can be easily translated into the piecewise linear category. Just as in the smooth category, piecewise linear manifolds and cobordisms admit handle decompositions. Using these, the proof of the h-cobordism theorem translates without difficulty. One may then define manifold structures, Poincaré complexes, and the normal structure set in the analogous way. The bundle theoretic arguments that we set forth work as well, provided that one replaces smooth vector bundles with PL block bundles. (See [Cas96] for a review of PL block bundles.) Our proof of the surgery exact sequence can then be translated into this category, giving us the PL surgery exact sequence

$$L_{m+1}(\mathbb{Z}) \to \mathcal{S}_{\mathrm{PL}}(X) \to \mathcal{N}_{\mathrm{PL}}(X) \to L_m(\mathbb{Z}).$$

The arguments from the last section that express $\mathcal{N}_{\mathrm{PL}}(X)$ homotopy theoretically may be repeated as well, but the classifying spaces are different. Here, there exists a space BPL that classifies PL block bundles, and there exists a map $J_{\mathrm{PL}}\colon BPL \to BG$ that corresponds to taking the spherical fibration associated to a PL block bundle. We then define G/PL to be the homotopy fiber of J_{PL}, so that $G/PL \to BPL \to BG$ is a fibration.

There are slight technical difficulties in describing the H-space PL as a stand-alone geometric object. Intuitively, PL should be thought of as the PL self-homeomorphisms of $\mathbb{R}^n$ that send 0 to 0. However, due to the somewhat discrete nature of the PL category, the compact-open topology is no longer the correct way to topologize this space. Instead, one describes it as a simplicial complex; see [Cas96] for details. Fortunately, we never actually use the space PL by itself, provided we are willing to accept the somewhat abstract homotopy theoretic definition of G/PL given above, so this does not complicated our arguments. As such, we obtain without incident the analogous unnatural correspondence $\mathcal{N}_{\mathrm{PL}}(X) \cong [X, G/PL]$, which becomes a natural correspondence $\mathcal{N}_{\mathrm{PL}}(M) \cong [M, G/PL]$ if we fix a manifold structure on our Poincaré complex X.

The way in which the smooth and piecewise linear theories truly differ is that the generalized Poincaré conjecture holds in the PL category, whereas it fails in the smooth one. This means that $\mathcal{S}_{PL}(S^n)$ consists of a single element, whereas $\mathcal{S}(S^n)$ can be quite complicated. (In fact, as shown by Kervaire and Milnor in [KM63], computing its size depends on the computation of the homotopy groups of spheres, and thus cannot be solved with presently existing methods.) When we run this information through the respective surgery exact sequences, we see that the homotopy groups $[S^n, G/PL]$ are quite simple, whereas those of G/PL are extremely complicated. The simplicity of the homotopy groups of G/PL will allow us in the next section to obtain a relatively simple description of its homotopy type. This will permit us to actually compute many PL structure sets without too much difficulty. However, the fact that we can't even understand the homotopy groups of G/O renders hopeless a similar pursuit for G/O, and the general computation of smooth structure sets remains unsolved.

As a final note about the different categories, we remark that this entire theory can be extended into the category of topological manifolds. Whereas the translation into the PL category is completely straightforward, the translation into the topological category is very hard. The problems arise at the very beginnings of the theory, where one loses any obvious notion of transversality, as well as the handle decomposition of manifolds and cobordisms. It turns out that enough of this structure can be recovered, but only with a serious amount of work. This was accomplished by Kirby and Siebenmann in 1969 and published in their landmark book [KS77]. We shall not touch on this theory, but we note that the surgery exact sequence does in fact carry over to this category.

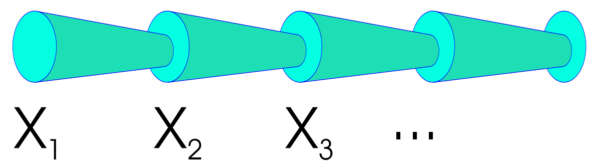

Figure 5.1: Constructing a mapping telescope

5.3 The Homotopy Type of G/PL

We have now reduced the classification of simply connected manifolds to homotopy theory. Here we shall actually solve some of this homotopy theory in the PL case by computing the homotopy type of G/PL. As mentioned before, computing the homotopy groups of G/O requires a knowledge of the homotopy groups of spheres. Since the homotopy groups of spheres are not currently understood, the homotopy type of G/O remains beyond our reach, and we cannot fully solve the homotopy theory in the smooth case.

At this point, an apology is possibly in order. We deviate somewhat in this section from our professed goal of keeping the treatment elementary, and we assume a bit more homotopy theory than we had been assuming thus far. In particular, we assume knowledge of Postnikov systems, spectra, and generalized homology and cohomology theories. We also provide only a brief introduction to the theory of localization, of which we make essential use. For a good treatment of Postnikov systems, see [Spa66]. For spectra and generalized homology and cohomology theories, see [Ada74]. For a more in-depth discussion of localization, see [Sul74] or [Sel91].

We temper this apology with the explanation that this is probably the most elementary proof of the homotopy type of G/PL available in the literature. In fact, tracking down any proof of it is a major task. It was originally proven by Sullivan in the 1960's, but he never published a full proof. Madsen and Milgram sketch a proof in [MM79], but they skip many of the details, and their discussion is difficult for all but the most seasoned homotopy theorists to penetrate. As such, we feel that a simple explanation with most of the details written out is worth including, even if it deviates slightly from the rest of the thesis in the level of background that it assumes. We begin with a short discussion of the localization of a homotopy type. We assume the reader already to be familiar with the algebraic localization of a $\mathbb{Z}$-module.

5.3.1 Localization

In algebra and algebraic geometry, it is often advantageous to localize away the effects of certain primes in a ring so that one can isolate information about those that remain. Once one understands the ring locally, he can piece the local information back together to understand the ring globally.

In this section, we will discuss an operation that one can perform on many topological spaces that has the effect of localizing (as $\mathbb{Z}$-modules) the homotopy and homology groups. Just as in algebra, the local objects are often much simpler than the global ones. It is therefore sometimes useful to analyze the local homotopy types of spaces and then use this to obtain global information.

Not all spaces can be localized, and the operation is difficult to describe in general for those that can be. However, all H-spaces can be localized, and the operation for doing so is quite simple. Since they are the only spaces that we shall localize in the sequel, we restrict our discussion to them.

This construction makes use of mapping telescopes, which generalize mapping cylinders. Let

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n$$

be a sequence of maps of topological spaces. Let M_i be the mapping cylinder

$$((X_i \times I) \amalg X_{i+1}) \,/\, ((x \times \{1\}) \sim f_i(x))$$

of f_i. We can attach the mapping cylinder of f_{i+1} to that of f_i by by gluing $X_{i+1} \subseteq M_i$, to $X_{i+1} \times \{0\} \subseteq M_{i+1}$. (See Figure 5.1.) We can thus attach the mapping cylinders end-to-end to obtain a single space, which we call the *mapping telescope* of the sequence of maps. We can clearly form the mapping telescope of an infinite sequence of maps as well in the obvious way.

Now, let X be an H-space (with unit), and for simplicity assume that X is simply connected. Let S be a multiplicative set of integers, and let $\mathbb{Z}_S$ denote $\mathbb{Z}$ with the elements of S inverted. For an abelian group G, let G_S denote its localization $G \otimes \mathbb{Z}_S$. Furthermore, for $n \in \mathbb{Z}$, let $n\colon X \to X$ denote the operation of raising an element in X to the n^{th} power. We note that the pushforward map $n_*\colon \pi_k(X) \to \pi_k(X)$ is multiplication by n.

