Very Weak Turbulence for Certain Dispersive Equations

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Dispersive equations

Question: Why certain PDE are called dispersive equations?

Because, these PDE, when globally defined in space, admit solutions that are wave that spread out spatially while maintaining constant mass and energy.

Probably the best well known examples are the Schrödinger and the KdV equations and a large literature has been compiled about the multiple aspects of these equations and their solutions.

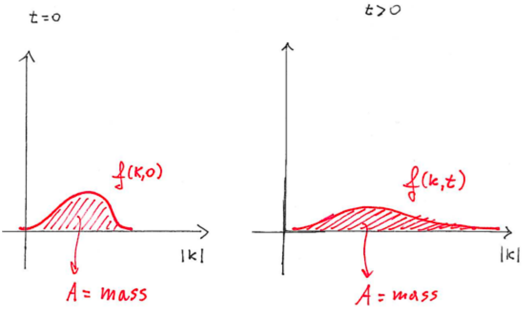
In these two lectures I will consider the situation in which existence, uniqueness, stability of solutions are available globally in time (global well-posedness) and our goal is to investigate if a certain phenomena physically relevant and already studied experimentally or numerically can be proved also mathematically: the Forward Cascade or Weak Turbulence.

Notion of Forward Cascade

Assume that $u(x, t)$ is a smooth wave solution to a certain *nonlinear* dispersive PDE defined for all times t.

How do frequency components of this wave interact in time at different scales due to the presence of nonlinearity?

Consider the function $f(k, t) := |\hat{u}(k, t)|^2$ and its subgraph at times $t = 0$ and $t > 0$



Notion of Weak Turbulence

We know that from conservation of mass and Plancherel's theorem,

$$\int |\hat{u}(k,t)|^2 dk = \text{Constant},$$

that is the subgraph of the function $f(k,t) := |\hat{u}(k,t)|^2$ has a constant volume. But how is its shape?

Expectation: when dispersion is limited by imposing boundary conditions (i.e. periodic case), a *migration* from low frequencies to high ones could happen for certain solutions.

Definition

For us today weak turbulence is the phenomenon of global-in-time solutions shifting toward increasingly high frequencies.

This is the reason why this phenomenon is also called forward cascade.

How do we capture forward cascade?

• How can we capture mathematically a low-to-high frequency cascade or *weak turbulence*?

One way to capture this phenomenon is by analyzing the growth of high Sobolev norms. In fact by using Plancherel's theorem we see that

$$\|u(t)\|_{H^s}^2 = \int |\hat{u}(k,t)|^2 \langle k \rangle^{2s} dk$$

weighs the higher frequencies more as s becomes larger, and hence its growth in time t gives us a quantitative estimate for how much of the support of $\widehat{u}$ has transferred from the low to the high frequencies k.

Weak Turbulence, Scattering & Integrability

Weak turbulence is incompatible with scattering or complete integrability.

- Scattering: In this context scattering (at $+\infty$) means that for any global solution $u(t,x) \in H^s$ there exists $u_0^+ \in H^s$ such that, if $S(t)$ is the linear Schrödinger operator, then

$$\lim_{t\to+\infty} \|u(t,x) - S(t)u_0^+(x)\|_{H^s} = 0.$$

  Since $\|S(t)u_0^+\|_{H^s} = \|u_0^+\|_{H^s}$, it follows that $\|u(t)\|^2_{\dot{H}^s}$ cannot grow.
- Complete Integrability: For example the 1D equation

$$(i\partial_t - \Delta)u = -|u|^2 u$$

  is integrable in the sense that it admits infinitely many conservation laws. Combining them in the right way one gets that $\|u(t)\|^2_{\dot{H}^s} \le C_s$ for all times.

The trivial exponential estimate

For the problems we consider we have a very good local well-posedness theory that allows us to say that for a given initial datum u_0 there exists a constant $C > 1$ and a time constant $\delta > 0$, depending only on the energy of the system (hence on u_0) such that for all t:

$$\|u(t+\delta)\|_{H^s} \leq C\|u(t)\|_{H^s}. \tag{2.1}$$

Iterating (2.1) yields the exponential bound:

$$\|u(t)\|_{H^s} \leq C_1 e^{C_2|t|}. \tag{2.2}$$

Here, $C_1, C_2 > 0$ again depend only on u_0.

From exponential to polynomial bounds

The first significant improvement over the exponential (trivial) bound is due to Bourgain. The key estimate is to improve the local bound in (2.1) to:

$$\|u(t+\delta)\|_{H^s} \leq \|u(t)\|_{H^s} + C\|u(t)\|_{H^s}^{1-r}. \tag{2.3}$$

As before, $C, \tau_0 > 0$ depend only on u_0 and $r \in (0,1)$ and usually satisfies $r \sim \frac{1}{s}$. One can show then that (2.3) implies that for all $t \in \mathbb{R}$:

$$\|u(t)\|_{H^s} \leq C(u_0)(1+|t|)^{\frac{1}{r}}.$$

How to obtain the improved local estimate

1. **Bourgain**: used the *Fourier multiplier method*, together with the *WKB method* from semiclassical analysis.
2. **Colliander, Delort, Kenig and S.** and **S.**: used multilinear estimates in an $X^{s,b}$-space with negative first index.
3. **Catoire and W. Wang** and **Zhong**: analyzed the local estimate in the context of compact Riemannian manifolds following the analysis in the work of Burq, Gérard, and Tzvetkov.
4. **Sohinger**: used the *upside-down I-method*,
5. **Collinder, Kwon and Oh** : combined the upside-down I-method with normal for reduction.

A linear equation with potential

In the case of the linear Schrödinger equation with potential on $\mathbb{T}^d$:

$$iu_t + \Delta u = Vu. \tag{2.4}$$

better results are known.

1. **Bourgain**: Assume $d = 1, 2$, smooth V with uniformly bounded partial derivatives. Then for all $\epsilon > 0$ and all $t \in \mathbb{R}$:

$$\|u(t)\|_{H^s} \lesssim_{s,u_0,\epsilon} (1 + |t|)^\epsilon \tag{2.5}$$

   The proof of (2.5) is based on separation properties of the eigenvalues of the Laplace operator on $\mathbb{T}^d$.

2. **W. Wang**: She improved the bound from $(1 + |t|)^\epsilon$ to $\log t$.
3. **Delort**: Proved (2.5) for any d-dimensional torus, and for the linear Schrödinger equation on any Zoll manifold, i.e. on any compact manifold whose geodesic flow is periodic.

Open Problems

The results above listed do not complete the whole picture. For example one would like to prove

$$\|u(t)\|_{H^s} \lesssim_{s,u_0,\epsilon} (1+|t|)^{\epsilon}$$

• for the linear Schrödinger equation with potential in $\mathbb{R}^d$ when scattering is not available.

• for some nonlinear dispersive equations on $\mathbb{T}^d$ or in any other manifold that prevents scattering.

• Can one exhibit a solution for either NLS or KdV which Sobolev norms grow at least as $\log t$?

Can one show growth of Sobolev norms?

About the last open problem listed one should recall the following result of **Bourgain**:

Theorem

Given $m, s \gg 1$ *there exist* $\tilde{\Delta}$ *and a global solution* $u(x,t)$ *to the modified wave equation*

$$(\partial_{tt} - \tilde{\Delta})u = u^p$$

such that

$$\|u(t)\|_{H^s} \sim |t|^m.$$

The weakness of this result is in the fact that one needs to modify the equation in order to make a solution exhibit a cascade.

