Index theory on pin manifolds

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Let's begin with the question: Why pin manifolds?

Geometric structures

Definition: A *symmetry type* (of dim n) is a homomorphism of Lie groups $\lambda\colon G_n \to \mathrm{GL}_n\mathbb{R}$

Examples:

- $\mathrm{O}_n \hookrightarrow \mathrm{GL}_n\mathbb{R}$ (Riemannian geometry)
- $\mathrm{Spin}_n \to \mathrm{GL}_n\mathbb{R}$ (spin geometry)
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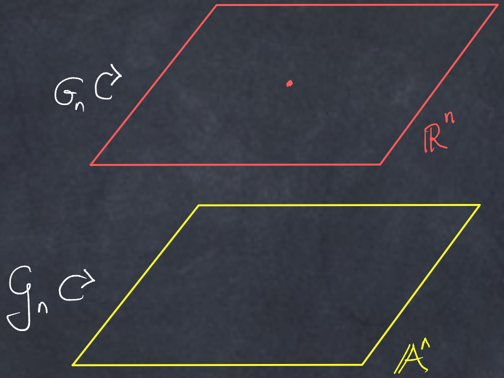
Affine symmetry group:

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & K & = & K & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbb{R}^n & \longrightarrow & \mathcal{G}_n & \dashrightarrow & G_n & \longrightarrow & 1 \\
 & & \| & & \downarrow{\scriptstyle \tilde{\lambda}} & & \downarrow{\scriptstyle \lambda} & & \\
1 & \longrightarrow & \mathbb{R}^n & \longrightarrow & \mathrm{Aff}_n & \longrightarrow & \mathrm{GL}_n\mathbb{R} & \longrightarrow & 1
\end{array}$$



Symmetry $\rightsquigarrow$ *structure* on a smooth manifold M, encoded in a lift of the frame bundle:

$$\begin{array}{c} (P,\Theta) \\ \downarrow{\scriptstyle G_n} \\ M \end{array} \qquad \begin{array}{ccc} \mathcal{B}(M) & \xrightarrow[\cong]{\theta} & \lambda(P) \\ & {\scriptstyle \mathrm{GL}_n\mathbb{R}}\searrow \quad \swarrow{\scriptstyle \mathrm{GL}_n\mathbb{R}} & \\ & M & \end{array} \qquad (\Theta \text{ is a } G_n\text{-connection})$$

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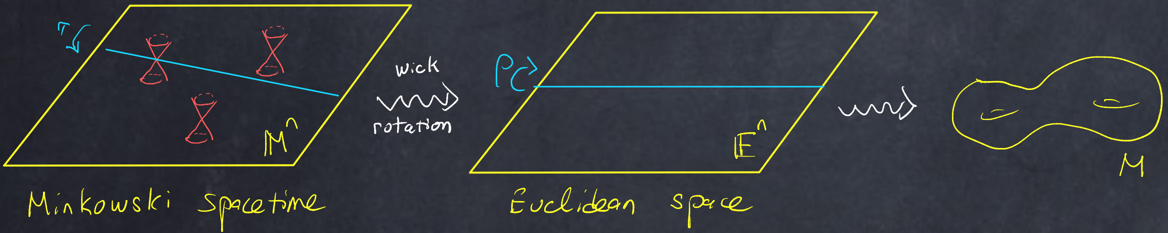

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Relativistic quantum theory with time-reversal symmetry and spinors $\rightsquigarrow$ pin manifolds

Pin groups and Clifford algebras

$\mathrm{Cliff}_{p,q} : e_1^2 = \cdots = e_p^2 = +1, \quad e_{p+1}^2 = \cdots = e_{p+q}^2 = -1$

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$\mathrm{Pin}_n^+ = \mathrm{Pin}_{n,0} \qquad \mathrm{Pin}_n^- = \mathrm{Pin}_{0,n} \qquad \mathrm{Cliff}_{+n} = \mathrm{Cliff}_{n,0} \qquad \mathrm{Cliff}_{-n} = \mathrm{Cliff}_{0,n}$

