# Microlocal analysis near null infinity on asymptotically flat spacetimes[†]

[†]This talk would not have been possible without the groundbreaking work of Richard Melrose

András Vasy, joint work with Peter Hintz

MIT, Cambridge, MA; May 2024

The Minkowski metric is $g_m = -dx_0^2 + \sum_{j=1}^n dx_j^2$ on $\mathbb{R}^{n+1}$.

Goal: understand long time behavior, including Fredholm and invertibility theory, for wave equations on spacetimes asymptotically like this. Moreover, do it microlocally, i.e. locally in phase space: helps with more complicated phenomena elsewhere (Kerr!) and QFT...

Question: what does 'like' mean?

While locally manifolds are just modelled on Euclidean space, and one does not need to specify any structures, just talks of 'differential operators' or 'tangent bundle', near infinity one needs to be more careful as one size does not fit all.

- Two structures on a vector space: dilations and translations.
- A different way: compactifications with bundle structures.

Case study: Riemannian version: $g_e = \sum_{j=1}^n dx_j^2$ on $\mathbb{R}^n$. In the polar decomposition one can see that this is asymptotically conic: $x = r\omega$, $\omega \in \mathbb{S}^{n-1}$, $g_e = dr^2 + r^2 g_{\mathbb{S}^{n-1}}$. Note that this emphasizes the dilation structure on $\mathbb{R}^n$.

If we want to 'bring in infinity', let $\rho = 1/r$, so the metric is

$$g_e = \frac{d\rho^2}{\rho^4} + \frac{g_{\mathbb{S}^{n-1}}}{\rho^2}$$

which is a 'scattering metric' in the sense of Melrose, and now the cross section can be generalized. One can add $\rho = 0$ as a boundary, with $\rho$ as the boundary defining function, i.e. near $\rho = 0$ the space is $[0,1)_\rho \times \mathbb{S}^{n-1}$, with interior identified with $\{r > 1\}$ in $\mathbb{R}^n$.

This compactifies the space, hard coding the behavior near infinity.

One can also then consider metrics that are asymptotically like this (coefficients smooth, or at least conormal on the compactification), with the cross section generalized (full scattering metric class).

One can also then consider metrics that are asymptotically like this (coefficients smooth, or at least conormal on the compactification), with the cross section generalized (full scattering metric class). These are fiber inner products on the corresponding tangent bundle ${}^{\mathrm{sc}}T\overline{\mathbb{R}^n}$, called the scattering tangent bundle, corresponding to the dual vector fields $\rho^2\partial_\rho$ and $\rho\partial_{y_j}$.

- Smooth sections of ${}^{\mathrm{sc}}T\overline{\mathbb{R}^n}$ are of the form

$$a_0(\rho, y)\rho^2\partial_\rho + \sum_j a_j(\rho, y)\rho\partial_{y_j},$$

  with all $a_i$ smooth on $\overline{\mathbb{R}^n}$, i.e. down to $\rho = 0$. (Same as the span of $\partial_{x_j}$ with $C^\infty(\overline{\mathbb{R}^n})$ coefficients, so ${}^{\mathrm{sc}}T\overline{\mathbb{R}^n} = \overline{\mathbb{R}^n} \times \mathbb{R}^n$!) Conormal regularity would mean $|(\rho\partial_\rho)^{\alpha_0}\partial_y^{\alpha'} a_i| \leq C_\alpha \rho^k$.
- Dually, smooth sections of ${}^{\mathrm{sc}}T^*\overline{\mathbb{R}^n}$ are of the form

$$b_0(\rho, y)\,\frac{d\rho}{\rho^2} + \sum_j b_j(\rho, y)\,\frac{dy_j}{\rho},$$

- so inner products on ${}^{\mathrm{sc}}T\overline{\mathbb{R}^n}$ are indeed of the stated form.

Minkowski metric is similarly a 'Lorentzian cone' as it is homogeneous of degree 2 under dilations. *One* suitable generalization goes towards 'Lorentzian scattering metrics', which basically would be Lorentzian signature fiber inner products on ${}^{\mathrm{sc}}T\overline{\mathbb{R}^{n+1}}$.