More references

Recently Gerard and Grellier obtained some growth results for Sobolev norms of solutions to the periodic 1D cubic Szegö equation:

$$i\partial_t u = \Pi(|u|^2 u),$$

where $\Pi(\sum_k \hat{f}(k)e^{xk}) = \sum_{k>0} \hat{f}(k)e^{xk}$ is the Szegö projector.

- Physics: Weak turbulence theory due to Hasselmann and Zakharov.
- Numerics (d=1): Majda-McLaughlin-Tabak; Zakharov et. al.
- Probability: Benney and Newell, Benney and Saffman.

To show how far we are from actually solving the open problems proposed above I will present what is known so far for the 2D cubic defocusing NLS in $\mathbb{T}^2$.

The 2D cubic NLS Initial Value Problem in $\mathbb{T}^2$

We consider the defocusing initial value problem:

(4.1) $$\begin{cases} (-i\partial_t + \Delta)u = |u|^2 u \\ u(0,x) = u_0(x), \ \text{where} \ x \in \mathbb{T}^2. \end{cases}$$

We have (see Bourgain)

Theorem (Global well-posedness for smooth data)

For any data $u_0 \in H^s(\mathbb{T}^2)$, $s \geq 1$ there exists a unique global solution $u(x,t) \in C(\mathbb{R}, H^s)$ to the Cauchy problem (4.1).

We also have

$$\text{Mass} = M(u) = \|u(t)\|^2 = M(0)$$

$$\text{Energy} = E(u) = \int (\frac{1}{2}|\nabla u(t,x)|^2 + \frac{1}{4}|u(x,t)|^4)\,dx = E(0).$$

Two Theorems

Consider again the IVP

$$(4.2) \qquad \begin{cases} (-i\partial_t + \Delta)u = |u|^2 u \\ u(0,x) = u_0(x), \text{ where } x \in \mathbb{T}^2, \end{cases}$$

Theorem (Bourgain, Zhong, Sohinger)

For the smooth global solutions of the periodic IVP (4.2) *we have:*

$$\|u(t)\|_{\dot{H}^s} \leq C_s |t|^{s+}.$$

Theorem (Colliander-Keel-S.-Takaoka-Tao)

Let $s > 1$, $K \gg 1$ and $0 < \sigma < 1$ be given. Then there exist a global smooth solution $u(x,t)$ to the IVP (4.2) *and $T > 0$ such that*

$$\|u_0\|_{H^s} \leq \sigma \quad \text{and} \quad \|u(T)\|^2_{\dot{H}^s} \geq K.$$

On the proof of Theorem 1

Here I will propose a recent proof given by Sohinger (Ph.D. Thesis 2011). In this approach the iteration bound comes from an *almost conservation law*, which is reminiscent of the work of Colliander-Keel-S.-Takaoka-Tao (I-Team). In other words, given a frequency threshold N, one can construct a “energy” $\tilde{E}(u)$, which is related to $\|u(t)\|_{H^s}$, and can find $\delta > 0$, depending only on the initial data such that, for some $\alpha > 0$ and all $t \in \mathbb{R}$:

$$\tilde{E}(u(t+\delta)) \leq C\big(1 + \frac{1}{N^\alpha}\big)\tilde{E}(u(t)). \tag{5.1}$$

This type of iteration bound can be iterated $O(N^\alpha)$ times without obtaining exponential growth. We note that this method doesn’t require s to be a positive integer (needed by Bourgain and Zhong).

Upside-down I-operator

We construct an *Upside-down I-operator*. This operator is defined as a Fourier multiplier operator.

Suppose $N \geq 1$ is given. Let $\theta : \mathbb{Z}^2 \to \mathbb{R}$ be given by:

$$\theta(n) := \begin{cases} \left(\frac{|n|}{N}\right)^s, \text{if } |n| \geq N \\ 1, \text{ if } |n| \leq N \end{cases}$$

Then, if $f : \mathbb{T}^2 \to \mathbb{C}$, we define $\mathcal{D}f$ by:

$$\widehat{\mathcal{D}f}(n) := \theta(n)\hat{f}(n).$$

We observe that:

$$\|\mathcal{D}f\|_{L^2} \lesssim_s \|f\|_{H^s} \lesssim_s N^s\|\mathcal{D}f\|_{L^2}.$$

Our goal is to estimate $\|\mathcal{D}u(t)\|_{L^2}$, from which we can then estimate $\|u(t)\|_{H^s}$.

Good Local estimates

We first define the space $X^{s,b}$ as:

$$f(x,t) \in X^{s,b} \quad \text{iff} \quad \int \sum_k |\hat{f}(k,\tau)|^2 \langle k \rangle^{2s} \langle \tau - |k|^2 \rangle^{2b} \, d\tau < \infty.$$

Theorem

There exist $\delta = \delta(s, E(u_0), M(u_0))$, $C = C(s, E(u_0), M(u_0)) > 0$, *which are continuous in energy and mass, such that for all* $t_0 \in \mathbb{R}$, *there exists a globally defined function* $v : \mathbb{T}^2 \times \mathbb{R} \to \mathbb{C}$ *such that:*

$$|v|_{[t_0, t_0+\delta]} = |u|_{[t_0, t_0+\delta]}.$$

$$\|v\|_{X^{1,\frac{1}{2}+}} \leq C(s, E(u_0), M(u_0))$$

$$\|\mathcal{D}v\|_{X^{0,\frac{1}{2}+}} \leq C(s, E(u_0), M(u_0)) \|\mathcal{D}u(t_0)\|_{L^2}.$$

Definition of E^1

We then define the *modified energy:*

$$E^1(u(t)) := \|\mathcal{D}u(t)\|_{L^2}^2.$$

Differentiating in time, and using an appropriate symmetrization, we obtain that for some $c \in \mathbb{R}$, one has:

$$\frac{d}{dt}E^1(u(t)) = ic \sum_{n_1+n_2+n_3+n_4=0} \left(\theta^2(n_1) - \theta^2(n_2) + \theta^2(n_3) - \theta^2(n_4)\right)$$

$$\times \widehat{u}(n_1)\widehat{\bar{u}}(n_2)\widehat{u}(n_3)\widehat{\bar{u}}(n_4).$$

Definition of E^2

We now consider the *higher modified energy*, by adding an appropriate *quadrilinear correction* to E^1:

$$E^2(u) := E^1(u) + \lambda_4(M_4; u).$$

Some notation: Given k, an even integer, The quantity M_k is taken to be a function on the hyperplane

$$\Gamma_k := \{(n_1, \dots, n_k) \in (\mathbb{Z}^2)^k,\ n_1 + \cdots + n_k = 0\},$$

and:

$$\lambda_k(M_k; u) := \sum_{n_1+\cdots+n_k=0} M_k(n_1, \dots, n_k)\widehat{u}(n_1)\widehat{\bar{u}}(n_2)\cdots\widehat{\bar{u}}(n_k).$$

Reason: We are adding the multilinear correction to cancel the quadrilinear contributions from $\frac{d}{dt}E^1(u(t))$ and "replace" it with a new term with the same order of derivatives, but more factors of u to distribute these derivatives better. Hence, we expect $E^2(u(t))$ to *vary more slowly* than $E^1(u(t))$.