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In low dimensions there are special isomorphisms:

n	Spin_n	Pin_n^+	Pin_n^-
1	μ_2	$\mu_2 \times \mu_2$	μ_4
2	$\mathbb{T}$	$\mu_2 \ltimes \mathbb{T}$	$(\mu_4 \ltimes \mathbb{T})/\mu_2$
3	SU_2	$(\mu_4 \times \mathrm{SU}_2)/\mu_2$	$\mu_2 \times \mathrm{SU}_2$

Key observation: There exist embeddings

$$\begin{aligned}
\mathrm{Pin}_n^+ &\hookrightarrow \mathrm{Spin}_{n,1} &\subset \mathrm{Cliff}^0_{n,1} &\cong [\mathrm{Cliff}_{+n} \otimes \mathrm{Cliff}_{-1}]^0 \\
\mathrm{Pin}_n^- &\hookrightarrow \mathrm{Spin}_{n+1} &\subset \mathrm{Cliff}^0_{+(n+1)} &\cong [\mathrm{Cliff}_{+n} \otimes \mathrm{Cliff}_{+1}]^0
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$$\begin{aligned}
g &\longmapsto g \otimes 1, && g \in \mathrm{Spin}_n \\
g &\longmapsto g \otimes e, && g \in \mathrm{Pin}_n^{\pm} \setminus \mathrm{Spin}_n
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Note the Morita equivalence

$$\mathrm{Cliff}_{n,1} \cong \mathrm{Cliff}_{+(n-1)} \otimes \mathrm{Cliff}_{1,1} \overset{\text{Morita}}{\simeq} \mathrm{Cliff}_{+(n-1)}$$

and so the opposite shifts

A 10-fold way

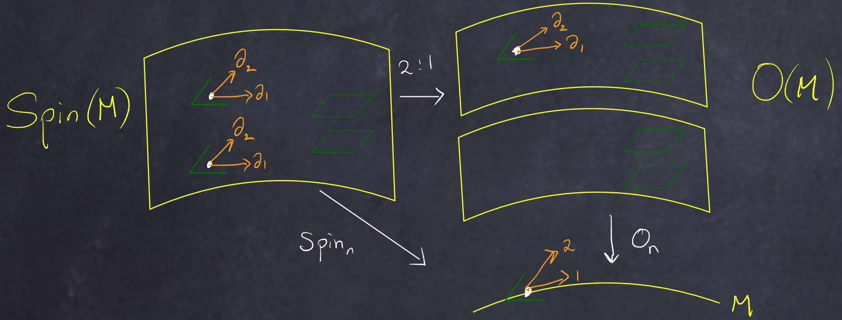
Theorem: There are embeddings $H_n(s) \hookrightarrow \text{Cliff}_{+n} \otimes D(s)$ compatible with Clifford multiplication

s	H^c	K	Cartan	D
0	Spin^c	$\mathbb{T}$	A	$\mathbb{C}$
1	Pin^c	$\mathbb{T}$	AIII	$\text{Cliff}^{\mathbb{C}}_{-1}$

s	H	K	Cartan	D
0	Spin	μ_2	D	$\mathbb{R}$
-1	Pin^+	μ_2	DIII	Cliff_{-1}
-2	$\text{Pin}^+ \ltimes_{\{\pm 1\}} \mathbb{T}$	$\mathbb{T}$	AII	Cliff_{-2}
-3	$\text{Pin}^- \times_{\{\pm 1\}} \text{Sp}_1$	Sp_1	CII	Cliff_{-3}
4	$\text{Spin} \times_{\{\pm 1\}} \text{Sp}_1$	Sp_1	C	$\mathbb{H}$
3	$\text{Pin}^+ \times_{\{\pm 1\}} \text{Sp}_1$	Sp_1	CI	Cliff_{+3}
2	$\text{Pin}^- \ltimes_{\{\pm 1\}} \mathbb{T}$	$\mathbb{T}$	AI	Cliff_{+2}
1	Pin^-	μ_2	BDI	Cliff_{+1}