A key difference emerges immediately if one looks at the associated PDE:

- In the Riemannian setting they are elliptic in the standard differential sense, so if one places them in an pseudodifferential algebra in which they are elliptic at infinity as well in a strong sense ('fully elliptic'), one gets a Fredholm theory easily.
- In the pseudo-Riemannian setting on the other hand there are propagation phenomena along bicharacteristics within the characteristic set, and for a Fredholm theory one needs to obtain control in some places so that one can propagate it. Naturally this will be at infinity.

A bit more concretely: for $\Delta_g$, $g$ Riemannian scattering metric, the 'obvious' Fredholm theory setting would be in the 'scattering category', i.e. $\Psi_{\mathrm{sc}}(X)$, e.g. $X = \overline{\mathbb{R}^n}$.

- This algebra is perhaps the most standard one, quantizing symbols $a \in S^{m,l}(\mathbb{R}^n_z \times \mathbb{R}^n_\zeta)$ with
$$|(D_z^\alpha D_\zeta^\beta a)(z,\zeta)| \leq C_{\alpha\beta}\langle z\rangle^{l-|\alpha|}\langle\zeta\rangle^{m-|\beta|}$$
as, for instance,
$$(\mathrm{Op}(a)u)(z) = (2\pi)^{-n}\int e^{i(z-z')\cdot\zeta}a(z,\zeta)u(z')\,dz'\,d\zeta,$$
to obtain $\mathrm{Op}(a) \in \Psi^{m,l} = \Psi^{m,l}_{\mathrm{sc}}(X)$.
- $\mathrm{Op}(a) \in \mathcal{L}(H_{sc}^{s,r}, H_{sc}^{s-m,r-l})$, $H_{sc}^{s,r} = \langle z\rangle^{-r}H^s(\mathbb{R}^n)$.
- $a$ is elliptic if there is $c > 0$ such that $|(z,\zeta)| \gg 1$ implies that
$$|a(z,\zeta)| \geq c\langle z\rangle^l\langle\zeta\rangle^m.$$
- If $a$ is elliptic, then $\mathrm{Op}(a)$ is Fredholm in all the above senses: symbol calculus using $S^{m,l}/S^{m-1,l-1}$ and $\Psi_{\mathrm{sc}}^{-\infty,-\infty}$ is compact on all spaces.

- This actually works for $\Delta_g + 1$, for its principal symbol is $|\xi|_g^2 + 1$, with 1 the same order as $|\xi|_g^2$ in the sense of spatial decay, so this is fully elliptic at $\partial X$, where finite $\xi$ also matters.
- However, this is not the case for $\Delta_g$, for its principal symbol, $|\xi|_g^2$, has a quadratic degeneracy at the 0-section.
- Most conceptual way of handling this: blow up the singular locus, i.e. the 0-section at the boundary; this is 2-microlocalization.

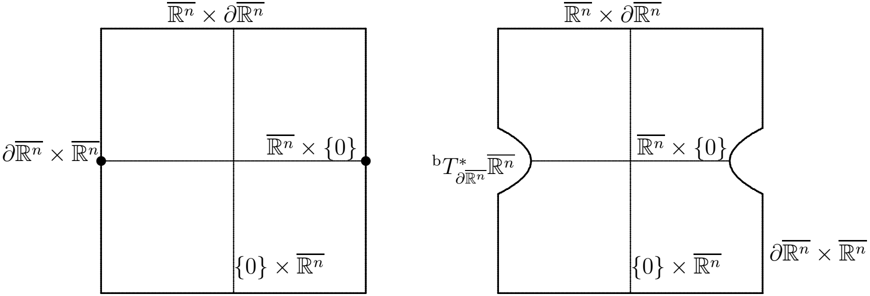



Figure: Left: $\overline{{}^{\mathrm{sc}}T^*\mathbb{R}^n} = \overline{\mathbb{R}^n} \times \overline{(\mathbb{R}^n)^*}$. The principal symbol of $\Delta$ vanishes quadratically at the zero section at the boundary. Right: the blow up of this, allowing refined, non-degenerate, estimates. It is simpler, however, the work with the b-algebra, introduced by Melrose, which is a blow-down of this.