We denote $n_{ij} := n_i + n_j, n_{ijk} := n_i + n_j + n_k$. If we fix a multiplier M_4, we obtain:

$$\frac{d}{dt}\lambda_4(M_4; u) = -i\lambda_4(M_4(|n_1|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2); u)$$

$$-i \sum_{n_1+n_2+n_3+n_4+n_5+n_6=0} \Big[M_4(n_{123}, n_4, n_5, n_6) - M_4(n_1, n_{234}, n_5, n_6)+$$

$$+M_4(n_1, n_2, n_{345}, n_6) - M_4(n_1, n_2, n_3, n_{456})\Big]$$

$$\times\widehat{u}(n_1)\widehat{\bar{u}}(n_2)\widehat{u}(n_3)\widehat{\bar{u}}(n_4)\widehat{u}(n_5)\widehat{\bar{u}}(n_6).$$

The choice of M_4

To cancel the forth linear term in $\frac{d}{dt}E^1(u)$ we would like to take

$$M_4(n_1, n_2, n_3, n_4) := C\frac{(\theta^2(n_1) - \theta^2(n_2) + \theta^2(n_3) - \theta^2(n_4))}{|n_1|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2}$$

but we have to make sure that this expression is well defined. There is the problem of *small denominators*

$$|n_1|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2$$

which in fact become zero in the resonant set of four wave interaction.

For $(n_1, n_2, n_3, n_4) \in \Gamma_4$, one has:

$$|n_1|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2 = 2n_{12} \cdot n_{14}.$$

This quantity vanishes not only when $n_{12} = n_{14} = 0$, but also when n_{12} and n_{14} are orthogonal. Hence, on Γ_4, it is possible for

$$|n_1|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2 = 0$$

but

$$\theta^2(n_1) - \theta^2(n_2) + \theta^2(n_3) - \theta^2(n_4) \neq 0,$$

hence our first choice for M_4 is not suitable in our 2D case!

The fix

We remedy this by canceling the *non-resonant part* of the quadrilinear term. A similar technique was used in work of the I-Team. More precisely, given $\beta_0 \ll 1$, which we determine later, we decompose:

$$\Gamma_4 = \Omega_{nr} \sqcup \Omega_r.$$

Here, the set Ω_{nr} of *non-resonant* frequencies is defined by:

$$\Omega_{nr} := \{(n_1, n_2, n_3, n_4) \in \Gamma_4; n_{12}, n_{14} \neq 0, |cos\angle(n_{12}, n_{14})| > \beta_0\}$$

and the set Ω_r of *resonant* frequencies is defined to be its complement in Γ_4. In the sequel, we choose:

$$\beta_0 \sim \frac{1}{N}.$$

The final choice of M_4

We now define the multiplier M_4 by:

$$M_4(n_1, n_2, n_3, n_4) := \begin{cases} c\frac{(\theta^2(n_1)-\theta^2(n_2)+\theta^2(n_3)-\theta^2(n_4))}{|n_1|^2-|n_2|^2+|n_3|^2-|n_4|^2} \text{ in } \Omega_{nr} \\ 0 \text{ in } \Omega_r \end{cases}$$

Let us now define the multiplier M_6 on Γ_6 by:

$$\begin{aligned} M_6(n_1, n_2, n_3, n_4, n_5, n_6) &:= M_4(n_{123}, n_4, n_5, n_6) - M_4(n_1, n_{234}, n_5, n_6) \\ &+ M_4(n_1, n_2, n_{345}, n_6) - M_4(n_1, n_2, n_3, n_{456}) \end{aligned}$$

We obtain:

$$\frac{d}{dt}E^2(u) =$$

$$\sum_{\Omega_r} \left(\theta^2(n_1) - \theta^2(n_2) + \theta^2(n_3) - \theta^2(n_4)\right)\widehat{u}(n_1)\widehat{\bar{u}}(n_2)\widehat{u}(n_3)\widehat{\bar{u}}(n_4) +$$

$$+ \sum_{n_1+\ldots+n_6=0} M_6(n_1, \ldots, n_6)\widehat{u}(n_1)\widehat{\bar{u}}(n_2)\widehat{u}(n_3)\widehat{\bar{u}}(n_4)\widehat{u}(n_5)\widehat{\bar{u}}(n_6)$$

It is crucial to prove pointwise bounds on the multiplier M_4. We dyadically localize the frequencies, i.e, we find dyadic integers N_j s.t. $|n_j| \sim N_j$. We then order the N_j's to obtain:

$$N_1^* \geq N_2^* \geq N_3^* \geq N_4^*.$$

The bound we prove is:

Bound on M_4

Lemma (Pointwise bounds on M_4)

With notation as above,

$$M_4 \sim \frac{N}{(N_1^*)^2}\theta(N_1^*)\theta(N_2^*).$$

This bound allows us to deduce for example the equivalence of E^1 and E^2:

Lemma

One has that:

$$E^1(u) \sim E^2(u)$$

Here, the constant is independent of N as long as N is sufficiently large.

The main lemma

But more importantly, for $\delta > 0$, we are interested in estimating:

$$E^2(u(t_0+\delta)) - E^2(u(t_0)) = \int_{t_0}^{t_0+\delta} \frac{d}{dt} E^2(u(t))dt$$

The iteration bound that one show is:

Lemma

For all $t_0 \in \mathbb{R}$, one has:

$$\left|E^2(u(t_0+\delta)) - E^2(u(t_0))\right| \lesssim \frac{1}{N^{1-}} E^2(u(t_0)).$$

In the proof of this lemma the key elements are the local-in-time bounds for the solution, the pointwise multiplier bounds for M_4, and the known Strichartz Estimates on $\mathbb{T}^2$.

Conclusion of the proof

To finish the proof we now observe that the estimate

$$E^2(u(t_0+\delta)) \leq (1+\frac{C}{N^{1-}})E^2(u(t_0))$$

can be iterated $\sim N^{1-}$ times without getting any exponential growth. We hence obtain that for $T \sim N^{1-}$, one has:

$$\|\mathcal{D}u(T)\|_{L^2} \lesssim \|\mathcal{D}u_0\|_{L^2}.$$

It follows that:

$$\|u(T)\|_{H^s} \lesssim N^s \|u_0\|_{H^s}$$

and hence:

$$\|u(T)\|_{H^s} \lesssim T^{s+}\|u_0\|_{H^s} \lesssim (1+T)^{s+}\|u_0\|_{H^s}.$$

This proves Theorem 1 for times $t \geq 1$. The claim for times $t \in [0,1]$ follows by local well-posedness theory.

Theorem 2 and the elements of its proof

We recall that Theorem 2 states:

Theorem (Colliander-Keel-S.-Takaoka-Tao)

Let $s > 1$, $K \gg 1$ and $0 < \sigma < 1$ be given. Then there exist a global smooth solution $u(x,t)$ to the IVP

$$\begin{cases} (-i\partial_t + \Delta)u = |u|^2 u \\ u(0,x) = u_0(x), \quad \textit{where } x \in \mathbb{T}^2, \end{cases}$$

and $T > 0$ such that

$$\|u_0\|_{H^s} \leq \sigma \quad \textit{and} \quad \|u(T)\|^2_{\dot{H}^s} \geq K.$$

1. Reduction to a resonant problem *RFNLS*
2. Construction of a special finite set Λ of frequencies
3. Truncation to a resonant, finite-d Toy Model
4. “Arnold diffusion” for the Toy Model
5. Approximation result via perturbation lemma
6. A scaling argument

2. Finite Resonant Truncation of *NLS*

We consider the gauge transformation

$$v(t,x) = e^{-i2Gt}u(t,x),$$

for $G \in \mathbb{R}$. If u solves *NLS* above, then v solves the equation

$$((NLS)_G) \qquad (-i\partial_t + \Delta)v = (2G + v)|v|^2.$$

We make the ansatz

$$v(t,x) = \sum_{n \in \mathbb{Z}^2} a_n(t) e^{i(\langle n,x\rangle + |n|^2 t)}.$$

Now the dynamics is all recast trough $a_n(t)$:

$$-i\partial_t a_n = 2Ga_n + \sum_{n_1 - n_2 + n_3 = n} a_{n_1}\overline{a_{n_2}}a_{n_3}e^{i\omega_4 t}$$

where $\omega_4 = |n_1|^2 - |n_2|^2 + |n_3|^2 - |n|^2$.