The Clifford linear Dirac operator

M	Riemannian spin manifold
$\mathrm{O}(M) \longrightarrow M$	bundle of orthonormal frames
$\partial_1, \dots, \partial_n$	tautological horizontal vector fields
$\mathrm{Spin}(M) \longrightarrow \mathrm{O}(M) \longrightarrow M$	lift to principal Spin_n-bundle
$\mathrm{Spin}_n \subset \mathrm{Cliff}_{+n} \circlearrowright \mathrm{Cliff}_{+n} \circlearrowleft \mathrm{Cliff}_{+n}$	left regular Cliff_{+n}-module



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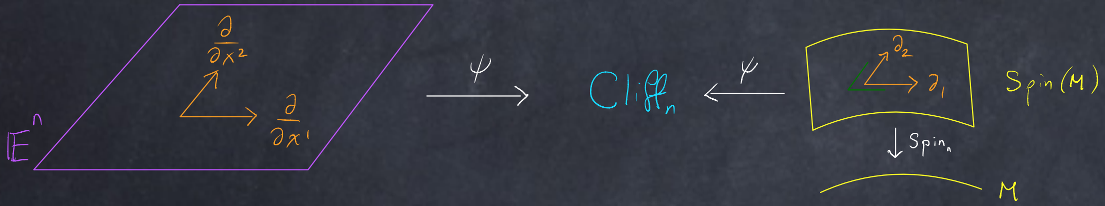

$\mathbb{E}^n$: $\quad D = \gamma^1 \frac{\partial}{\partial x^1} + \cdots + \gamma^n \frac{\partial}{\partial x^n} \circlearrowright \left(\psi\colon \mathbb{E}^n \longrightarrow \mathrm{Cliff}_{+n}\right) \circlearrowleft \mathrm{Cliff}_{+n}$

M : $\quad D = \gamma^1 \partial_1 + \cdots + \gamma^n \partial_n \circlearrowright \left(\psi\colon \mathrm{Spin}(M) \longrightarrow \mathrm{Cliff}_{+n}\right) \circlearrowleft \mathrm{Cliff}_{+n}$

Modification for a Pin manifold:

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is equivalent to commuting left actions

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Therefore, the Dirac operators

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$$M : \qquad D = \gamma^1 \partial_1 + \cdots + \gamma^n \partial_n \circlearrowright \left(\psi \colon \mathrm{Pin}^+(M) \longrightarrow \mathrm{Cliff}_{n,1}\right)$$

have a commuting left $\mathrm{Cliff}_{1,n} \overset{\text{Morita}}{\simeq} \mathrm{Cliff}_{-(n-1)}$ action

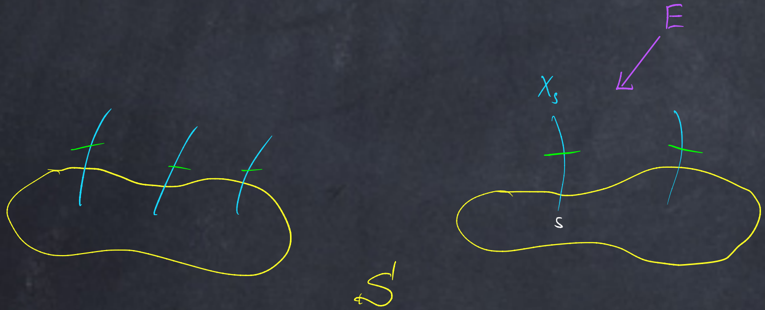
Atiyah-Singer index theorem for Dirac operators

$\pi\colon X \longrightarrow S$ proper fiber bundle of relative dimension n

relative spin structure

Riemannian structure on π (relative metric + horizontal distribution)

$E \longrightarrow X$ orthogonal vector bundle with compatible ∇



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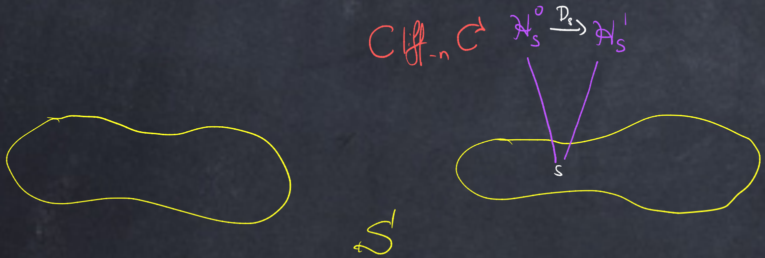