Let $\{n_i\}$ be a sequence of integers that is cofinal in S under multiplication. (That is, if $s \in S$, there exists some i such that s divides n_i.) Now let the *localization* X_S be the mapping telescope of the sequence of maps

$$X \xrightarrow{n_1} X \xrightarrow{n_2} X \xrightarrow{n_3} \cdots$$

We have the following intuitive result:

Theorem 5.3.1. *The homotopy and homology groups of X_S obey*

$$\pi_n(X_S) = (\pi_n(X))_S$$

and

$$H_n(X_S) = (H_n(X))_S.$$

Proof. See [Sul74] or [Sel91]. □

Localization is functorial, so it commutes nicely with maps of spaces and induced maps of homotopy and homology.

We now need a result that will let us assemble the different pieces of local information to get global information. Suppose $\mathbb{Z}_S$ and $\mathbb{Z}_T$ are two different localizations of $\mathbb{Z}$. Let $\langle S, T\rangle$ denote the multiplicative set generated by the elements of S and T. We have the diagram of rings

$$\begin{tikzcd} \mathbb{Z}_{S\cap T} \arrow[r] \arrow[d] & \mathbb{Z}_S \arrow[d] \\ \mathbb{Z}_T \arrow[r] & \mathbb{Z}_{\langle S,T\rangle} \end{tikzcd},$$

which gives rise to an analogous diagram of spaces.

Theorem 5.3.2. *Let X be a simply connected H-space. (In fact, this holds for any simply connected X if one uses a more general definition that allows one to localize it.) The diagram*

$$\begin{tikzcd} X_{S\cap T} \arrow[r] \arrow[d] & X \arrow[d] \\ X_T \arrow[r] & X_{\langle S,T\rangle} \end{tikzcd}$$

is a fiber and cofiber square. Furthermore, a map $f\colon Y \to X_{S\cap T}$, where Y is a finite CW complex, is determined up to homotopy by the compositions $Y \to X_{S\cap T} \to X_S$ and $Y \to X_{S\cap T} \to X_T$.

Proof. See [Sul74]. □

We shall only localize with three multiplicative sets in that which follows, so we define a special notation for them. We let $\mathbb{Z}[2]$ denote the integers with all odd primes inverted, and we let $\mathbb{Z}[\frac{1}{2}]$ denote the integers with 2 inverted. If X is an H-space, we let $X[2]$ and $X[\frac{1}{2}]$ denote its corresponding localizations. We refer to the former as the *even* or *2-local* homotopy type of X, and to the latter as the *odd* homotopy type of X. Finally, if we take our multiplicative set to consist of all nonzero elements of $\mathbb{Z}$, we obtain the rationals. We denote the corresponding localization of X by $X_{\mathbb{Q}}$.

We now use localization to analyze the homotopy type of G/PL. As we shall see, the even and odd homotopy types of G/PL are quite different, but they are both relatively simple. The first splits as a product of Eilenberg-Maclane spaces, whereas the second is the same as the odd homotopy type of BO.

5.3.2 The Two-Local Homotopy Type of G/PL

Theorem 5.3.3 (Sullivan). *The two-local homotopy type of G/PL satisfies*

$$G/PL[2] \simeq F \times \prod_{n>1} K(\mathbb{Z}[2]), 4n) \times K(\mathbb{Z}/2\mathbb{Z}, 4n-2),$$

where F is the two-stage Postnikov system with nontrivial homotopy groups $\pi_2(F) = \mathbb{Z}/2\mathbb{Z}$ and $\pi_4(F) = \mathbb{Z}[2]$ and k-invariant $\beta Sq^2 \in H^5(K(\mathbb{Z}/2\mathbb{Z}, 2); \mathbb{Z}[2])$, where β is the Bockstein operator.

Lemma 5.3.4. *The homotopy groups of G/PL are given by:*

$$\pi_n(G/PL) = L_n(\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n \equiv 0 \pmod 4, \\ \mathbb{Z}/2\mathbb{Z} & \text{if } n \equiv 2 \pmod 4, \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. For $n \geq 5$, we have the surgery exact sequence

$$L_{n+1}(\mathbb{Z}) \to \mathcal{S}_{\mathrm{PL}}(S^n) \to [S^n, G/PL] \to L_n(\mathbb{Z}).$$

$\mathcal{S}_{\mathrm{PL}}(S^n)$ has exactly one element for such n by the generalized Poincaré conjecture, so $\pi_n(G/PL) = [S^n, G/PL]$ injects into $L_n(\mathbb{Z})$. By Milnor's plumbing theorem (Theorem 4.5.2), there is a normal map to (D^n, S^{n-1}) with surgery obstruction 1. Coning off the boundary produces a normal map to S^n with surgery obstruction 1, so this the map is surjective as well. The claim thus follows for $n \geq 5$. Somewhat different methods are required for $n < 5$; see [Lev83]. □

Remark 5.3.5. For $n = 4$, it is perhaps more proper to write that $\pi_4(G/PL) = 2L_4(\mathbb{Z})$. Rochlin's theorem ([Roc86]) tells us that every 4-dimensional spin manifold has signature divisible 16, which can easily be seen to imply that the surgery obstruction of every element of $\mathcal{N}(S^4) = \pi_4(G/PL)$ is even.

The localization $G/PL[2]$ thus has

$$\pi_n(G/PL[2]) = \begin{cases} \mathbb{Z}[2] & \text{if } n \equiv 0 \pmod 4, \\ \mathbb{Z}/2\mathbb{Z} & \text{if } n \equiv 2 \pmod 4, \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Proposition 5.3.6. *For each $n > 1$ there exist maps*

$$\phi_{4n}\colon G/PL[2] \to K(\mathbb{Z}[2], 4n)$$

and

$$\psi_{4n-2}\colon G/PL[2] \to K(\mathbb{Z}/2\mathbb{Z}, 4n-2)$$

that induce isomorphisms on π_{4n} and π_{4n-2} respectively.

The proof of this proposition makes essential use of the following result of Thom:

Theorem 5.3.7 (Thom). *Let MO and MSO be the spectra that classify unoriented and oriented bordism, respectively, and let $K(G)$ for an abelian group G be the Eilenberg-Maclane spectrum whose n^{th} space is $K(G,n)$. There exist constants a_i, b_i, and c_i such that:*

$$MO \simeq \prod_{i=0}^{\infty} \Sigma^{a_i} K(\mathbb{Z}/2\mathbb{Z})$$
$$MSO[2] \simeq \prod_{i=0}^{\infty} \Sigma^{b_i} K(\mathbb{Z}/2\mathbb{Z}) \times \prod_{j=0}^{\infty} \Sigma^{c_j} K(\mathbb{Z}[2]).$$

Proof. See [Sto68]. □

Corollary 5.3.8. *Let*

$$\alpha_{MO}\colon MO_k(X) \to H_k(X; \mathbb{Z}/2\mathbb{Z})$$

and

$$\alpha_{MSO}\colon MSO_k(X) \to H_k(X; \mathbb{Z})$$

be given by

$$\{M, f\} \mapsto f_*([M]),$$

and let

$$\alpha_{MSO[2]}\colon MSO[2]_k(X) \to H_k(X; \mathbb{Z}[2])$$

be induced by α_{MSO}. The maps α_{MO} and $\alpha_{MSO[2]}$ have sections.

Proof. This follows immediately from the construction used in the proof of Theorem 5.3.7. The maps of spectra

$$\widetilde{\alpha}_{MO}\colon MO \to K(Z/2\mathbb{Z})$$

and

$$\widetilde{\alpha}_{MSO[2]}\colon MSO[2] \to K(\mathbb{Z}[2])$$

induced by α_{MO} and $\alpha_{MSO[2]}$ respectively are in fact the homotopy equivalences described in the theorem composed with projections onto individual factors. As such, they obviously have sections. □

Remark 5.3.9. MSO does not itself split up as a product of suspensions of Eilenberg-Maclane spectra, and thus an assertion about it analogous to Corollary 5.3.8 does not hold. As we shall see, this is what forces us to localize to 2 and prevents our proof of Theorem 5.3.3 from applying to (the unlocalized) G/PL.

Notation. If S is a spectrum and X is a CW complex, we shall use $S_n(X)$ to denote the n^{th} group in the homology theory associated to S.