The *FNLS* system

By choosing

$$G = -\|v(t)\|_{L^2}^2 = -\sum_k |a_k(t)|^2$$

which is constant from the conservation of the mass, one can rewrite the equation above as

$$-i\partial_t a_n = -a_n|a_n|^2 + \sum_{n_1, n_2, n_3 \in \Gamma(n)} a_{n_1} \overline{a_{n_2}} a_{n_3} e^{i\omega_4 t}$$

where

$$\Gamma(n) = \{n_1, n_2, n_3 \in \mathbb{Z}^2 \;/\; n_1 - n_2 + n_3 = n; n_1 \neq n; n_3 \neq n\}.$$

From now on we will be refering to this system as the *FNLS* system, with the obvious connection with the original *NLS* equation.

The *RFNLS* system

We define the set

$$\Gamma_{res}(n) = \{n_1, n_2, n_3 \in \Gamma(n) \,/\, \omega_4 = 0\},$$

where again $\omega_4 = |n_1|^2 - |n_2|^2 + |n_3|^2 - |n|^2$.
The geometric interpretation for this set is the following: If n_1, n_2, n_3 are in $\Gamma_{res}(n)$, then these four points represent the vertices of a rectangle in $\mathbb{Z}^2$.
We finally define the Resonant Truncation *RFNLS* to be the system

$$-i\partial_t b_n = -b_n|b_n|^2 + \sum_{n_1, n_2, n_3 \in \Gamma_{res}(n)} b_{n_1} \overline{b_{n_2}} b_{n_3}.$$

Finite dimensional resonant truncation

- A finite set $\Lambda \subset \mathbb{Z}^2$ is closed under resonant interactions if

$$n_1, n_2, n_3 \in \Gamma_{res}(n), n_1, n_2, n_3 \in \Lambda \ =: n = n_1 - n_2 + n_3 \in \Lambda.$$

- A Λ-finite dimensional resonant truncation of *RFNLS* is

$$(RFNLS_\Lambda) \qquad -i\partial_t b_n = -b_n|b_n|^2 + \sum_{(n_1,n_2,n_3)\in\Gamma_{res}(n)\cap\Lambda^3} b_{n_1}\overline{b}_{n_2}b_{n_3}.$$

- $\forall$ resonant-closed finite $\Lambda \subset \mathbb{Z}^2$, $RFNLS_\Lambda$ is an ODE.

We will construct a **special set** Λ of frequencies.

3. Abstract Combinatorial Resonant Set Λ

Our goal is to have a resonant-closed $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_N$ with the properties below.

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- Existence and uniqueness of spouse and children: $\forall\ 1 \leq j < N$ and $\forall\ n_1 \in \Lambda_j\ \exists$ unique nuclear family such that $n_1, n_3 \in \Lambda_j$ are parents and $n_2, n_4 \in \Lambda_{j+1}$ are children.

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- Existence and uniqueness of siblings and parents: $\forall\ 1 \leq j < N$ and $\forall\ n_2 \in \Lambda_{j+1}\ \exists$ unique nuclear family such that $n_2, n_4 \in \Lambda_{j+1}$ are children and $n_1, n_3 \in \Lambda_j$ are parents.

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Our goal is to have a resonant-closed $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_N$ with the properties below. Define a nuclear family to be a rectangle (n_1, n_2, n_3, n_4) where the frequencies n_1, n_3 (the 'parents') live in generation Λ_j and n_2, n_4 ('children') live in generation Λ_{j+1}.

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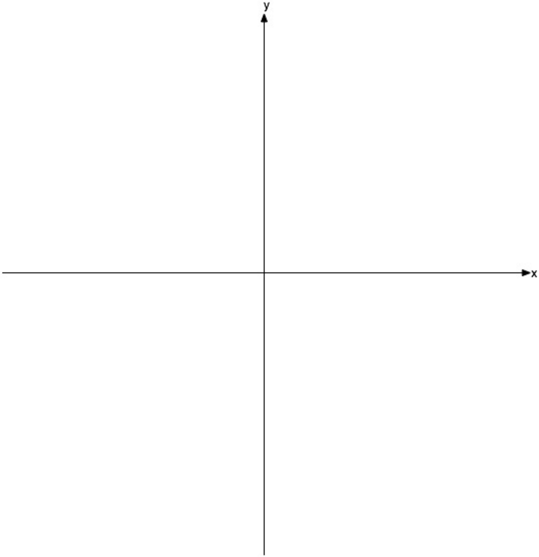
Cartoon Construction of Λ



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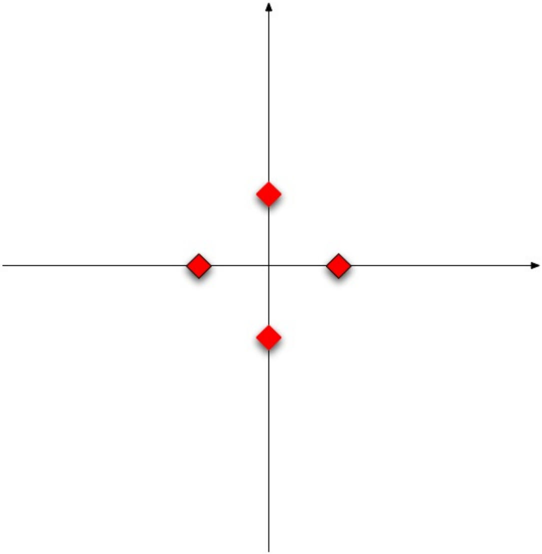

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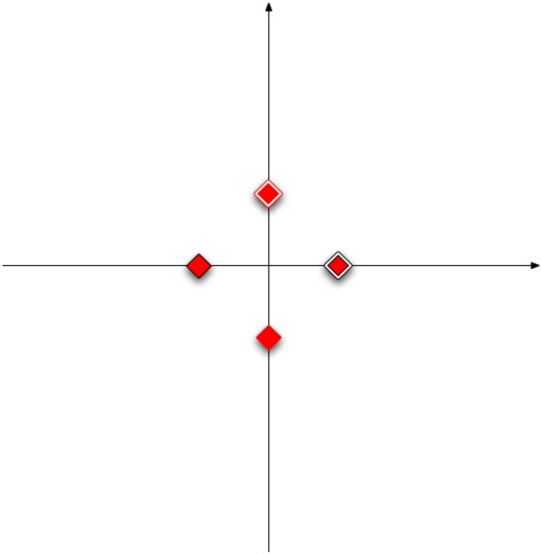

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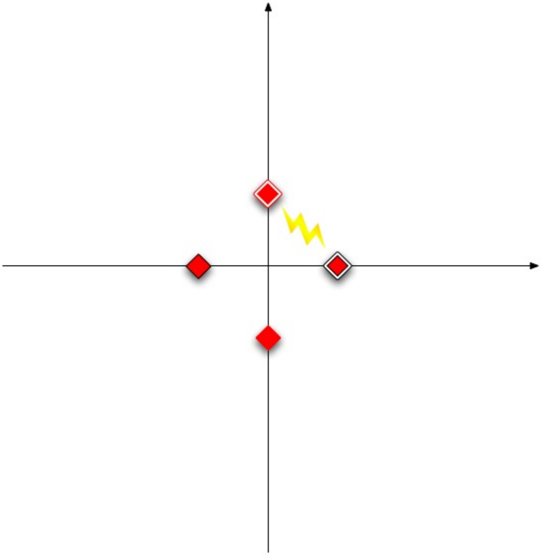