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Modification for Pin^+

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relative pin^+ structure

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To extract more information we turn to differential K-theory and secondary invariants

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Precursors: Deligne cohomology (1971) and Cheeger-Simons differential characters (1973)

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This only scratches the surface; see the recent survey by Debray and Amabel-Debray-Haine

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$$\check{H}^2(M) \cong \{\text{principal } \mathbb{R}/\mathbb{Z}\text{-connections } (P,\Theta) \longrightarrow M\} \;/\; \text{isomorphism}$$

$$\begin{array}{ccc}
\check{H}^q(M) & \xrightarrow{\text{curvature}} & \Omega^q_{\mathbb{Z}}(M) \\
{\scriptstyle \pi_0}\downarrow & & \downarrow{\scriptstyle \text{de Rham}} \\
H^q(M;\mathbb{Z}) & \longrightarrow & H^q(M;\mathbb{R})
\end{array}$$

This is a *commutative* square of abelian groups, but not a *pullback* square: try $q = 1$ and $M = \text{pt}$

$$
\begin{array}{ccc}
\check{H}^q(M) & \xrightarrow{\text{curvature}} & \Omega^q_{\mathbb{Z}}(M) \quad \omega \\
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c \quad H^q(M;\mathbb{Z}) & \longrightarrow & H^q(M;\mathbb{R}) \quad h
\end{array}
$$

$$
\begin{cases} c \in C^q(M;\mathbb{Z}) \\ \omega \in \Omega^q(M) \\ h \in C^{q-1}(M;\mathbb{R}) \end{cases}
\qquad
\begin{cases} \delta c = 0 \\ d\omega = 0 \\ \delta h = \omega - c \end{cases}
$$

One constructs $\check{H}^q(M)$ by a *homotopy* pullback

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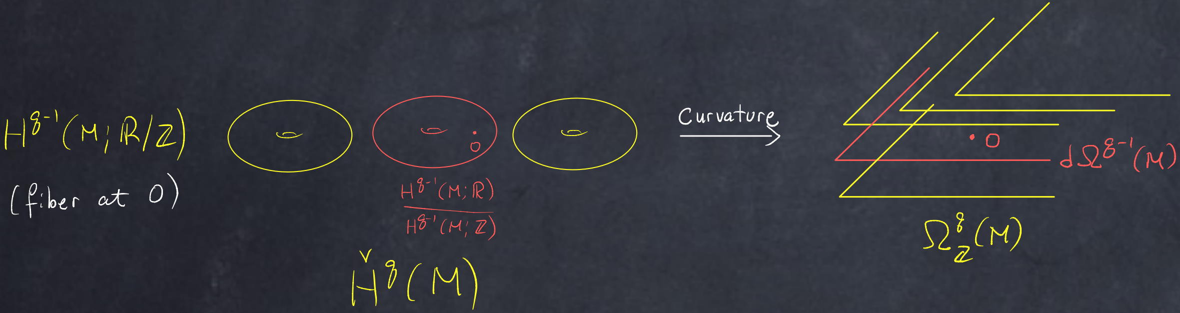

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- calculus of *differential cocycles* lifts calculus of differential forms
- interplay of π_0 and ι gives topological information beyond cohomology

Differential KO-theory

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KO^q(M) & \longrightarrow & H\big(M;\mathbb{R}[v,v^{-1}]\big)^q
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$$\iota\colon \mathbb{R}/\mathbb{Z} \cong K^0(\mathrm{pt};\mathbb{R}/\mathbb{Z}) \xhookrightarrow{\cong} \check{K}^1(\mathrm{pt})$$

Differential K-theory and η-invariants

Theorem (Klonoff): Let M be a closed n-dimensional spin^c manifold, and suppose $E \to M$ is a unitary vector bundle with covariant derivative. Then the pushforward $\check{\pi}_!^M : \check{K}^0(M) \longrightarrow \check{K}^{-n}(\mathrm{pt})$ is

$$\check{\pi}_!^M([\check{E}]) = \begin{cases} \operatorname{ind} D_M(E), & n \text{ even} \\ \xi_M(E) \pmod 1, & n \text{ odd} \end{cases}$$