Simpler version (closely related): think of $\Delta_g \in \Psi_b^{2,-2}(X)$. Totally characteristic, or b-, vector fields are those tangent to the boundary, i.e. in local coordinates $\rho\partial_\rho$, $\partial_{y_j}$. Then

$$\Delta_g \sim \rho^2((\rho D_\rho)^2 + \Delta_{g_{\partial X}}).$$

Then not only is $\Delta_g$ elliptic in the differential sense, up to the boundary (automatic as the dual metric function is non-degenerate) but there is a Mellin-transformed normal operator (factoring out the overall $\rho^2$), roughly $\sigma^2 + \Delta_{g_{\partial X}}$, and then invertibility of this family on $\partial X$ with $\operatorname{Im}\sigma = -l$ gives rise to Fredholm theory on b-Sobolev spaces with weight $l$.

Note that this works for Riemannian scattering metrics, i.e. asymptotic cones, in general.

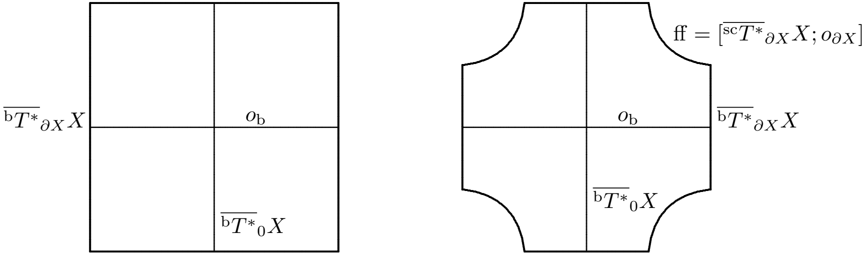
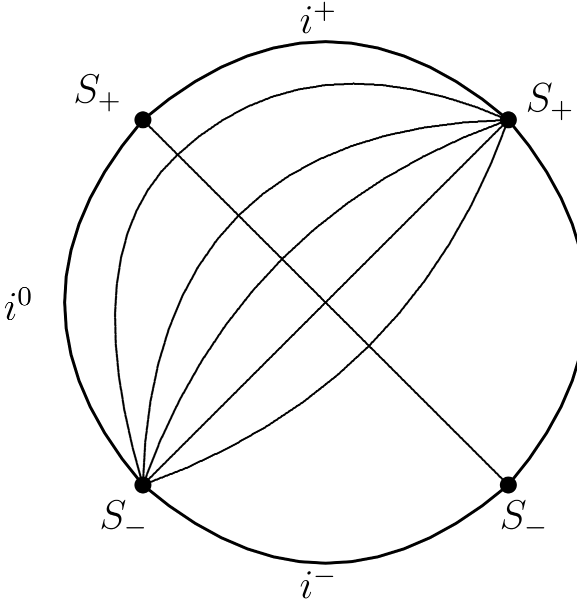



Figure: Left: the fiber compactified b-cotangent bundle. The principal symbol of $\Delta$ now lives only at fiber infinity, and is elliptic. A normal operator captures the full Fredholm theory. Right: the blow up of the corner creates the previous resolved scattering cotangent bundle. While geometrically equivalent to the previous construction, this is the analytically better approach.

The complication in the pseudo-Riemannian setting is that estimates propagate, so how does one start them?

- In the Lorentzian setting one possibility is Cauchy hypersurfaces, but then one needs to work between two Cauchy hypersurfaces for Fredholm theory as the adjoint is also needed.
- In general (not even just in the geometric settings) the best case scenario is a normal source/sink structure, i.e. there are submanifolds so that all bicharacteristics start/end at them.
- This is indeed the case for translation invariant metrics for *any* non-Riemannian signature on a vector space once one compactifies it, and then one can allow well-behaved coefficients on this compactification: the sources/sinks lie above the light cone at the boundary of the compactification. (Also, Anosov flows!)


