II. Dynamical zeta functions

1. General results

- $\varphi^t : M \circlearrowleft$ Anosov flow
- A closed orbit is a pair $\gamma = (x_0, T)$ where $T > 0$ (called the period) and $x_0 \in M$, $\varphi^T(x_0) = x_0$

We identify (x_0, T) with $(\varphi^t(x_0), T)$ $\forall t \in \mathbb{R}$

- The primitive period of γ is the minimal $T^\# > 0$ s.t. $\varphi^{T^\#}(x_0) = x_0$.

If $T = T^\#$ we say γ is a primitive closed orbit

- Poincaré map: for γ a closed orbit,

$$P_\gamma := d\varphi^{-T}(x_0)\Big|_{E_u(x_0) \oplus E_s(x_0)}.$$

Since φ^t is an Anosov flow, $\det(I - P_\gamma) \neq 0$.

Ruelle zeta function:

$$\zeta(\lambda) = \prod_{\gamma^\#} \left(1 - e^{-\lambda T_{\gamma^\#}}\right), \quad \mathrm{Re}\,\lambda \gg 1$$

where the product is over primitive closed orbits $\gamma^\#$ and $T_{\gamma^\#}$ is the period

The above product <u>converges</u> for $\mathrm{Re}\,\lambda \gg 1$ as $\#(\text{closed orbits of period} \leq T)$ grows at most exponentially for Anosov flows

<u>Example:</u> $M = \mathbb{R}^2_{x,y} \times \mathbb{S}^1_\theta$, $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$

$X = x\partial_x - y\partial_y + \partial_\theta$ (again, not compact...)

Only 1 primitive orbit $\{x=y=0\}$ of period 1. So

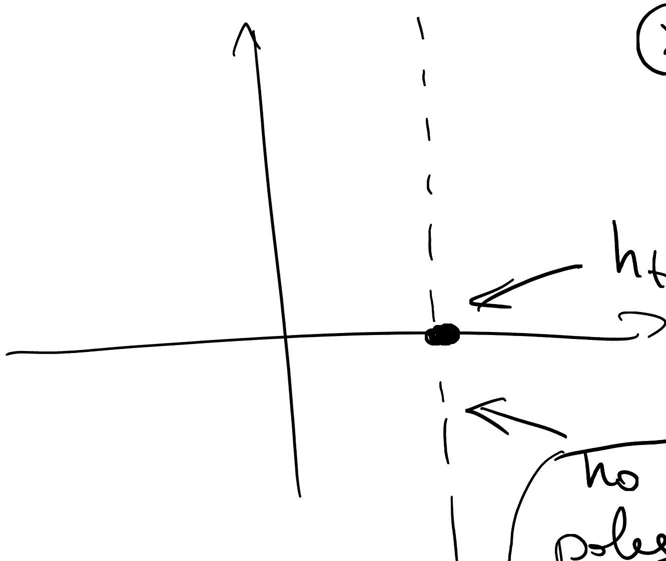
$\zeta(\lambda) = 1 - e^{-\lambda}$. <u>Zeros:</u> $\lambda = 2\pi i k$, $k \in \mathbb{Z}$

Thm [Meromorphic continuation]

The function $\zeta(\lambda)$ admits a meromorphic extension to $\lambda \in \mathbb{C}$.

References:
Conjectured by Smale '67
Proved by Giulietti–Liverani–Pollicott '13
We use the later proof: Dyatlov–Zworski '16

The structure of the singularities (zeroes/poles) of $\zeta(\lambda)$ is as follows:



λ

$h_{top} > 0$, the topological entropy of the flow φ_t

no other poles with $\mathrm{Re}\,\lambda > h_{top}$

Can be used to show

Thm [Prime Orbit Thm] $\boxed{\text{Margulis '69}}$

Under the same assumptions as the Mixing Thm

$$\#\left(\substack{\text{primitive closed orbits} \\ \text{of period} \le T}\right) \sim \frac{e^{h_{top} T}}{T} \quad \text{as } T \to \infty$$

(2) Sketch of proof of meromorphic extension of ζ

Step 1: After some manipulation (easier if E_u is an orientable bundle...)

we can write $\zeta(\lambda)^{\pm 1} = \prod_{k=0}^{n-1} \zeta_k(\lambda)^{(-1)^k}$

where $n = \dim M$

and $\zeta(\lambda)$ is a more complicated looking zeta-function corresponding to k-forms

We just write down ζ_0, or in fact, its log-derivative:

$$\frac{\zeta_0'(\lambda)}{\zeta_0(\lambda)} = \sum_{\gamma} \frac{T_\gamma^\# e^{-\lambda T_\gamma}}{|\det(I - P_\gamma)|}$$

where the sum is over <u>all</u> (not just primitive) closed orbits

Step 2: use the

Atiyah-Bott-Guillemin trace formula:

the *flat trace* of $e^{-tX}: f \mapsto f \circ \varphi^{-t}$

is given by

delta function

$$\operatorname{tr}^\flat e^{-tX} = \sum_{\gamma} \frac{T_\gamma^\# \, \delta(t - T_\gamma)}{|\det(I - P_\gamma)|}$$

all closed orbits

Here the flat trace of an operator A is defined as the integral of the restriction of the integral kernel of A to the diagonal:

if $Af(x) = \int_M K(x,x') f(x') \, d\mu(x')$ then

$$\operatorname{tr}^\flat A = \int_M K(x,x) \, d\mu(x).$$

Here $K \in \mathcal{D}'(M \times M)$ is a distribution, so we can't always define $K|_{\{x = x'\}}$.