$\xi = \frac{\eta + \dim\ker}{2}$ is the Atiyah-Patodi-Singer η-invariant and $\check{K}^{-n}(\mathrm{pt}) \cong \begin{cases} \mathbb{Z}, & n \text{ even} \\ \mathbb{R}/\mathbb{Z}, & n \text{ odd} \end{cases}$

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- A thorough development of geometric index theory using $\check{K}$- and $\widecheck{KO}$-theory is needed

Differential KO-theory and η-invariants on spin manifolds

$$\widecheck{KO}^{-n}(\mathrm{pt}) \cong \begin{cases} \mathbb{R}/\mathbb{Z}, & n \equiv 7 \pmod 8 \\ 0, & n \equiv 6 \pmod 8 \\ 0, & n \equiv 5 \pmod 8 \\ \mathbb{Z}, & n \equiv 4 \pmod 8 \\ \mathbb{R}/\mathbb{Z}, & n \equiv 3 \pmod 8 \\ \mathbb{Z}/2\mathbb{Z}, & n \equiv 2 \pmod 8 \\ \mathbb{Z}/2\mathbb{Z}, & n \equiv 1 \pmod 8 \\ \mathbb{Z}, & n \equiv 0 \pmod 8 \end{cases}$$

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For $n \equiv 3 \pmod 8$ the correct invariant is $\xi/2 \pmod 1$

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Key point: the shift by one implies the characteristic differential form that computes the variation has odd degree, so it vanishes

Example: (n=4) The pushforward $M \longmapsto \check{\pi}_!^M(1)$ induces an isomorphism

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Remark: Another example of this phenomenon: Let $E \longrightarrow M$ be a rank r real vector bundle, r odd. The usual Euler class is $\beta_{\mathbb{Z}}\big(w_{r-1}(E)\big) \in H^r(M;\mathbb{Z})$. The Euler class in differential cohomology is $w_{r-1}(E)$, interpreted as an element of order 2 in $H^{r-1}(M;\mathbb{R}/\mathbb{Z})$, the flat subgroup of $\check{H}^r(M)$. There is a similar statement for the Euler class in K-theory and KO-theory.

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Problem: Produce topological formulas for $\xi_M(E)$ and $\xi_M(E)/2$

M-Theory from 11d supergravity

Fields in M-theory ($\mathcal{F}$): ρ pin$^+$ structure

g Riemannian metric

ψ Rarita-Schwinger field

C local 3-form, field strength is global closed 4-form

SUPERGRAVITY THEORY IN 11 DIMENSIONS

E. CREMMER, B. JULIA and J. SCHERK

Laboratoire de Physique Théorique de l'Ecole Normale Supérieure+
Paris, France

Abstract : We present the action and transformation laws of supergravity in 11 dimensions which is expected to be closely related to the O(8) theory in 4 dimensions after dimensional reduction.

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The Lagrangian we find is the following :

$$\mathcal{L} = -\frac{V}{4\kappa^2} R(\omega) - \frac{iV}{2} \bar{\psi}_\mu \Gamma^{\mu\nu\rho} D_\nu\left(\frac{\omega+\hat{\omega}}{2}\right)\psi_\rho - \frac{V}{48} F_{\mu\nu\rho\sigma} F^{\mu\nu\rho\sigma}$$
$$+ \frac{\kappa V}{192}\left(\bar{\psi}_\mu \Gamma^{\mu\nu\alpha\beta\gamma\delta}\psi_\nu + 12\bar{\psi}^\alpha \Gamma^{\gamma\delta}\psi^\beta\right)\left(F_{\alpha\beta\gamma\delta} + \hat{F}_{\alpha\beta\gamma\delta}\right)$$
$$+ \frac{2\kappa}{(144)^2} \varepsilon^{\alpha_1\alpha_2\alpha_3\alpha_4\beta_1\beta_2\beta_3\beta_4\mu\nu\rho} F_{\alpha_1\alpha_2\alpha_3\alpha_4} F_{\beta_1\beta_2\beta_3\beta_4} A_{\mu\nu\rho}$$