$i^+$
$S_+$
$S_+$
$i^0$
$S_-$
$S_-$
$i^-$

- More precisely, the dynamics takes place in ${}^bT^*\overline{\mathbb{R}^{n+1}} \setminus o$ and is homogeneous in the fibers, so (after a renormalization to make it homogeneous of degree zero) it is even better to consider it taking place in ${}^bS^*\overline{\mathbb{R}^{n+1}} = ({}^bT^*\overline{\mathbb{R}^{n+1}} \setminus o)/\mathbb{R}^+$, or better yet as identified as 'fiber infinity' in ${}^b\overline{T^*\mathbb{R}^{n+1}}$ (boundary of the compactified fibers); the flow is tangent to this boundary.
- The b-conormal bundles of the light cones at the boundary form the normal sources/sinks: in the Minkowski case there are two components of the characteristic set, and in each over future and past light cone at the boundary there is such a normal source/sink.

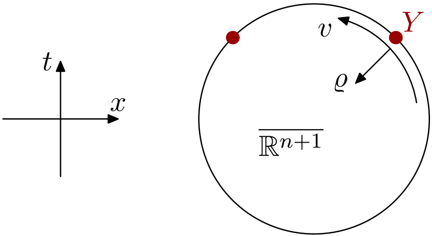



Figure: The compactification of $\mathbb{R}^{n+1}$, and the light cone at infinity $S = Y \cong \mathbb{S}^{n-1}$, in the case $n = 1$.

- More precisely, if we let $\rho$ to be a (local) boundary defining function, $v$ define the light cone at the boundary, $S$, so for instance $\rho = (t+r)^{-1}$, $v = \frac{t-r}{t+r}$, the metric has the form
$$-v\frac{d\rho^2}{\rho^4} + \left(\frac{d\rho}{\rho^2} \otimes \frac{\alpha}{\rho} + \frac{\alpha}{\rho} \otimes \frac{d\rho}{\rho^2}\right) + \frac{k}{\rho^2},$$
where $\alpha = \frac{1}{2}\,dv$ at $S$, and $k$ is positive definite on the annihilator of $\mathrm{Span}(d\rho, dv)$. (This general class is the Lorentzian scattering metric definition of Baskin-V-Wunsch; in fact any metric with the same source/sink structure works.)
- Then with $(\xi, \gamma, \eta)$ the dual variables of $(\rho, v, y)$ the source sink at fiber infinity is at $\rho = v = 0$, $\xi = 0$, $\eta = 0$, which is the b-conormal bundle of $\rho = v = 0$, i.e. the image of $(*)\,d\rho + (*)\,dv$ in the b-cotangent bundle.
- To turn in this into global estimates, need to be able to trace all bicharacteristics to (in both the forward and backward directions: duality!) these sources/sinks: have non-trapping bicharacteristic flow. (Upgrades with mild trapping.)

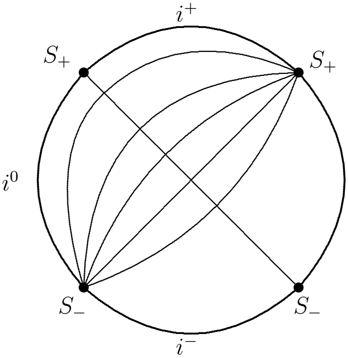
- The point from which propagation takes place is marked by high regularity, to which is by low regularity. (Variable order, or anisotropic, spaces.)
- For the adjoint it is in the opposite direction relative to the direct problem.
- There are $2^2$ Fredholm problems: in each component of the characteristic set we can propagate estimates forwards/backwards: two causal and Feynman/anti-Feynman. (V, Baskin-V-Wunsch, Gell-Redman-Haber-V.)
- For each a discrete set of weights, corresponding to the indicial roots/resonances, i.e. poles of the Mellin-transformed family's inverse, need to be avoided.
- Note that the Mellin-transformed family is a glued together version of the hyperbolic/de Sitter Laplace/d'Alembert spectral families, corresponding to the projective compactification (Klein model).