Proof of Proposition 5.3.6. Let h be the Hurewicz map from homotopy to homology, and let γ_{2n} be a generator of $\pi_{2n}(G/PL[2])$. By tracing out the maps in the isomorphism $[G/PL[2], K(G,n)] \cong H^n(G/PL[2]; G)$, we see that finding the desired ϕ_{4n} and ψ_{4n-2} is equivalent to finding cohomology classes

$$\rho_{4n} \in H^n(G/PL[2]; \mathbb{Z}[2])$$

and

$$\tau_{4n-2} \in H^n(G/PL[2]; \mathbb{Z}/2\mathbb{Z})$$

for each $n > 1$ such that

$$\rho_{4n}(h(\gamma_{4n})) = 1 \tag{5.2}$$

and

$$\tau_{4n-2}(h(\gamma_{4n-2})) = 1. \tag{5.3}$$

We shall actually construct elements

$$\overline{\rho}_{4n} \in \mathrm{Hom}(H_{4n}(G/PL[2]; \mathbb{Z}[2]), \mathbb{Z}[2])$$

and

$$\overline{\tau}_{4n-2} \in \mathrm{Hom}(H_{4n-2}(G/PL[2]; \mathbb{Z}/2\mathbb{Z}), \mathbb{Z}/2\mathbb{Z})$$

that take $h(\gamma_k)$ to 1. We shall then have

$$\mathrm{Ext}(H_{4n-1}(G/PL[2]; \mathbb{Z}[2]), \mathbb{Z}[2]) \longrightarrow H^{4n}(G/PL[2]; \mathbb{Z}[2]) \longrightarrow \mathrm{Hom}(H_{4n}(G/PL[2]; \mathbb{Z}[2]), \mathbb{Z}[2]) \longrightarrow 0$$

and

$$\mathrm{Ext}(H_{4n-1}(G/PL[2]; \mathbb{Z}/2\mathbb{Z}), Z/2\mathbb{Z}) \longrightarrow H^{4n}(G/PL[2]; Z/2\mathbb{Z}) \longrightarrow \mathrm{Hom}(H_{4n}(G/PL[2]; \mathbb{Z}/2\mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) \longrightarrow 0$$

by the universal coefficient theorem, which will allow us to lift ρ_{4n} and τ_{4n-2} to cohomology classes ρ and τ satisfying (5.2) and (5.3).

The first step in the construction of $\overline{\rho}_{4n}$ and $\overline{\tau}_{4n-2}$ is the realization that the surgery obstruction $\sigma_n \colon [S^n, G/PL] \to L_n(\mathbb{Z})$ is bordism invariant, and thus it is oriented bordism invariant as well. This gives us the factorizations

$$\begin{tikzcd} \pi_{4n}(G/PL[2]) \arrow[rr, "\sigma_{4n}", "\cong"'] \arrow[dr, "h_{MSO[2]}"'] & & \mathbb{Z}[2] \\ & MSO[2]_{4n}(G/PL[2]) \arrow[ur, "\overline{\sigma}_{4n}"'] & \end{tikzcd} \tag{5.4}$$

and

$$\begin{tikzcd} \pi_{4n-2}(G/PL[2]) \arrow[rr, "\sigma_{4n-2}", "\cong"'] \arrow[dr, "h_{MO}"'] & & \mathbb{Z}/2\mathbb{Z}\,, \\ & MO_{4n-2}(G/PL[2]) \arrow[ur, "\overline{\sigma}_{4n-2}"'] & \end{tikzcd} \tag{5.5}$$

where $h_{MSO[2]}$ and h_{MO} are the Hurewicz maps that take $f \colon S^n \to G/PL[2]$ to the pair $\{S^n, f\}$ in the given bordism groups.

Claim 5.3.10. *The maps $h_{MSO[2]}$ and h_{MO} factor through homology:*

$$\begin{tikzcd}
\pi_{4n}(G/PL[2]) \arrow[rr, "h_{MSO[2]}"] \arrow[dr, "h"'] & & MSO[2]_{4n}(G/PL[2]) \\
& H_{4n}(G/PL[2];\mathbb{Z}[2]) \arrow[ur, "\eta"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
\pi_{4n-2}(G/PL[2]) \arrow[rr, "h_{MO}"] \arrow[dr, "h"'] & & MO_{4n-2}(G/PL[2]) . \\
& H_{4n-2}(G/PL[2];\mathbb{Z}/2\mathbb{Z}) \arrow[ur, "\theta"'] &
\end{tikzcd}$$

Proof of Claim 5.3.10. Let

$$\alpha_{MSO[2]}\colon MSO[2]_{4n}(G/PL[2]) \to H_{4n}(G/PL[2];\mathbb{Z}[2])$$

and

$$\alpha_{MO}\colon MO_{4n-2}(G/PL[2]) \to H_{4n-2}(G/PL[2];\mathbb{Z}/2\mathbb{Z})$$

be as defined in the statement of Corollary 5.3.8. These maps fit into the commutative diagrams

$$\begin{tikzcd}
\pi_{4n}(G/PL[2]) \arrow[rr, "h_{MSO[2]}"] \arrow[dr, "h"'] & & MSO[2]_{4n}(G/PL[2]) \arrow[dl, "\alpha_{MSO[2]}"] \\
& H_{4n}(G/PL[2];\mathbb{Z}[2]) &
\end{tikzcd} \tag{5.6}$$

and

$$\begin{tikzcd}
\pi_{4n-2}(G/PL[2]) \arrow[rr, "h_{MO}"] \arrow[dr, "h"'] & & MO_{4n-2}(G/PL[2]) . \arrow[dl, "\alpha_{MO}"] \\
& H_{4n-2}(G/PL[2];\mathbb{Z}/2\mathbb{Z}) &
\end{tikzcd}$$

Both $\alpha_{MSO[2]}$ and α_{MO} have sections by Corollary 5.3.8. Setting η and θ equal to these sections gives us the desired factorizations. ☐

Combining Claim 5.3.10 with equations (5.4) and (5.5) gives us the diagrams:

$$\begin{tikzcd}
& \pi_{4n}(G/PL[2]) \arrow[rr, "\sigma_{4n}", "\cong"'] \arrow[dl, "h"'] \arrow[dr, "h_{MSO[2]}"] & & \mathbb{Z}[2] \\
H_{4n}(G/PL[2];\mathbb{Z}[2]) \arrow[rr, "\alpha_{MSO[2]}"] & & MSO[2]_{4n}(G/PL[2]) \arrow[ur, "\overline{\sigma}_{4n}"] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
& \pi_{4n-2}(G/PL[2]) \arrow[rr, "\sigma_{4n-2}", "\cong"'] \arrow[dl, "h"'] \arrow[dr, "h_{MO}"] & & \mathbb{Z}/2\mathbb{Z} , \\
H_{4n-2}(G/PL[2];\mathbb{Z}/2\mathbb{Z}) \arrow[rr, "\alpha_{MO}"] & & MO_{4n-2}(G/PL[2]) \arrow[ur, "\overline{\sigma}_{4n-2}"] &
\end{tikzcd}$$

The surgery obstructions σ_{4n} and σ_{4n+2} take generators of $\pi_k(G/PL[2])$ to 1 in their respective codomains. It thus follows from the diagrams that the compositions $\overline{\sigma}_{4n} \circ \alpha_{MSO[2]}$ and $\overline{\sigma}_{4n-2} \circ \alpha_{MO}$ give us members of $\mathrm{Hom}(H_{4n}(G/PL[2]; \mathbb{Z}[2]), \mathbb{Z}[2])$ and $\mathrm{Hom}(H_{4n-2}(G/PL[2]; \mathbb{Z}/2\mathbb{Z}), \mathbb{Z}/2\mathbb{Z})$ with the claimed properties, from which Proposition 5.3.6 now follows. □

Proof of Theorem 5.3.3. By attaching cells of dimension six and higher to $G/PL[2]$, we can produce a complex X^4 with $\pi_k(X^4) = 0$ for all $k > 4$. The inclusion $\iota\colon G/PL[2] \hookrightarrow X^4$ induces isomorphisms on π_j for all $j \leq 4$.