$F = dA$

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Definition: Let M be a pin$^+$ manifold. An $\mathfrak{m}_c$ *structure* on M is a w_1-twisted integer lift of $w_4(M)$. Compare: spin^c structure = integer lift of $w_2(M)$

The Rarita-Schwinger anomaly

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This motivated us to find topological formulas for this invariant

Generators of the $\mathfrak{m}_c$ bordism group

Theorem: The following six $\mathfrak{m}_c$-manifolds generate the group $\pi_{12} M\mathfrak{m}_c \otimes \mathbb{Z}_2$:

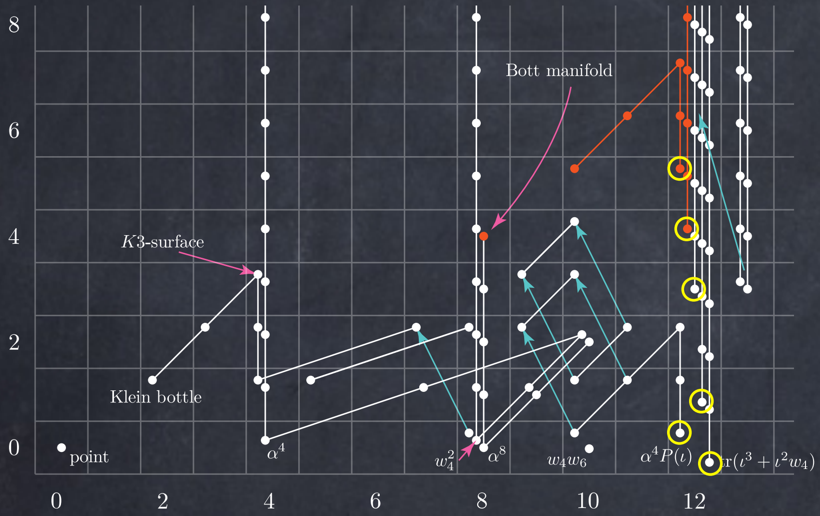
$$(W_0', c_0), \quad (W_0'', 0), \quad (W_1, \lambda)$$

$$(K \times \mathbb{HP}^2, \lambda), \quad (\mathbb{RP}^4, c_{\mathbb{RP}^4}) \times B, \quad (\mathbb{RP}^4 \# \mathbb{RP}^4, 0) \times B.$$

K	$K3$ surface
$\mathbb{HP}^2$	quaternionic projective plane
B	Bott manifold
$\mathbb{HP}^2 \# \mathbb{HP}^2 \longrightarrow W_0' \longrightarrow \mathbb{RP}^4$	$S^4 \times (\mathbb{HP}^2 \# \mathbb{HP}^2) \xrightarrow{2:1} W_0'$
$\mathbb{RP}^8 \longrightarrow W_0'' = \mathbb{P}(K_{\mathbb{R}}^{\oplus 2} \oplus \underline{\mathbb{R}}) \xrightarrow{\rho} S^4$	$K_{\mathbb{R}} \to S^4$ generating $\mathbb{H}$-line bundle
$\mathbb{HP}^2 \longrightarrow W_1 \longrightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$	$\mathcal{B}_{SO}\big(\mathcal{O}(1,1)_{\mathbb{R}} \oplus \underline{\mathbb{R}} \to \mathbb{CP}^1 \times \mathbb{CP}^1\big)$
	$SO_3 \cong \mathbb{P}\,Sp_1 \circlearrowright \mathbb{HP}^2$

Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_{\mathbf{A}}^{s,t}(H^* M\mathfrak{m}_c, \mathbb{Z}/2\mathbb{Z}) \Rightarrow \pi_{t-s} M\mathfrak{m}_c \otimes \mathbb{Z}_2$$



Topological computations of $\xi_M(E)/2$

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Method 2 (Stolz): $\pi\colon \widehat{M} \longrightarrow M$ orientation double cover with free orientation-reversing involution $\sigma\colon \widehat{M} \to \widehat{M}$. Suppose $\widehat{M} = \partial Z$, Z compact pin$^+$, and $\pi^*E \longrightarrow \widehat{M}$ extends over Z, as does σ. If the extension has a finite set $\{f\}$ of fixed points then (based on APS, Donnelly)

$$\xi_M(E)/2 = \sum_f \frac{\epsilon_f \tau_f}{2^8}, \qquad \epsilon_f = \pm 1, \quad \tau_f = \text{trace of involution on fiber}$$