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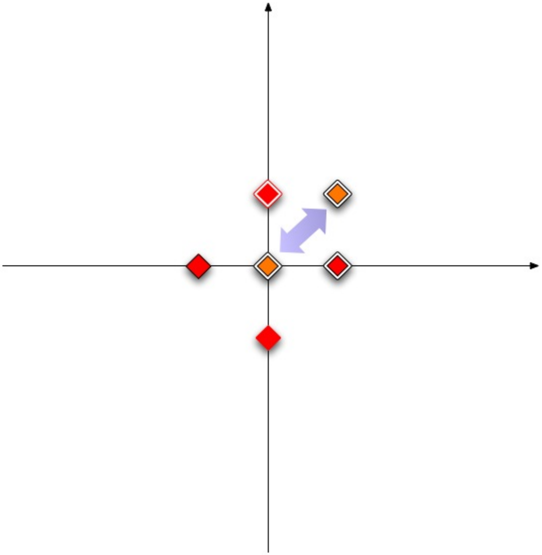

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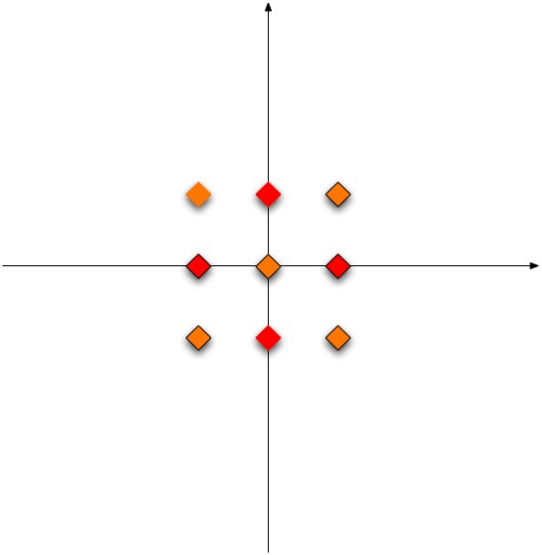

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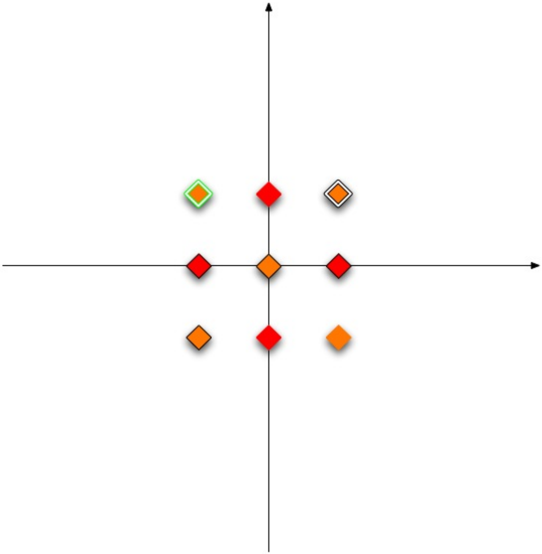

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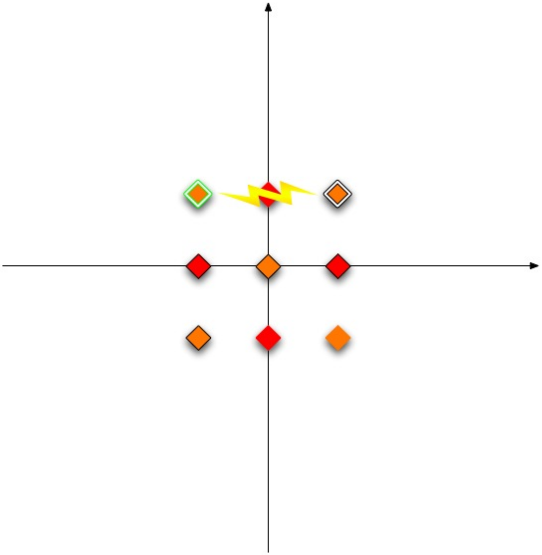

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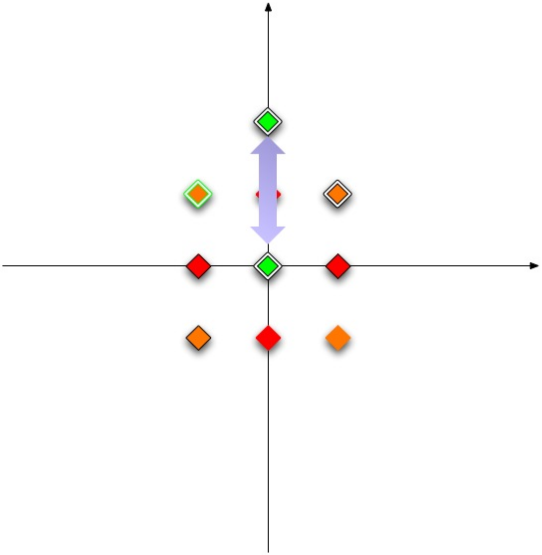

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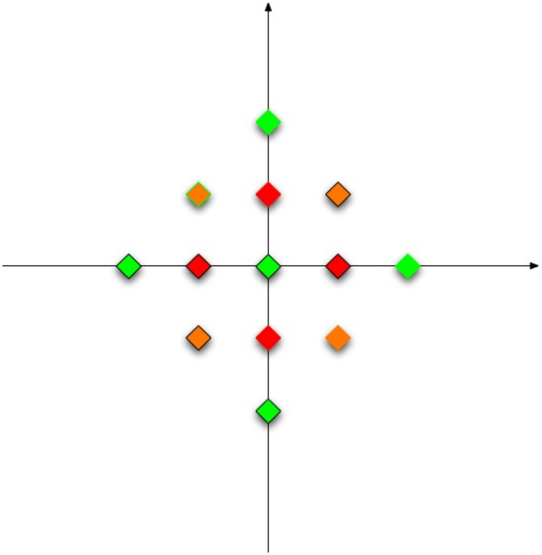

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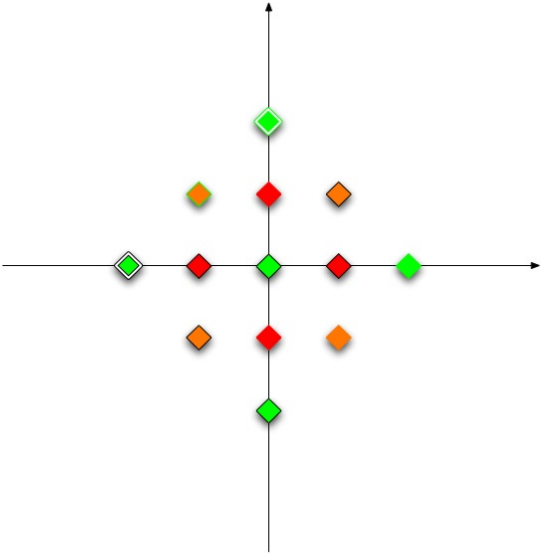