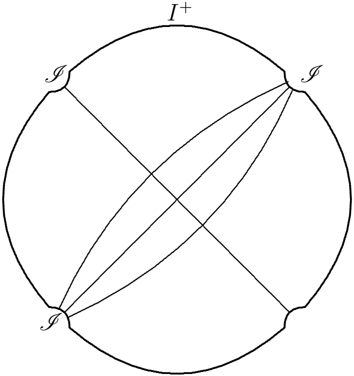
So we have a microlocal understanding without dealing with null infinity, what is missing?

The issue is that for nonlinear applications one needs to consider coefficients that are produced by solutions of the (say, linearized) equations. Unlike in the elliptic setting, where elliptic regularity gives conormality to the boundary, i.e. being in an infinite differential order weighted b-Sobolev space, the analogous statement is *false* in the pseudo-Riemannian setting.

What do we get instead? A 'b-conormal distribution associated to the b-conormal bundle of the boundary of the light cone, $\rho = v = 0$', i.e. the source/sink.

Simpler put, encoding it in the base, one blows up $\rho = v = 0$ to obtain $[X; S]$ and the front face $\mathscr{I}$, and solutions are now conormal to the boundary hypersurfaces on this space.


$i^+$
$S_+$
$S_+$
$i^0$
$S_-$
$S_-$
$i^-$


$I^+$
$\mathscr{I}$
$\mathscr{I}$
$\mathscr{I}$

Simpler put, encoding it in the base, one blows up $\rho = v = 0$ to obtain a front face $\mathscr{I}$, and solutions are now conormal to the boundary hypersurfaces on this space.

We then allow coefficients that have exactly this behavior, with orders that match that of Minkowski space in the sense we discuss, and indeed decay (at any rate) at $\mathscr{I}$.

But if we do this, we lose the b-cotangent bundle analysis we relied on! Can we salvage anything?

There are two potential ways.

1.) This is a geometric 3-body type wave operator for which there is a scattering pseudodifferential algebra (V.), and indeed we can even salvage the b-behavior away from $\mathscr{I}$ thanks to its 3-body-b refinement (Hintz). The issue is that from this perspective one has operator valued symbols, depending on some partial momenta, and it is quite degenerate at some points.

Note that from this perspective one expects some interaction between different parts of the b-flow, corresponding to diffractive phenomena both in scattering theory and also for wave propagation (V.).

In the Lorentzian setting this perspective turns out to be more useful for allowing metric singularities in the time-like infinity, for instance to deal with Kerr-like geometries (Hintz).

2.) A more adventurous perspective, taking advantage of the Lorentzian nature of the metric is to replace the defining function of $\mathscr{I}$ by its square root, so use

$$x_{\mathscr{I}} = \sqrt{\frac{t-r}{r}},\ \rho_+ = \frac{1}{t-r},\ y,\ \ t-r > 1.$$

The metric is then an edge-b metric on $[X; S]_{1/2}$: edge (Mazzeo!) at $\mathscr{I}$, b (Melrose!) at the lifted boundary.

What does this mean? There is a blow down map, $\beta$, so we can talk about vector fields tangent to its fibers. Locally this means that

$$\rho_+\partial_{\rho_+},\ x_{\mathscr{I}}\partial_{x_{\mathscr{I}}},\ x_{\mathscr{I}}\partial_{y_j}$$

span these vector fields; cf. Klainerman's vector field method. However, we want to work in phase space for instance to be able to deal with microlocal issues elsewhere.

Concretely,

$$2\rho_+^{-2}x_{\mathscr{I}}^{-2}\Box = (x_{\mathscr{I}}\partial_{x_{\mathscr{I}}} - (n-1))(x_{\mathscr{I}}\partial_{x_{\mathscr{I}}} - 2\rho_+\partial_{\rho_+}) + 2x_{\mathscr{I}}^2\Delta_k$$

shows that it fits into this structure. Note that this *does* rely on the detailed metric behavior at $S$, i.e. is not true for a general non-trapping Lorentzian metric in the scattering category (more general than the BVW definition).