Generators of the $\mathfrak{m}_c$ bordism group

Theorem: The following six $\mathfrak{m}_c$-manifolds generate the group $\pi_{12} M\mathfrak{m}_c \otimes \mathbb{Z}_2$:

$$(W_0', c_0), \quad (W_0'', 0), \quad (W_1, \lambda)$$

$$(K \times \mathbb{HP}^2, \lambda), \quad (\mathbb{RP}^4, c_{\mathbb{RP}^4}) \times B, \quad (\mathbb{RP}^4 \# \mathbb{RP}^4, 0) \times B.$$

K — $K3$ surface

$\mathbb{HP}^2$ — quaternionic projective plane

B — Bott manifold

$\mathbb{HP}^2 \# \mathbb{HP}^2 \longrightarrow W_0' \longrightarrow \mathbb{RP}^4$ — $S^4 \times (\mathbb{HP}^2 \# \mathbb{HP}^2) \xrightarrow{2:1} W_0'$

$\mathbb{RP}^8 \longrightarrow W_0'' = \mathbb{P}(K_{\mathbb{R}}^{\oplus 2} \oplus \underline{\mathbb{R}}) \xrightarrow{\rho} S^4$ — $K_{\mathbb{R}} \to S^4$ generating $\mathbb{H}$-line bundle

$\mathbb{HP}^2 \longrightarrow W_1 \longrightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$ — $\mathcal{B}_{SO}(\mathcal{O}(1,1)_{\mathbb{R}} \oplus \underline{\mathbb{R}} \to \mathbb{CP}^1 \times \mathbb{CP}^1)$

$\mathrm{SO}_3 \cong \mathbb{P}\,\mathrm{Sp}_1 \circlearrowright \mathbb{HP}^2$

problem child

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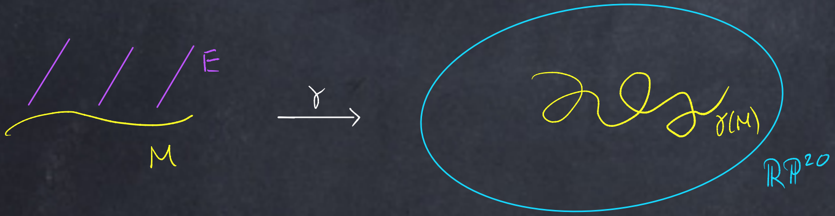

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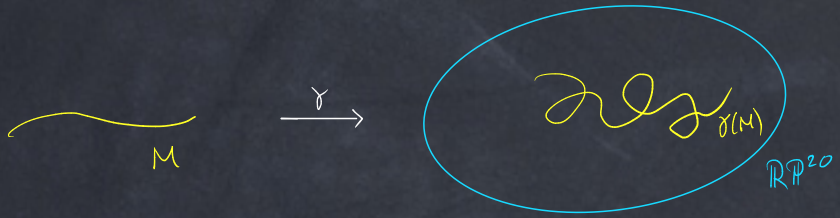
Problem: Give a topological proof of Theorem

Possible approach

$$\begin{array}{ccc} M & \xrightarrow{\gamma} & \mathbb{RP}^{20} \\ & \pi^M \searrow \quad \swarrow \pi^{\mathbb{RP}^{20}} & \\ & \mathrm{pt} & \end{array}$$



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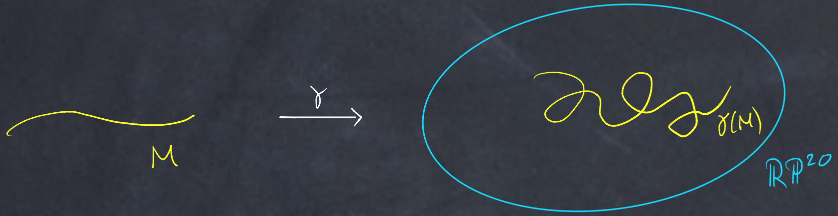