Let

$$\Pi = \prod_{n>1} K(\mathbb{Z}[2], 4n) \times K(\mathbb{Z}/2\mathbb{Z}, 4n-2).$$

The maps given by Proposition 5.3.6 give us a map $\Phi\colon G/PL[2] \to \Pi$ that induces isomorphisms on π_k for $k > 4$, and $\pi_j(\Pi) = 0$ for all $j \leq 4$.

Putting everything together, we thus get a map

$$\iota \times \Phi\colon G/PL[2] \to X^4 \times \Pi$$

that induces isomorphisms on all homotopy groups, so it is a homotopy equivalence by Whitehead's theorem. It just remains to check that X^4 is homotopy equivalent to the space F described in the statement of Theorem 5.3.3.

By its homotopy groups, we know that X^4 is a two-stage Postnikov system with the same homotopy groups as F, so we just need to compute its k-invariant $k^4 \in H^5(K(\mathbb{Z}[2], 2), \mathbb{Z}[2]) = \mathbb{Z}/4\mathbb{Z}$. We claim that it is the element βSq^2, which corresponds to $2 \in \mathbb{Z}/4\mathbb{Z}$. $G/PL[\frac{1}{2}]$ is an H-space, so its k-invariants are primitive with respect to its multiplication [Kah63]. (That is, if $m\colon G/PL[\frac{1}{2}] \times G/PL[\frac{1}{2}] \to G/PL[\frac{1}{2}]$ is the multiplication map, then $m^*(k^4) = 1 \times k^4 + k^4 \times 1$.) A direct calculation shows that 0 and 2 are the only primitive elements of $H^5(K(\mathbb{Z}[2], 2), \mathbb{Z}[2])$, so it suffices to prove:

Claim 5.3.11. *The k-invariant $k^4 \neq 0$.*

Proof. This is equivalent to showing that X^4 is not homotopy equivalent to $Y = K(\mathbb{Z}/2\mathbb{Z}, 2) \times K(\mathbb{Z}[2], 4)$. We note that $\pi_4(Y) = \mathbb{Z}[2]$. By the Künneth formula, $H_4(Y) = \mathbb{Z}[2] \oplus H^4(K(\mathbb{Z}/2\mathbb{Z}), 2)$, and the Hurewicz map is an isomorphism onto the first direct summand. We show that the Hurewicz homomorphism $h\colon \pi_4(F) \to H_4(F)$ is not an isomorphism onto a direct summand, from which the claim follows.

A version of Rochlin's theorem ([Roc86])states that any manifold that admits a degree one map to S^4 has signature divisible by 16, so that the surgery obstruction $\sigma_4\colon \pi_4(G/PL) \to \mathbb{Z}$ is not an isomorphism but is instead multiplication by 2. Localizing this fact and combining it with the factorization of h through $h_{MSO[2]}$ given by equation (5.6) yields the diagram:

$$\begin{tikzcd}
\mathbb{Z}[2] = \pi_4(G/PL[2]) \arrow[r, "h_{MSO[2]}"] \arrow[rr, bend left, "h"] \arrow[dr, "\sigma_4 = \cdot 2"'] & MSO[2]_4(G/PL[2]) \arrow[r, "\alpha_{MSO[2]}"] \arrow[d, "\sigma_4"] & H_4(G/PL[2], \mathbb{Z}[2]) . \\
& \mathbb{Z}[2] &
\end{tikzcd}$$

If we can show that $\sigma_4\colon MSO[\frac{1}{2}]_4(G/PL[2]) \to \mathbb{Z}$ is a surjection, this will imply our desired claim that h is not an isomorphism onto a direct summand. For this, it suffices to construct a PL manifold

M^4 and a map $f\colon M \to G/PL$ with surgery invariant 1. This is equivalent to finding a PL manifold N^4 and a degree one normal map $g\colon N \to M$ such that $\sigma_4(g) = (\sigma(M) - \sigma(N))/8 = 1$.

Let $M = \mathbb{CP}^2$, and let $N = \mathbb{CP}^2 \# 8\overline{\mathbb{CP}}^2 = \mathbb{CP}^2 \# \overline{\mathbb{CP}}^2 \# \cdots \# \overline{\mathbb{CP}}^2$, where $\overline{\mathbb{CP}}^2$ is $\mathbb{CP}^2$ with the conjugate complex structure. The signatures $\sigma(\mathbb{CP}^2) = 1$ and $\sigma(\overline{\mathbb{CP}}^2) = -1$, so $\sigma(N) = 1 - 8 = -7$. As such, a degree one normal map $f\colon N \to M$ would have surgery obstruction $\sigma(f) = (1-(-7))/8 = 1$, as desired, so it suffices to construct such a map.

Let $\gamma = (3, 1, 1, 1, 1, 1, 1, 1, 1) \in H^2(N, \mathbb{Z}) = \mathbb{Z}^9$, where the first coordinate corresponds to the $\mathbb{CP}^2 \subseteq N$. An element of $H^2(N, \mathbb{Z})$ gives rise to a map $N \to K(\mathbb{Z}, 2) = \mathbb{CP}^\infty$, which we may assume lands in the four-skeleton $\mathbb{CP}^2 \subseteq \mathbb{CP}^\infty$. We thus get a map $f\colon M \to \mathbb{CP}^2$. Let $\beta \in H^2(\mathbb{CP}^2, \mathbb{Z})$ be a generator, so that $\beta \cup \beta = [\mathbb{CP}^2]$. Furthermore, let e_i, $i \in \{0, \ldots, 9\}$, be the element of $H^2(N, \mathbb{Z}) = \mathbb{Z}^9$ with a one in the i^{th} spot and zeros elsewhere. The cup product pairing on $H^2(N, \mathbb{Z})$ is:

$$e_i \cup e_j = \begin{cases} \delta_{ij}[N] & \text{if } i = 0, \\ -\delta ij[N] & \text{if } i \neq 0. \end{cases}$$

This implies that

$$f^*([\mathbb{CP}^2]) = f^*(\beta) \cup f^*(\beta) = (3, 1, 1, 1, 1, 1, 1, 1, 1) \cup (3, 1, 1, 1, 1, 1, 1, 1, 1) = 9[N] - 8[N] = [N],$$

so $\deg(f) = 1$.

We now show f to be a normal map. Let O be the direct limit of $O(1) \hookrightarrow O(2) \hookrightarrow \cdots$. As a CW complex, $\mathbb{CP}^2$ consists of a single 4-cell glued to its 2-skeleton, which is just S^2. We note that $\pi_1(O) = \mathbb{Z}/2\mathbb{Z}$, so stable vector bundles on S^2 are classified by their first Stiefel-Whitney class. $\mathbb{CP}^2$ is orientable, so its stable tangent bundle has $w_1(\tau_{\mathbb{CP}^2}) = 0$ and is therefore trivial on the 2-skeleton. Stable vector bundles on $\mathbb{CP}^2$ that are trivial on the 2-skeleton are in one-to-one correspondence with stable bundles on $\mathbb{CP}^2$ mod its two-skeleton, which is S^4. These bundles are classified by $\pi_3(O) = \mathbb{Z}$, and this classification is given by the first Pontrjagin class. By the Hirtzebruch signature theorem, $p_1(\tau_{\mathbb{CP}^2}) = 3[\mathbb{CP}^2]$. Since $N_{\mathbb{CP}^2} \oplus \tau_{\mathbb{CP}^2} = \epsilon^\infty$, the stable normal bundle has $w_1(N_{\mathbb{CP}^2}) = 0$ and $p_1(N_{\mathbb{CP}^2}) = -3[\mathbb{CP}^2]$.