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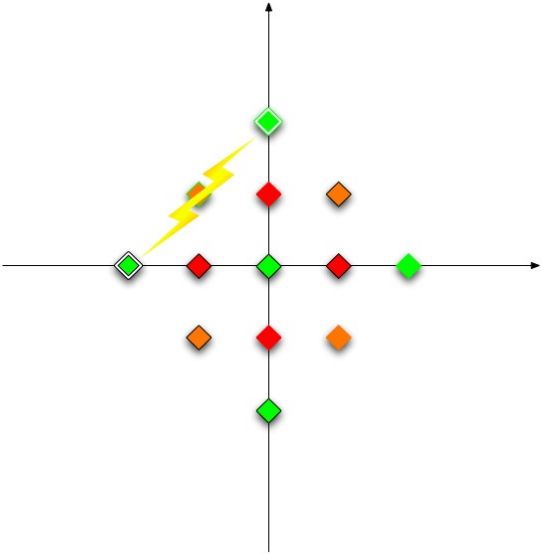

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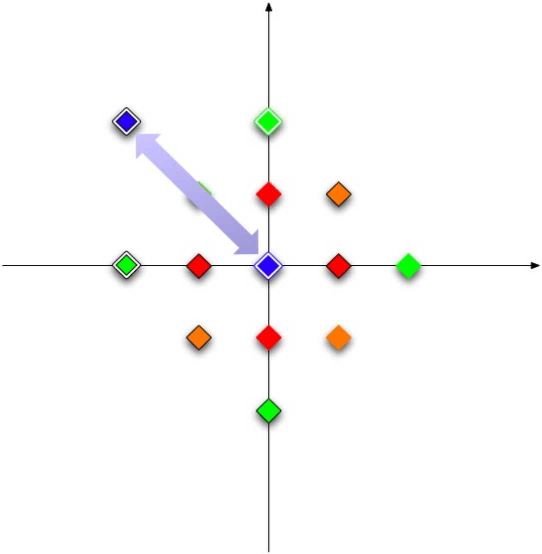

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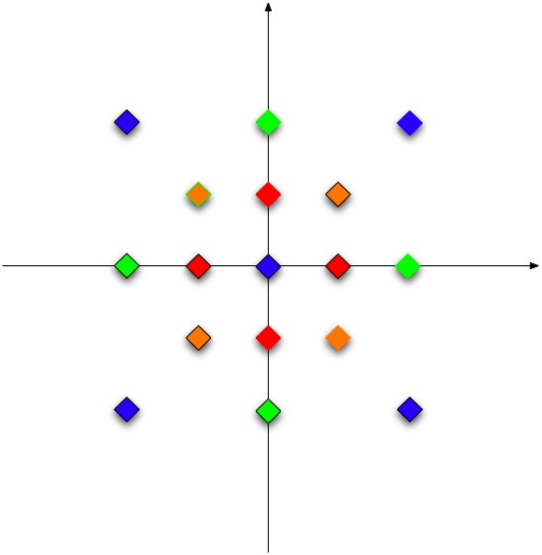

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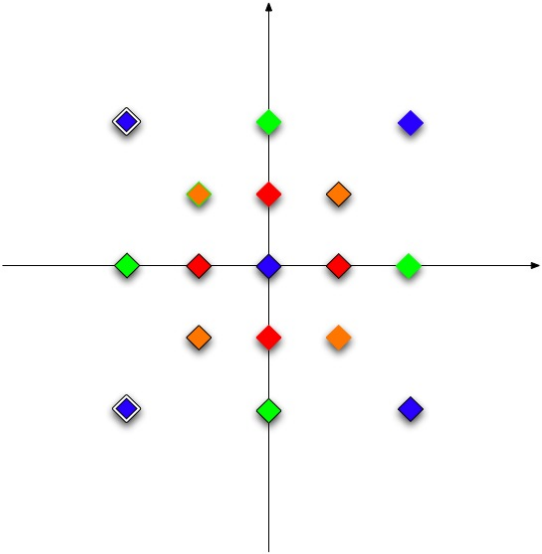

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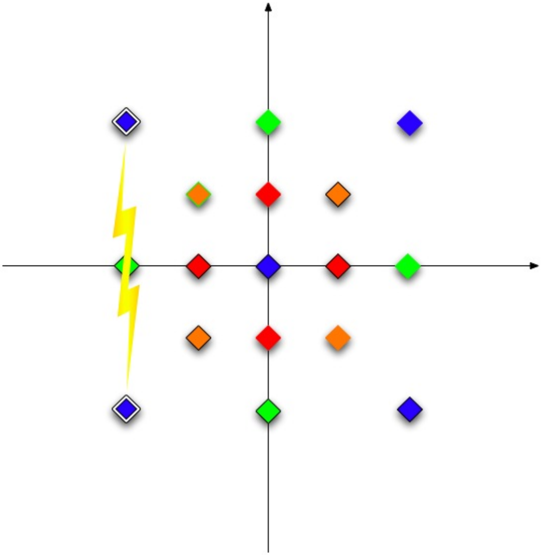

More properties for the set Λ

- **Multiplicative Structure:** If $N = N(\sigma, K)$ is large enough then Λ consists of $N \times 2^{N-1}$ disjoint frequencies n with $|n| > N = N(\sigma, K)$, the first frequency in Λ_1 is of size N and the last frequency in Λ_N is of size $C(N)N$. We call N the Inner Radius of Λ.
- **Wide Diaspora:** Given $\sigma \ll 1$ and $K \gg 1$, there exist M and $\Lambda = \Lambda_1 \cup .... \cup \Lambda_N$ as above and

$$\sum_{n \in \Lambda_N} |n|^{2s} \geq \frac{K^2}{\sigma^2} \sum_{n \in \Lambda_1} |n|^{2s}.$$

- **Approximation:** If $\text{spt}(a_n(0)) \subset \Lambda$ then $FNLS$-evolution $a_n(0) \longmapsto a_n(t)$ is nicely approximated by $RFNLS_\Lambda$-ODE $a_n(0) \longmapsto b_n(t)$.
- Given ϵ, s, K, build Λ so that $RFNLS_\Lambda$ has weak turbulence.

4. The Toy Model

- The truncation of *RFNLS* to the constructed set Λ is the ODE

$$(RFNLS_\Lambda) \qquad -i\partial_t b_n = -b_n|b_n|^2 + \sum_{(n_1,n_2,n_3)\in\Lambda^3\cap\Gamma_{res}(n)} b_{n_1} b_{n_2} b_{n_3}.$$

- The intergenerational equality hypothesis ($n \longmapsto b_n(0)$ is constant on each generation Λ_j.) persists under $RFNLS_\Lambda$:

$$\forall\ m, n \in \Lambda_j,\ b_n(t) = b_m(t).$$

- $RFNLS_\Lambda$ may be reindexed by generation number j.
  The recast dynamics is the Toy Model (ODE):

$$-i\partial_t b_j(t) = -b_j(t)|b_j(t)|^2 - 2b_{j-1}(t)^2\overline{b_j(t)} - 2b_{j+1}(t)^2\overline{b_j(t)},$$

  with the boundary condition

$$(BC) \qquad b_0(t) = b_{N+1}(t) = 0.$$

Conservation laws for the *ODE* system

The following are conserved quantities for (*ODE*)

$$\textit{Mass} = \sum_j |b_j(t)|^2 = C_0$$

$$\textit{Momentum} = \sum_j |b_j(t)|^2 \sum_{n\in\Lambda_j} n = C_1,$$

and if

$$\textit{Kinetic Energy} = \sum_j |b_j(t)|^2 \sum_{n\in\Lambda_j} |n|^2$$

$$\textit{Potential Energy} = \frac{1}{2}\sum_j |b_j(t)|^4 + \sum_j |b_j(t)|^2|b_{j+1}(t)|^2,$$

then

$$\textit{Energy} = \textit{Kinetic Energy} + \textit{Potential Energy} = C_2.$$

Toy model traveling wave solution

[1] Maybe dynamical systems methods are useful here?