Need a condition on the wavefront set $WF(K)$.

Example of the trace formula:

Same example as before

$(X = x\partial_x - y\partial_y + \partial_\theta)$.

Closed trajectories : at $x = y = 0$

with period $T_\gamma = \ell \in \mathbb{N}$

primitive period $T_\gamma^\# = 1$

Poincaré map $P_\gamma = \begin{pmatrix} e^{-\ell} & 0 \\ 0 & e^{\ell} \end{pmatrix}$

$\det(I - P_\gamma) = (1 - e^{-\ell})(1 - e^{\ell})$.

Integral kernel of e^{-tX} :

$$e^{-tX} f(z) = f(\varphi^{-t}(z))$$

$$= \int_M \delta(z' - \varphi^{-t}(z))\, f(z')\, dz'$$

So $K(z, z') = \delta(z' - \varphi^{-t}(z))$

In our example,

$$K(x, y, \theta, x', y', \theta')$$
$$= \delta(x' - e^{-t}x) \cdot \delta(y' - e^{t}y) \cdot \delta(\theta' - \theta - t \bmod \mathbb{Z})$$

Compute the flat trace:

$$\operatorname{tr}^{b} e^{-tX} = \int_{\mathbb{R}^2_{x,y} \times S^1_\theta} K(x, y, \theta, x, y, \theta)\, dx\, dy\, d\theta$$

$$= \int_{\mathbb{R}^2 \times S^1} \delta((1-e^{-t})x)\, \delta((1-e^{t})y)\, \delta(t \bmod \mathbb{Z})\, dx\, dy$$

$$= \sum_{\ell \geq 0} \frac{\delta(t-\ell)}{|1-e^{-\ell}| \cdot |1-e^{\ell}|}$$

which is the RHS in the trace formula

Here we use that $\delta(ax) = \frac{1}{|a|}\delta(x)$

and $\int_{\mathbb{R}^2} \delta(x)\delta(y) = 1$

Step 3: use the resolvent

We have for $\mathrm{Re}\,\lambda \gg 1$

$$\frac{\zeta_0'}{\zeta_0}(\lambda) = \sum_{\gamma} \frac{T_\gamma^{\#} e^{-\lambda T_\gamma}}{|\det(I - P_\gamma)|}$$

$$= \int_0^\infty e^{-\lambda t}\, \mathrm{tr}^\flat (e^{-tX})\, dt$$

$0 \leftarrow$ (should replace by $\varepsilon > 0$ but let's not bother)

The strategy is to take the flat trace out of the $\int$:

$$\frac{\zeta_0'(\lambda)}{\zeta_0(\lambda)} = \mathrm{tr}^\flat \int_0^\infty e^{-\lambda t} e^{-tX} dt = \mathrm{tr}^\flat R(\lambda)$$

where $R(\lambda) = (X+\lambda)^{-1}$.

We then use the meromorphic continuation $R(\lambda) : C^\infty \to \mathcal{D}'$, $\lambda \in \mathbb{C}$ and the fact that $WF(R(\lambda))$ does satisfy the condition needed to make sense of $\mathrm{tr}^\flat$ to meromorphically continue $\frac{\zeta_0'}{\zeta_0}$ and then ζ_0.

Note: the formula

$$\frac{\zeta_0'(\lambda)}{\zeta_0(\lambda)} = \mathrm{tr}^b (X+\lambda)^{-1}$$

means that we can formally interpret $\zeta_0(\lambda)$ as the "characteristic polynomial" of X (on the "right anisotropic spaces"):

$$\boxed{\zeta_0(\lambda) \text{ "="} \det(X+\lambda).}$$

In general we get

$$\zeta_k(\lambda) \text{ "="} \det(\mathcal{L}_X + \lambda)$$

with the Lie derivative $\mathcal{L}_X$ acting on differential k-forms ω on M which satisfy $\iota_X \omega = 0$.

3) Connections to topology

Here we look at the order of vanishing $m_R(0)$ of $\zeta(\lambda)$ at $\lambda = 0$,
i.e. $\lambda^{-m_R(0)}\zeta(\lambda)$ is holomorphic and $\neq 0$ at $\lambda = 0$

Fried '86 : if φ^t is the geodesic flow on a hyperbolic surface Σ
then $m_R(0) = -\chi(\Sigma)$

Euler characteristic

Also did higher dimensional hyperbolic manifolds and related the value $\zeta(0)$ (for twisted ζ functions...)
to the analytic/Reidemeister torsion which is a topological invariant

Shen '16 : a similar result (Fried's conjecture) for all compact locally symmetric spaces (real rank 1)

What happens for more general negatively curved manifolds?

Dyatlov-Zworski '17:

$m_R(0) = -\chi(\Sigma)$ for any negatively curved surface Σ

Dang-Guillarmou-Rivière-Shen '20:

Fried's conjecture on $\zeta(0)$ (when $m_R(0)=0$) when Σ is a nearly hyperbolic 3-manifold

Cekić-Delarue-Dyatlov-Paternain '22:

for nearly hyperbolic 3-manifolds $m_R(0)$ is different generically than for exactly hyperbolic ones.