The stable normal bundle thus pulls back to a bundle $E = f^*(N_{\mathbb{CP}^2})$ that is trivial on the 2-skeleton of N. Like $\mathbb{CP}^2$, N is obtained by attaching a single 4-cell to its 2-skeleton, so such bundles are classified by their first Pontrjagin classes. We have $p_1(E) = f^*(-3[\mathbb{CP}^2]) = -3[N]$. By the Hirtzebruch signature theorem, $p_1(\tau_N) = -7 \cdot 3 = -21[N]$, so $p_1(N_N) = 21[N]$.

We claim that the stable spherical fibrations associated to these two bundles are the same, which will show f to be a normal map. Since both are trivial on the 2-skeleton, it suffices to show that the stable spherical fibrations over S^4 associated to the bundles with first Pontrjagin classes $-3[S^4]$ and $21[S^4]$ are the same. This is equivalent to showing that the stable spherical fibration associated the vector bundle with Pontrjagin class $24[S^4]$ is trivial.

The tautological bundle ξ over $\mathbb{CP}^2$ has $p_1(\xi) = 1$, so $\eta = 24\xi = \xi \oplus \cdots \oplus \xi$ has $p_1(\eta) = 24[S^4]$. Stable spherical fibrations over S^4 are classified by $[S^4, G] = \pi_4(G) = \pi_3(G) = \pi_3^S = \mathbb{Z}/24\mathbb{Z}$. As such, 24 times the spherical fibration associated to ξ is trivial, so f is a degree one normal map with $\sigma_4(f) = 1$, and thus the Hurewicz map is not an isomorphism onto a direct summand, from which Claim 5.3.11 follows. □

This completes the proof of Theorem 5.3.3. □

5.3.3 The Odd Homotopy Type of G/PL

We now compute the odd homotopy type of G/PL:

Theorem 5.3.12. *The odd homotopy type of G/PL is given by*

$$G/PL[\frac{1}{2}] \simeq BO[\frac{1}{2}],$$

where BO is the classifying space for real stable vector bundles.

The proof of this theorem is similar in spirit to the proof of the 2-local homotopy type of G/PL. We use the connections between homotopy, bordism, and (this time) K-theory to obtain an element of $\mathrm{Hom}(KO[\frac{1}{2}]_0(G/PL), \mathbb{Z}[\frac{1}{2}])$, and we then use a universal coefficient theorem to lift this to an element of the cohomology theory $KO[\frac{1}{2}]^0(X)$, which gives rise to a map from $G/PL[\frac{1}{2}]$ to $BO[\frac{1}{2}]$ that we check to be a homotopy equivalence.

We thus need an analogue of Thom's theorem that allows us to relate odd real K-theory to one of our other theories, and we need a version of the universal coefficient theorem for odd real K-theory. These theorems are somewhat more difficult than the corresponding ones for the even case, and we state them without proof.

Theorem 5.3.13. *If X is a CW complex,*

$$KO[\frac{1}{2}]_k(X) \cong MSO[\frac{1}{2}]_{4*+k}(X) \otimes_{MSO[\frac{1}{2}]_*(pt.)} \mathbb{Z}[\frac{1}{2}], \quad k \in \mathbb{Z}/4\mathbb{Z}.$$

Proof. See [CF66]. □

Theorem 5.3.14. *If X is a CW complex, we have the exact sequence*

$$\mathrm{Ext}(KO[\tfrac{1}{2}]_{n-1}(X), \mathbb{Z}[\tfrac{1}{2}]) \longrightarrow KO[\tfrac{1}{2}]^n(X) \longrightarrow \mathrm{Hom}(KO[\tfrac{1}{2}]_n(X), \mathbb{Z}[\tfrac{1}{2}]) \longrightarrow 0\ , \quad n \in \mathbb{Z}/4\mathbb{Z}.$$

Proof. See [Yos75]. □

We now use these theorems to prove Theorem 5.3.12.

Proof of Theorem 5.3.12. It follows from Lemma 5.3.4 that

$$\pi_n(G/PL[\frac{1}{2}]) = \begin{cases} \mathbb{Z}[\frac{1}{2}] & \text{if } n \equiv 0 \pmod 4, \\ 0 & \text{if } n \equiv 1,2,3 \pmod 4). \end{cases}$$

Furthermore, Bott's computation of the homotopy groups of BO shows that $BO[\frac{1}{2}]$ has these homotopy groups as well. Accordingly, we restrict our attention to dimensions divisible by 4.

As in the even case, we use the bordism invariance of the surgery obstruction to factor it through MSO and then localize, giving us the diagram

$$\begin{tikzcd} \pi_{4*}(G/PL[\frac{1}{2}]) \arrow[rr, "\sigma_{4*}", "\cong"'] \arrow[dr, "h_{MSO[\frac{1}{2}]}"'] & & \mathbb{Z}[\frac{1}{2}]\ , \\ & MSO_{4*}(G/PL[\frac{1}{2}]) \arrow[ur, "\overline{\sigma}_{4*}"'] & \end{tikzcd}$$

where $h_{MSO[\frac{1}{2}]}$ is the Hurewicz map induced from h_{MSO} by localization.

Now, while in the even case we factored the Hurewicz map through our homology theory, this time we shall factor $\overline{\sigma}_{4*}$ through it. By Theorem 5.3.13, there exists ν_{4*} such that

$$\begin{tikzcd} \pi_{4*}(G/PL[\frac{1}{2}]) \arrow[rr, "\sigma_{4*}", "\cong"'] \arrow[d, "h_{MSO[\frac{1}{2}]}"'] & & \mathbb{Z}[\frac{1}{2}] \\ MSO_{4*}(G/PL[\frac{1}{2}]) \arrow[rr, "\mathrm{Id}\otimes 1"] \arrow[urr, "\overline{\sigma}_{4*}"] & & KO[\frac{1}{2}]_0(G/PL[\frac{1}{2}]) \arrow[u, "\nu_{4*}"'] \end{tikzcd} \tag{5.7}$$

commutes.

Remark 5.3.15. There is a detail to check here that we have omitted: for our map to factor, we need to know that $\overline{\sigma}_{4*}$ kills the everything in the kernel of μ_{4*}. To check this, we would need to know explicitly what the map in Theorem 5.3.13 is, and describing it here would take us too far afield. However, this detail can be easily checked once this map is written out. See [MM79, p.89].

By Theorem 5.3.14, $\nu_{4*} \in \mathrm{Hom}(KO[\frac{1}{2}]_0(G/PL[\frac{1}{2}]), \mathbb{Z}[\frac{1}{2}])$ lifts to an element $\widetilde{\nu} \in KO[\frac{1}{2}]^0(G/PL[\frac{1}{2}])$, which we treat as an element of the reduced cohomology theory $\widetilde{KO}[\frac{1}{2}]^0(G/PL[\frac{1}{2}])$. This may be interpreted as a map $\widetilde{\nu}\colon G/PL[\frac{1}{2}] \to BO[\frac{1}{2}]$, since the spectrum $KO[\frac{1}{2}]$ has a copy of $BO[\frac{1}{2}] \times \mathbb{Z}[\frac{1}{2}]$ in each dimension divisible by four. We claim this to be a homotopy equivalence, which we show by proving that it induces isomorphisms on π_{4k} for all k. This is essentially by construction; we just have to unwind all of our maps.

We first write out the pairing between $\widetilde{KO}[\frac{1}{2}]^0$ and $\widetilde{KO}[\frac{1}{2}]_0$. Let $\gamma\colon X \to BO[\frac{1}{2}]$ represent an element of $\widetilde{KO}[\frac{1}{2}]^0(X)$, and let $\delta\colon S^{4n} \to X \wedge BO[\frac{1}{2}]$ represent an element of $\widetilde{KO}[\frac{1}{2}]_0(X)$. We have the composition of maps

$$S^{4n} \xrightarrow{\delta} X \wedge BO[\tfrac{1}{2}] \xrightarrow{\gamma\wedge Id} BO[\tfrac{1}{2}] \wedge BO[\tfrac{1}{2}] \xrightarrow{\otimes} BO[\tfrac{1}{2}]\,, \tag{5.8}$$

where the map labeled "$\otimes$" is induced by taking the tensor product of vector bundles. This gives us some element of $\pi_{4n}(BO[\frac{1}{2}]) \cong \mathbb{Z}[\frac{1}{2}]$, which gives us the desired pairing.