The Hamilton flow is complicated but well behaved at $\mathscr{I}$:

- There is a global radial set at $\mathscr{I}$ corresponding to the radiation field; this is a source/sink.
- There are two radial sets at $\mathscr{I} \cap I^0$, and one at $\mathscr{I} \cap I^+$; these are saddle points, corresponding to flow through $S$.
- All these are non-degenerate, so one can propagate estimates to/from/through them, subject to restrictions on the orders.
- Apart from one of saddle points in $\mathscr{I} \cap I^0$, these only use the decay orders.

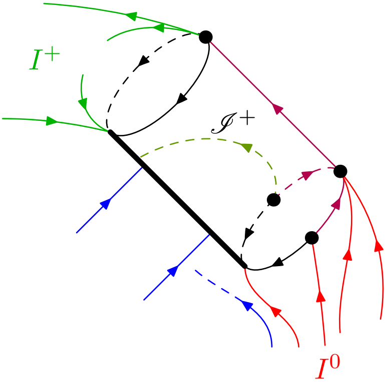



Figure: Structure of the null-bicharacteristic flow near null infinity in $2+1$ spacetime dimensions. The cross sections of the cylinder are cross sections of the future light cones inside of each fiber of the eb-phase space over $\mathscr{I}^+$. The thick black sets are the radial sets (the two antipodal points over $\mathscr{I}^+ \cap I^0$ forming a connected radial set in higher dimensions).

For a Fredholm theory two more parts are needed: gaining decay at the lift of the original boundary and at $\mathscr{I}$. These use the b-, respectively edge, normal operators.

The b-normal operators are actually the hyperbolic space and de Sitter ones, but here they are decoupled. The square root change of the boundary defining function places both in the 0-world, i.e. conformally compact spaces, as opposed to the projective compactification.

Since it is more delicate than the elliptic setting of Mazzeo, we treat the edge normal operator in a slightly more ad hoc fashion, working with a sufficiently dense set of model fibers and estimating the difference between the fiberwise and nearby models.

One could even upgrade the Fredholm theory to full invertibility, at least near $I^+$, by working on a sufficiently thin region near it (this gives a small parameter).

**Theorem (Global edge-b-regularity of waves, Hintz-V.)**

*Let $\Omega = \{t \geq 0\}$. Suppose $\alpha_+ + \frac{1}{2} < \alpha_{\mathscr{I}} < \min(-\frac{1}{2}, \alpha_0 + \frac{1}{2})$, and let $s \geq 0$. Let $f \in H_{eb}^{s-1,(\alpha_0+2,2\alpha_{\mathscr{I}}+2,\alpha_++2)}(\Omega)$. Then the forward solution $u$ of the wave equation $\Box_{g_0} u = f$ on asymptotically Minkowski space (with coefficients well-behaved on $[X; S]$) satisfies $u \in H_{eb}^{s,(\alpha_0,2\alpha_{\mathscr{I}},\alpha_+)}(\Omega)$. If $f$ enjoys additional $k$ orders of b-regularity, then $u$ enjoys additional $k$ degrees of b-regularity as well.*

Remark: Well-behaved=conormal, with leading term at $I^+$, $I^0$, decay to the model at $\mathscr{I}$, with globally non-trapping flow, and if no decay at $I^+$, $I^0$ to the model (just the same class) then a Mellin-transformed normal operator invertibility condition, satisfied for small perturbation of hyperbolic and de Sitter spaces.

- One can also look at the Klein-Gordon equation, which is much like $\Delta - 1$ in being less degenerate (works in the scattering pseudodifferential algebra, rather than 2-microlocalization, or b-).
- In particular, one can use a fully symbolic algebra: scattering/double edge.
- This was treated by Sussman in his 2023 MIT PhD thesis for the forward problem, under somewhat more restrictive assumptions. The fiber-infinity behavior is the same, but there are some additional flow complexities at finite scattering frequencies (but is symbolic, not operator valued!).
- Work in progress: Mikhail Molodyk is extending to the full Feynman framework. For K-G essentially in place; for the wave equation much more delicate.

Many happy returns, Richard!