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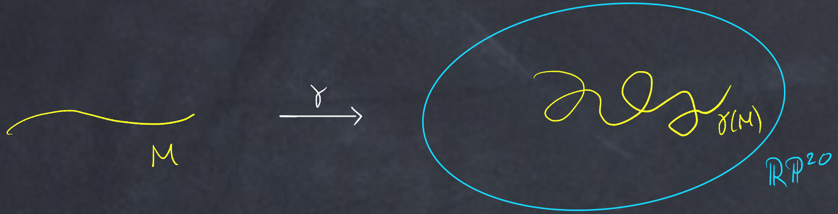

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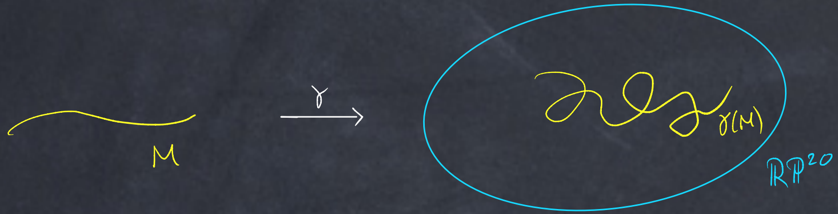

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This formulation illustrates the interplay of differential and topological aspects of differential cohomology, and should be an instance of a more general principle

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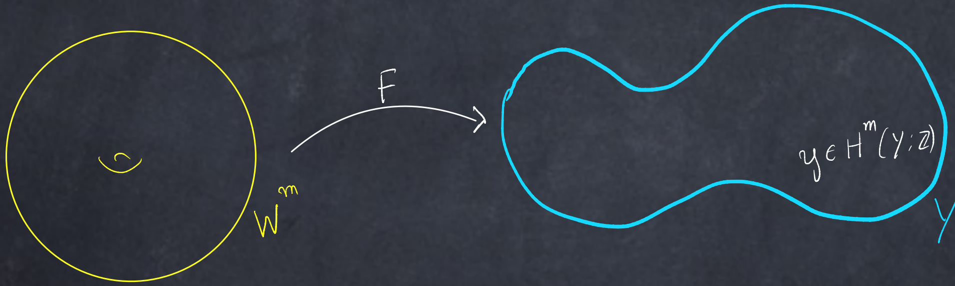

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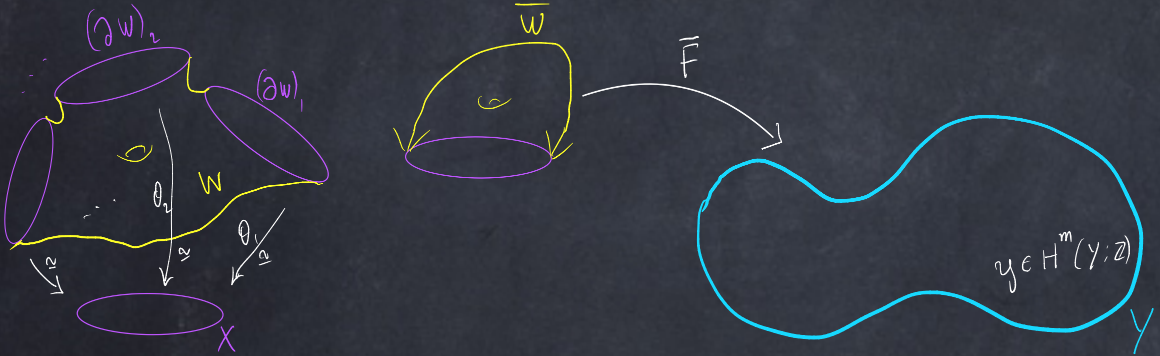

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Richard and I proved a $\mathbb{Z}/k\mathbb{Z}$ analog of the Atiyah-Singer index theorem that equates a-ind and t-ind for symbols of elliptic operators; it can be used to compute $\mathbb{Z}/k\mathbb{Z}$-periods of a K-theory class (Higson gave an alternative proof using C^*-algebras)

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Combine the mod k index theorem with Atiyah-Patodi-Singer for pin$^+$ manifolds to obtain

Theorem: $\xi_M(E)/2 = \pi_!\big([E]\big)$

Happy Birthday, Richard!