Combining Equations (5.7) and (5.8), we obtain the diagram

$$\begin{tikzcd} \pi_{4n}(G/PL[\frac{1}{2}]) \arrow[r, "\widetilde{\nu}_*"] \arrow[d] \arrow[ddr, "\sigma"] & \pi_{4n}(BO[\frac{1}{2}]) = \mathbb{Z}[\frac{1}{2}] \arrow[d] \arrow[dd, bend left=60, "Id"] \\ \pi_{4n}(G/PL[\frac{1}{2}] \wedge BO[\frac{1}{2}]) \arrow[r, "(\widetilde{\nu}\wedge Id)_*"] \arrow[dr, "\nu_{4n}"'] & \pi_{4n}(BO[\frac{1}{2}] \wedge BO[\frac{1}{2}]) \arrow[d, "\otimes"] \\ & \pi_{4n}(BO[\frac{1}{2}]) = \mathbb{Z}[\frac{1}{2}] \end{tikzcd} .$$

This shows that $\widetilde{\nu}_*\colon \mathbb{Z}[\frac{1}{2}] = \pi_{4n}(G/PL[\frac{1}{2}]) \to \pi_{4n}(BO[\frac{1}{2}]) = \mathbb{Z}[\frac{1}{2}]$ is given by the surgery obstruction σ. The surgery obstruction is the identity map from $\mathbb{Z}[\frac{1}{2}]$ to itself in dimensions $4k$, $k > 1$, and it is multiplication by 2 in dimension 4. In both cases, this is an isomorphism, which completes the proof. □

5.3.4 Combining Even and Odd

It is clear from the homotopy type of $G/PL[2]$ that if we invert all primes, the rational homotopy type of G/PL is simply

$$G/PL_{\mathbb{Q}} = \prod_{j\geq 1} K(\mathbb{Q}, 4j).$$

We thus obtain from Theorem 5.3.2:

Theorem 5.3.16. *The integral homotopy type of G/PL fits into the diagram*

$$\begin{tikzcd} G/PL \arrow[r] \arrow[d] & G/PL[2] \arrow[d] \\ BO[\frac{1}{2}] \arrow[r] & \prod_{j\geq 1} K(\mathbb{Q}, 4j) \end{tikzcd} .$$

From what we have done, the maps in this diagram are not completely obvious. In fact, they aren't even canonically determined, since we didn't canonically specify the lifts we obtained using the universal coefficient theorems. This can be remedied, but only with a lot of work. See [MM79] for a discussion of the literature available on how this may be done. We also note that these are all maps of topological spaces, but not necessarily of H-spaces. To get the H-space structure, one must again work a little bit harder; see [MM79]. However, what we have is sufficient for many calculations.

One particular fact that allows us to calculate effectively is the following. It follows immediately from the arguments we have given and the fact that $[X, G/PL[2]]$ and $\widetilde{KO}(X)$ have the same nontorsion components that and their images match up in $\prod_{j\geq 1} K(\mathbb{Q}, 4j)$.

Proposition 5.3.17. *Since G/PL is an H-space, $[X, G/PL]$ is a finitely generated abelian group when X is a finite CW complex. It is therefore a direct sum $\mathbb{Z}^k \oplus$* torsion. *The free part, $\mathbb{Z}^k$, is isomorphic to the direct sum of the free parts of $H^{4*}(X, \mathbb{Z})$.*

5.4 G/TOP

As a slight exercise in self-indulgence, we state a few results about the homotopy type of G/TOP. We recall that the nonexistence of a degree one normal map into S^4 with surgery obstruction 1 was what caused the nonzero k-invariant in the 2-local homotopy type of G/PL. Freedman proved, however, that there exists a topological degree one normal map with surgery obstruction 1. (This is easily seen to be equivalent to proving the existence of a spin 4-manifold with signature 8, which is the form in which Freedman originally proved this result. See [Fre82]). This causes the k-invariant to vanish, and we obtain the nicer result:

$$G/TOP[2] \simeq \prod_{i>0} K(\mathbb{Z}/2\mathbb{Z}, 4i-2) \times K(\mathbb{Z}[2], 4i).$$

G/PL and G/TOP have the same homotopy groups. There is a map $G/PL \to G/TOP$, and it induces isomorphisms on π_i, $i \neq 4$, and multiplication by two on π_4. This is, in essence, the only difference between the two categories. A stronger manifestation of this is the fact that there is a fibration

$$K(\mathbb{Z}/2\mathbb{Z}, 3) \to BPL \to BTOP,$$

so that

$$\mathrm{TOP/PL} \simeq K(Z/2\mathbb{Z}, 3).$$

This additional $\mathbb{Z}/2\mathbb{Z}$ in fact makes surgery theory work a little bit more nicely in the topological category, as is illustrated by the nicer for of the two-local homotopy type of G/TOP and by the work in [KS77].

Chapter 6

Examples

We may now reap the benefits of the machinery that we have developed by providing several simple examples and applications of this theory. Due to space constraints[1], we limit our examples to simple, fun applications of the theory. For a review of deeper applications, see [Ros00] or [Wal70].

Because it is covered well in several other places, we omit the classification of exotic spheres, even though it historically was the motivating example for surgery theory. We do, however, sketch the proof of the following weaker, yet still interesting, result:

Theorem 6.1.1 (Kervaire and Milnor). *If $n \geq 5$, there are finitely many smooth structures on S^n.*

Proof. We begin by (coarsely) analyzing $\pi_n(G/O)$. The group $\pi_n(G/O)$ fits into the exact sequence

$$\pi_n(O) \xrightarrow{J} \pi_n(G) \longrightarrow \pi_n(G/O) \longrightarrow \pi_{n-1}(O) \ . \tag{6.1}$$

$G(k)$ is the total space in the fibration

$$\Omega^{k-1}S^{k-1} \longrightarrow G(k) \xrightarrow{e} S^{k-1} \ ,$$

where e is the map that evaluates a given element of $G(k)$ (which is a homotopy equivalence $S^{k-1} \to S^{k-1}$) at a given fixed basepoint. It follows from examining the long exact sequence on homotopy groups arising from this fibration and taking limits that $\pi_n(G) = \pi_n^S$, where π_n^S is the n^{th} stable homotopy group of spheres. By a result of Serre, π_n^S is finite, so $\pi_n(G)$ is finite as well. (See [Hat] for a proof of this result.)

By a theorem of Bott, $\pi_n(O)$ is finite unless $n \equiv 3 \bmod 4$, in which case $\pi_n(O) = \mathbb{Z}$. Combining this with equation (6.1) and the finiteness of $\pi_n(G)$, we see that $\pi_n(G/O)$ is finite unless n is congruent to zero modulo four, in which case $\pi_n(G/O)$ consists of the direct sum of $\mathbb{Z}$ with some finite group.

We now plug this information into the surgery exact sequence, which, for S^n, reads:

$$L_{n+1}(\mathbb{Z}) \longrightarrow \mathcal{S}(S^n) \longrightarrow \pi_n(G/O) \longrightarrow L_n(\mathbb{Z}) \ .$$

The structure set of S^n is a group under connected sum, and it is not difficult to see that our sequence is now actually an exact sequence of groups.

[1] I suppose the fact that I have shamelessly exceeded the department's recommended page count removes some of the moral force from this concession.

We now have four cases, which we examine in increasing order of difficulty. Throughout that which follows, let A denote a (variable) finite abelian group.