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Using direct calculation[1], we will prove that our Toy Model ODE evolution $b_j(0) \longmapsto b_j(t)$ is such that:

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 & \cdot & \\
 & \cdot & \\
 & \cdot & \\
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Bulk of conserved mass is transferred from Λ_1 to Λ_N.

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$$

Bulk of conserved mass is transferred from Λ_1 to Λ_N. Weak turbulence lower bound follows from Wide Diaspora Property.

[1] Maybe dynamical systems methods are useful here?

Instability for the *ODE*: the set up

Global well-posedness for *ODE* is not an issue. Then we define

$$\Sigma = \{x \in \mathbb{C}^N \ / \ |x|^2 = 1\} \text{ and } W(t) : \Sigma \to \Sigma,$$

where $W(t)b(t_0) = b(t + t_0)$ for any solution $b(t)$ of *ODE*. It is easy to see that for any $b \in \Sigma$

$$\partial_t |b_j|^2 = 4\Re(i\bar{b}_j^{\,2}(b_{j-1}^2 + b_{j+1}^2)) \leq 4|b_j|^2.$$

So if

$$b_j(0) = 0 \quad \text{implies} \quad b_j(t) = 0, \ \text{ for all } \ t \in [0, T].$$

If moreover we define the torus

$$\mathbb{T}_j = \{(b_1, ..., b_N) \in \Sigma \,/\, |b_j| = 1, \ b_k = 0, \ k \neq j\}$$

then

$$W(t)\mathbb{T}_j = \mathbb{T}_j \ \text{ for all } j = 1, ..., N$$

($\mathbb{T}_j$ is invariant).

Instability for the *ODE*

Theorem (Sliding Theorem)

Let $N \geq 6$. Given $\epsilon > 0$ there exist x_3 within ϵ of $\mathbb{T}_3$ and x_{N-2} within ϵ of $\mathbb{T}_{N-2}$ and a time t such that

$$W(t)x_3 = x_{N-2}.$$

Remark

$W(t)x_3$ is a solution of total mass 1 arbitrarily concentrated near mode $j = 3$ at some time t_0 and then arbitrarily concentrated near mode $j = N - 2$ at later time t.

The sliding process

To motivate the theorem let us first observe that when $N = 2$ we can easily demonstrate that there is an orbit connecting $\mathbb{T}_1$ to $\mathbb{T}_2$. Indeed in this case we have the explicit "slider" solution

$$(11.1) \qquad b_1(t) := \frac{e^{-it}\omega}{\sqrt{1+e^{2\sqrt{3}t}}}; \quad b_2(t) := \frac{e^{-it}\omega^2}{\sqrt{1+e^{-2\sqrt{3}t}}}$$

where $\omega := e^{2\pi i/3}$ is a cube root of unity.

This solution approaches $\mathbb{T}_1$ exponentially fast as $t \to -\infty$, and approaches $\mathbb{T}_2$ exponentially fast as $t \to +\infty$. One can translate this solution in the j parameter, and obtain solutions that "slide" from $\mathbb{T}_j$ to $\mathbb{T}_{j+1}$. Intuitively, the proof of the Sliding Theorem for higher M should then proceed by concatenating these slider solutions......

This is a cartoon of what we have:

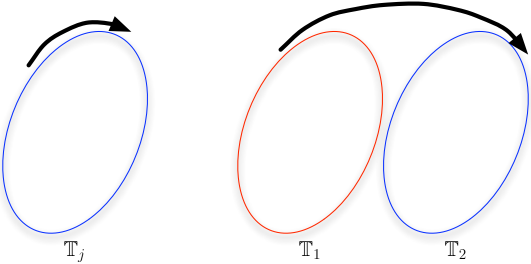

Figure: Explicit oscillator solution around $\mathbb{T}_j$ and the slider solution from $\mathbb{T}_1$ to $\mathbb{T}_2$

This though cannot work directly because each solution requires an infinite amount of time to connect one circle to the next, but it turns out that a suitably perturbed or "fuzzy" version of these slider solutions can in fact be glued together.

5. A Perturbation Lemma

Lemma

Let $\Lambda \subset \mathbb{Z}^2$ introduced above. Let $B \gg 1$ and $\delta > 0$ small and fixed. Let $t \in [0, T]$ and $T \sim B^2 \log B$. Suppose there exists $b(t) \in l^1(\Lambda)$ solving RFNLS$_\Lambda$ such that

$$\|b(t)\|_{l^1} \lesssim B^{-1}.$$

Then there exists a solution $a(t) \in l^1(\mathbb{Z}^2)$ of FNLS such that

$$a(0) = b(0), \quad \text{and} \quad \|a(t) - b(t)\|_{l^1(\mathbb{Z}^2)} \lesssim B^{-1-\delta},$$

for any $t \in [0, T]$.

Proof.

This is a standard perturbation lemma proved by checking that the “non resonant” part of the nonlinearity remains small enough. □

Recasting the main theorem

With all the notations and reductions introduced we can now recast the main theorem in the following way:

Theorem

For any $0 < \sigma \ll 1$ *and* $K \gg 1$ *there exists a complex sequence* (a_n) *such that*

$$\left(\sum_{n \in \mathbb{Z}^2} |a_n|^2 |n|^{2s} \right)^{1/2} \lesssim \sigma$$

and a solution $(a_n(t))$ *of* (FNLS) *and* $T > 0$ *such that*

$$\left(\sum_{n \in \mathbb{Z}^2} |a_n(T)|^2 |n|^{2s} \right)^{1/2} > K.$$

6. A Scaling Argument

In order to be able to use "instability" to move mass from lower frequencies to higher ones and start with a small data we need to introduce scaling.

Consider in $[0, \tau]$ the solution $b(t)$ of the system *RFNLS*$_\Lambda$ with initial datum b_0. Then the rescaled function

$$b^\lambda(t) = \lambda^{-1} b(\frac{t}{\lambda^2})$$

solves the same system with datum $b_0^\lambda = \lambda^{-1} b_0$.
We then first pick the complex vector $b(0)$ that was found in the "instability" theorem above. For simplicity let's assume here that $b_j(0) = 1 - \epsilon$ if $j = 3$ and $b_j(0) = \epsilon$ if $j \neq 3$ and then we fix

$$a_n(0) = \begin{cases} b_j^\lambda(0) \text{ for any } n \in \Lambda_j \\ 0 \text{ otherwise }. \end{cases}$$

Estimating the size of $(a(0))$

By definition

$$\left(\sum_{n\in\Lambda}|a_n(0)|^2|n|^{2s}\right)^{1/2}=\frac{1}{\lambda}\left(\sum_{j=1}^{N}|b_j(0)|^2\left(\sum_{n\in\Lambda_j}|n|^{2s}\right)\right)^{1/2}\sim\frac{1}{\lambda}Q_3^{1/2},$$

where the last equality follows from defining

$$\sum_{n\in\Lambda_j}|n|^{2s}=Q_j,$$

and the definition of $a_n(0)$ given above. At this point we use the proprieties of the set Λ to estimate $Q_3=C(N)N^{2s}$, where N is the inner radius of Λ. We then conclude that

$$\left(\sum_{n\in\Lambda}|a_n(0)|^2|n|^{2s}\right)^{1/2}=\lambda^{-1}C(N)N^s\sim\sigma.$$

Estimating the size of $(a(T))$

By using the perturbation lemma with $B = \lambda$ and $T = \lambda^2\tau$ we have

$$\|a(T)\|_{H^s} \geq \|b^\lambda(T)\|_{H^s} - \|a(T) - b^\lambda(T)\|_{H^s} = I_1 - I_2.$$

We want $I_2 \ll 1$ and $I_1 > K$. For the first

$$I_2 \leq \|a(T) - b^\lambda(T)\|_{l^1(\mathbb{Z}^2)} \left(\sum_{n\in\Lambda} |n|^{2s}\right)^{1/2} \lesssim \lambda^{-1-\delta} \left(\sum_{n\in\Lambda} |n|^{2s}\right)^{1/2}.$$

As above

$$I_2 \lesssim \lambda^{-1-\delta} C(N) N^s$$

At this point we need to pick λ and N so that

$$\|a(0)\|_{H^s} = \lambda^{-1} C(N) N^s \sim \sigma \quad \text{and} \quad I_2 \lesssim \lambda^{-1-\delta} C(N) N^s \ll 1$$

and thanks to the presence of $\delta > 0$ this can be achieved by taking λ and N large enough.