If $n \equiv 1 (\mathrm{mod}\, 4)$, the surgery exact sequence is:

$$\mathbb{Z}/2\mathbb{Z} \longrightarrow \mathcal{S}(S^n) \longrightarrow A \longrightarrow 0\,.$$

Since $\mathcal{S}(S^n)$ lies between two finite groups, it is finite.

If $n \equiv 2 (\mathrm{mod}\, 4)$, we have

$$0 \longrightarrow \mathcal{S}(S^n) \longrightarrow A \longrightarrow \mathbb{Z}/2\mathbb{Z}\,.$$

$\mathcal{S}(S^n)$ again lies between two finite groups, so it is again finite.

If $n \equiv 0 (\mathrm{mod}\, 4)$, the sequence is

$$0 \longrightarrow \mathcal{S}(S^n) \longrightarrow \mathbb{Z} \oplus A \xrightarrow{\sigma} \mathbb{Z}\,.$$

By Milnor's plumbing theorem (Theorem 4.5.2), there is a normal map $(M, \partial M) \to (D^n, S^{n-1})$ with surgery obstruction 1. By coning off its boundary, we can construct a normal map in $\pi_n(G/O) = \mathbb{Z} \oplus A$ with surgery obstruction 1. This guarantees that the kernel of σ is finite, so $\mathcal{S}(S^n)$ is finite as well.

If $m \equiv 3 (\mathrm{mod}\, 4)$, the sequence reads

$$\mathbb{Z} \xrightarrow{\omega} \mathcal{S}(S^n) \longrightarrow A \xrightarrow{\sigma} 0\,.$$

We need to show that the image of ω is finite, i.e., that it takes some nonzero element of $\mathbb{Z}$ to the standard smooth structure on the sphere. We omit this step, and simply cite a paper of Kervaire and Milnor in which it is explicitly shown that this occurs [KM58]. □

For a much more precise classification of exotic spheres, see [KM63].

Our next example is just a cute result that follows with essentially no work from our theory.

Proposition 6.1.2. *Let X be a simply connected 5-dimensional Poincaré complex. There are finitely many piecewise linear structures on X.*

Proof. By Poincaré duality, $H^4(X) \cong H_1(X) = 0$, by assumption. By our calculations of the local homotopy types of G/PL, this implies that $[X, G/PL[2]]$ and $[X, G/PL[\frac{1}{2}]] = [X, BO[\frac{1}{2}]]$ are both finite, so $[X, G/PL]$ is finite as well by Theorem 5.3.2. $L_6(\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ is obviously finite, so our desired result follows from the surgery exact sequence

$$\mathbb{Z}/2\mathbb{Z} \longrightarrow \mathcal{S}_{\mathrm{PL}}(X) \longrightarrow [X, G/PL] \longrightarrow 0\,.$$

□

Our last several examples deal with the application of our theory to products of spaces with spheres of dimension greater than or equal to two.

Proposition 6.1.3. *Let $f \times \mathrm{Id}\colon M^m \times S^n \to X \times S^n$ be a normal map of simply connected spaces, and suppose that $n \geq 2$ and $m + n \geq 5$. Then the surgery obstruction $\sigma(f \times \mathrm{Id}) = 0$.*

Proof. The map $f \times \mathrm{Id}$ is the restriction to the boundary of the normal map $f \times \mathrm{Id}\colon M^m \times D^{n+1} \to X \times D^{n+1}$. It therefore has surgery obstruction zero by Theorem 4.4.16. □

Remark 6.1.4. The restriction $n \geq 2$ is to preserve simple connectivity.

This proposition has several interesting corollaries.

Corollary 6.1.5. *If a simply connected Poincaré complex X of dimension greater than or equal to five admits a normal map, then $X \times S^k$, $k \geq 2$, is homotopy equivalent to a manifold.*

Proof. If $f\colon M \to X$ is a normal map, we have shown that $\sigma(f \times \mathrm{Id}) = 0$, so $f \times \mathrm{Id}$ is normally bordant to a homotopy equivalence. $\square$

Corollary 6.1.6. *Let $f\colon M \to N$ be a degree one map of simply connected manifolds of dimension greater than or equal to five such that the stable normal bundles obey $f^*(N_N) = N_M$. Then $M \times S^k$ is diffeomorphic to $N \times S^k$ for all $k \geq 2$.*

Proof. Since f pulls back the normal bundle of N to that of M, f is normally bordant to $\mathrm{Id}\colon N \to N$ by Theorem 3.4.7. As such, $f \times \mathrm{Id} : M \times S^k \to N \times S^k$ is normally bordant to $\mathrm{Id} \times \mathrm{Id}\colon N \times S^k \to N \times S^k$. But $\sigma(f \times \mathrm{Id}) = \sigma(\mathrm{Id} \times \mathrm{Id}) = 0$, so the two maps are h-bordant by the surgery exact sequence. The h-cobordism theorem now implies that $M \times S^k$ and $N \times S^k$ are diffeomorphic, as desired. $\square$

Remark 6.1.7. Proposition 6.1.3 and Corollaries 6.1.5 and 6.1.6 hold in the PL category by the same arguments.

Corollary 6.1.8. *There are multiple PL structures on $S^n \times S^m$ whenever $n + m \geq 5$ and either m or n is even.*

At first blush, this is somewhat remarkable, since the generalized Poincaré conjecture tells us that there is only one PL structure on S^n. In a sense, this is therefore the simplest example of a homotopy type with multiple PL (and, in fact, topological) structures.

Proof of Corollary 6.1.8. The normal structure set is given by

$$\mathcal{N}(S^n \times S^m) = [S^n \times S^m, G/PL].$$

G/PL is an infinite loop space, so maps into it (and its deloopings) form a generalized cohomology theory. Using the Künneth formula for generalized cohomology theories, we obtain

$$[S^n \times S^m, G/PL] = \pi_{n+m}(G/PL) \oplus \pi_n(G/PL) \oplus \pi_m(G/PL).$$

It is not difficult to see that the surgery obstruction is given by projection onto the first factor under the isomorphism $\pi_{n+m}(G/PL) \cong L_{n+m}(\mathbb{Z})$, giving us the isomorphism

$$\mathcal{S}(S^n \times S^m) \cong \pi_n(G/PL) \oplus \pi_m(G/PL) \cong L_n(\mathbb{Z}) \oplus L_m(\mathbb{Z}).$$

We can actually realize this map explicitly. By the Thom transversality theorem (Theorem 2.1.4), we can make f transverse to $S^n \subseteq S^n \times S^m$, and we set the image of f in $L_n(\mathbb{Z})$ to be the surgery obstruction of the map $f|_{f^{-1}(S^n)}$. We perform the analogous construction for the other direct summand, thereby giving us the asserted map.

However, we can show the surjectivity of $\mathcal{S}(S^n \times S^m) \to \pi_n(G/PL)$, and thus the nontriviality of $\mathcal{S}(S^n \times S^m)$, without resorting to the explicit assertions above. Indeed, we simply note that the map $f \times \mathrm{Id}$ has surgery obstruction zero by Proposition 6.1.3, from which the surjectivity follows immediately by the surgery exact sequence. $\square$

We note that the proof of surjectivity given in the last paragraph allows us to be a little bit more explicit in our description of these PL manifolds that are homotopy equivalent but not PL-isomorphic to $S^n \times S^m$, at least when $m \geq 5$. To construct one, start with a normal map of manifolds with boundary $(M, \partial M) \to (D^m, S^{m-1})$ with a given nonzero surgery obstruction, which exists by Milnor's plumbing theorem (Theorem 4.5.2). Now cone off the boundary to create a closed manifold N with the same (nonzero) signature, equipped with a normal map g to S^m. Since it has nonzero signature, N clearly is not homotopy equivalent to S^m. However, $g \times \mathrm{Id}\colon N \times S^n \to S^m \times S^n$ has surgery obstruction zero, so it is normally cobordant to a homotopy equivalence. The manifold homotopy equivalent to $S^n \times S^m$ that this produces is our desired manifold.

Bibliography

[Ada74] J.F. Adams. *Stable Homotopy and Generalized Homology.* Chicago University Press, Chicago, 1974.

[Ada78] J.F. Adams. *Infinite Loop Spaces*, volume 90 of *Annals of Mathematics Studies.* Princeton University Press, Princeton, New Jersey, 1978.

[Bro72] William Browder. *Surgery on Simply-Connected Manifolds.* Springer-Verlag, New York, 1972.

[Bro00] William Browder. Differential topology of higher dimensional manifolds. In Sylvain Cappell, Andrew Ranicki, and Jonathan Rosenberg, editors, *Surveys on Surgery Theory*, volume 1, pages 41–73. Princeton University Press, Princeton, New Jersey, 2000.

[BV73] J.M. Boardman and R.M. Vogt. *Homotopy Invariant Algebraic Structures on Topological Spaces*, volume 347 of *Lecture Notes in Mathematics.* Springer-Verlag, New York, 1973.

[Cas96] A.J. Casson. Generalisations and applications of block bundles. In *The Hauptvermutung Book*, number 1 in K-Monographs in Mathematics. Kluwer, 1996. Online at www.maths.ed.ac.uk/people/aar/books/haupt.pdf.

[CF66] P.E. Conner and E.E. Floyd. *The Relation of Cobordism to K-Theories.* Number 28 in Lecture Notes in Mathematics. Springer-Verlag, New York, 1966.

[CRR00a] Sylvain Cappell, Andrew Ranicki, and Jonathan Rosenberg, editors. *Surveys on Surgery Theory*, volume 1. Princeton University Press, Princeton, New Jersey, 2000.

[CRR00b] Sylvain Cappell, Andrew Ranicki, and Jonathan Rosenberg, editors. *Surveys on Surgery Theory*, volume 2. Princeton University Press, Princeton, New Jersey, 2000.

[Fre82] M.H. Freedman. The topology of four-dimensional manifolds. *Journal of Differential Geometry*, 17(3):357–453, 1982.

[Hat] Allen Hatcher. Spectral sequences in algebraic topology. Available online at www.math.cornell.edu/~hatcher.

[Hat02] Allen Hatcher. *Algebraic Topology.* Cambridge University Press, Cambridge, 2002. Online at www.math.cornell.edu/~hatcher.

[Kah63] D.W. Kahn. Induced maps for Postnikov systems. *Transactions of the American Mathematics Society*, 107:432–450, 1963.

[KM58] M. Kervaire and J. Milnor. Bernoulli numbers, homotopy groups, and a theorem of Rohlin. In *Proceedings of the International Congress of Mathematics*, Edinborough, 1958.

[KM63] M. Kervaire and J. Milnor. Groups of homotopy spheres: I. *The Annals of Mathematics*, 77(3):504–537, May 1963.

[Kos93] Antoni A. Kosinski. *Differential Manifolds.* Academic Press, Inc., New York, 1993.

[KS77] Robion C. Kirby and Laurence C. Siebenmann. *Foundational Essays on Topological Manifolds, Smoothings, and Triangulations.* Number 88 in Annals of Mathematics Studies. Princeton University Press, Princeton, New Jersey, 1977.

[Lev83] J. Levine. Lectures on groups of homotopy spheres. In *Algebraic and Geometric Topology Proceedings*, volume 1126 of *Lecture Notes in Mathematics*, pages 62–95, Rutgers, 1983. Springer-Verlag.

[Mil56] J. Milnor. On manifolds homeomorphic to the 7-sphere. *The Annals of Mathematics*, 64(2):399–405, September 1956.

[Mil58] J. Milnor. On simply connected 4-manifolds. In *Symp. Topologia Algebraica*, pages 122–128, Mexico, 1958.

[Mil61] J. Milnor. A procedure for killing the homotopy groups of differentiable manifolds. *Symposia in Pure Mathematics, American Mathematical Society*, 3:39–55, 1961.

[Mil63] J. Milnor. *Morse Theory.* Number 51 in Annals of Mathematics Studies. Princeton University Press, Princeton, New Jersey, 1963.

[Mil65] J. Milnor. *Lectures on the h-Cobordism Theorem.* Princeton University Press, Princeton, New Jersey, 1965.

[Mil00] J. Milnor. Classification of $(n-1)$-connected $2n$-dimensional manifolds and the discovery of exotic spheres. In Sylvain Cappell, Andrew Ranicki, and Jonathan Rosenberg, editors, *Surveys on Surgery Theory*, volume 1, pages 17–25. Princeton University Press, Princeton, New Jersey, 2000.

[MM79] Ib Madsen and R. James Milgram. *The Classifying Spaces for Surgery and Cobordism of Manifolds.* Number 92 in Annals of Mathematics Studies. Princeton University Press, Princeton, New Jersey, 1979.

[MS74] J. Milnor and J. Stasheff. *Characteristic Classes.* Number 76 in Annals of Mathematics Studies. Princeton University Press, Princeton, New Jersey, 1974.

[Nov64] S.P. Novikov. Homotopy equivalent smooth manifolds. *Dokl. Akad. Nauk SSSR*, 28:365–474, 1964. English Translation: *Translations of the American Mathematical Society* 48, 271-396(1965).

[Nov65] S.P. Novikov. Topological invariance of rational Pontrjagin classes. *Dokl. Akad. Nauk SSSR*, 28:298–300, 1965. English Translation: *Soviet Math Doklady* 6, 921-923.

[Ran02] Andrew Ranicki. *Algebraic and Geometric Surgery.* Oxford Mathematical Monographs. Oxford University Press, 2002. (To appear).

[RCS+96] A.A. Ranicki, A.J. Casson, D.P. Sullivan, M.A. Armstrong, C.P. Rourke, and G.E. Cooke. *The Hauptvermutung Book.* Number 1 in K-Monographs in Mathematics. Kluwer, 1996. Online at www.maths.ed.ac.uk/people/aar/books/haupt.pdf.

[Roc86] V.A. Rochlin. Quatre articles de V.A. Rohlin, IV. In L. Guillou and A. Marin, editors, *À la Recherche de la Topologie Perdue.* Birkhauser, Boston, 1986.

[Ros00] Jonathan Rosenberg. Surgery theory today—what it is and where it's going. In Sylvain Cappell, Andrew Ranicki, and Jonathan Rosenberg, editors, *Surveys on Surgery Theory*, volume 2, pages 3–49. Princeton University Press, Princeton, New Jersey, 2000.

[RS72] C.P. Rourke and B.J. Sanderson. *Introduction to Piecewise-Linear Topology.* Springer-Verlag, New York, 1972.

[Sel91] Paul Selick. *Introduction to Homotopy Theory.* Number 9 in Fields Institute Monographs. American Mathematical Society, Providence, Rhode Island, 1991.

[Sma61] Stephen Smale. Generalized Poincaré's conjecture in dimensions greater than four. *The Annals of Mathematics*, 74(2):391–406, September 1961.

[Spa66] Edwin H. Spanier. *Algebraic Topology.* Springer-Verlag, New York, 1966.

[Sta63] J. Stasheff. A classification theorem for fibrations. *Topology*, 2:239–246, 1963.

[Sto68] Robert E. Stong. *Notes on Cobordism Theory.* Princeton University Press, Princeton, New Jersey, 1968.

[Sul74] Dennis Sullivan. Genetics of homotopy theory and the Adams conjecture. *The Annals of Mathematics*, 100(1):1–79, July 1974.

[Wal70] C.T.C. Wall. *Surgery on Compact Manifolds.* Academic Press, Inc., London, 1970.

[Whi36] H. Whitney. Differentiable manifolds. *The Annals of Mathematics*, 37:647–680, 1936.

[Whi44] H. Whitney. The self-intersections of a smooth n-manifold in $2n$-space. *The Annals of Mathematics*, 45:220–246, 1944.

[Yos75] Z.I. Yosimura. Universal coefficient sequence for cohomology theories of CW spectra. *Osaka Journal of Mathematics*, 12:305–323, 1975.