Estimating I_1

It is important here that at time zero one starts with a fixed non zero datum, namely $\|a(0)\|_{H^s} = \|b^\lambda(0)\|_{H^s} \sim \sigma > 0$. In fact we will show that

$$I_1^2 = \|b^\lambda(T)\|_{H^s}^2 \geq \frac{K^2}{\sigma^2}\|b^\lambda(0)\|_{H^s}^2 \sim K^2.$$

If we define for $T = \lambda^2 t$

$$R = \frac{\sum_{n\in\Lambda} |b_n^\lambda(\lambda^2 t)|^2 |n|^{2s}}{\sum_{n\in\Lambda} |b_n^\lambda(0)|^2 |n|^{2s}},$$

then we are reduce to showing that $R \gtrsim K^2/\sigma^2$. Now recall the notation

$$\Lambda = \Lambda_1 \cup \ldots\ldots \cup \Lambda_N \quad \text{and} \quad \sum_{n\in\Lambda_j} |n|^{2s} = Q_j.$$

More on Estimating I_1

Using the fact that by the theorem on "instability" one obtains $b_j(T) = 1 - \epsilon$ if $j = N - 2$ and $b_j(T) = \epsilon$ if $j \neq N - 2$, it follows that

$$\begin{aligned} R &= \frac{\sum_{i=1}^{N} \sum_{n \in \Lambda_i} |b_i^{\lambda}(\lambda^2 t)|^2 |n|^{2s}}{\sum_{i=1}^{N} \sum_{n \in \Lambda_i} |b_i^{\lambda}(0)|^2 |n|^{2s}} \\ &\geq \frac{Q_{N-2}(1-\epsilon)}{(1-\epsilon)Q_3 + \epsilon Q_1 + ... + \epsilon Q_N} \sim \frac{Q_{N-2}(1-\epsilon)}{Q_{N-2}\left[(1-\epsilon)\frac{Q_3}{Q_{N-2}} + ... + \epsilon\right]} \\ &\gtrsim \frac{(1-\epsilon)}{(1-\epsilon)\frac{Q_3}{Q_{N-2}}} = \frac{Q_{N-2}}{Q_3} \end{aligned}$$

and the conclusion follows from "large diaspora" of Λ_j:

$$Q_{N-2} = \sum_{n \in \Lambda_{N-2}} |n|^{2s} \gtrsim \frac{K^2}{\sigma^2} \sum_{n \in \Lambda_3} |n|^{2s} = \frac{K^2}{\sigma^2} Q_3.$$

Where does the set Λ come from?

Here we do not construct Λ, but we construct Σ, a set that has a lot of the properties of Λ but does not live in $\mathbb{Z}^2$.

We define the *standard unit square* $S \subset \mathbb{C}$ to be the four-element set of complex numbers

$$S = \{0, 1, 1 + i, i\}.$$

We split $S = S_1 \cup S_2$, where $S_1 := \{1, i\}$ and $S_2 := \{0, 1 + i\}$. The combinatorial model Σ is a subset of a large power of the set S. More precisely, for any $1 \leq j \leq N$, we define $\Sigma_j \subset \mathbb{C}^{N-1}$ to be the set of all $N-1$-tuples $(z_1, \ldots, z_{N-1})$ such that $z_1, \ldots, z_{j-1} \in S_2$ and $z_j, \ldots, z_{N-1} \in S_1$. In other words,

$$\Sigma_j := S_2^{j-1} \times S_1^{N-j}.$$

Note that each Σ_j consists of 2^{N-1} elements, and they are all disjoint. We then set $\Sigma = \Sigma_1 \cup \ldots \cup \Sigma_N$; this set consists of $N2^{N-1}$ elements. We refer to Σ_j as the j^{th} *generation* of Σ.

For each $1 \leq j < N$, we define a *combinatorial nuclear family connecting generations* Σ_j, Σ_{j+1} to be any four-element set $F \subset \Sigma_j \cup \Sigma_{j+1}$ of the form

$$F := \{(z_1, \ldots, z_{j-1}, w, z_{j+1}, \ldots, z_N) : w \in S\}$$

where $z_1, \ldots, z_{j-1} \in S_2$ and $z_{j+1}, \ldots, z_N \in S_1$ are fixed. In other words, we have

$$F = \{F_0, F_1, F_{1+i}, F_i\} = \{(z_1, \ldots, z_{j-1})\} \times S \times \{(z_{j+1}, \ldots, z_N)\}$$

where $F_w = (z_1, \ldots, z_{j-1}, w, z_{j+1}, \ldots, z_N)$.

It is clear that

- F is a four-element set consisting of two elements F_1, F_i of Σ_j (which we call the *parents* in F) and two elements F_0, F_{1+i} of Σ_{j+1} (which we call the *children* in F).
- For each j there are 2^{N-2} combinatorial nuclear families connecting the generations Σ_j and Σ_{j+1}.

Properties of Σ

One easily verifies the following properties:

- Existence and uniqueness of spouse and children: For any $1 \leq j < N$ and any $x \in \Sigma_j$ there exists a unique combinatorial nuclear family F connecting Σ_j to Σ_{j+1} such that x is a parent of this family (i.e. $x = F_1$ or $x = F_i$). In particular each $x \in \Sigma_j$ has a unique spouse (in Σ_j) and two unique children (in Σ_{j+1}).
- Existence and uniqueness of sibling and parents: For any $1 \leq j < N$ and any $y \in \Sigma_{j+1}$ there exists a unique combinatorial nuclear family F connecting Σ_j to Σ_{j+1} such that y is a child of the family (i.e. $y = F_0$ or $y = F_{1+i}$). In particular each $y \in \Sigma_{j+1}$ has a unique sibling (in Σ_{j+1}) and two unique parents (in Σ_j).
- Nondegeneracy: The sibling of an element $x \in \Sigma_j$ is never equal to its spouse.

Example:

If $N = 7$, the point $x = (0, 1 + i, 0, i, i, 1)$ lies in the fourth generation Σ_4. Its spouse is $(0, 1 + i, 0, 1, i, 1)$ (also in Σ_4) and its two children are $(0, 1 + i, 0, 0, i, 1)$ and $(0, 1 + i, 0, 1 + i, i, 1)$ (both in Σ_5). These four points form a combinatorial nuclear family connecting the generations Σ_4 and Σ_5. The sibling of x is $(0, 1 + i, 1 + i, i, i, 1)$ (also in Σ_4, but distinct from the spouse) and its two parents are $(0, 1 + i, 1, i, i, 1)$ and $(0, 1 + i, i, i, i, 1)$ (both in Σ_3). These four points form a combinatorial nuclear family connecting the generations Σ_3 and Σ_4. Elements of Σ_1 do not have siblings or parents, and elements of Σ_7 do not have spouses or children.