EXERCISES FOR THE MINICOURSE ON FRACTAL UNCERTAINTY PRINCIPLE (WITH SOLUTIONS)

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Abstract. These are companion exercises to the minicourse given at the Spring School on Transfer Operators, organized by the Bernoulli Center, Lausanne, in March 2021.

1. Describe all the elements $\gamma\in \mathrm{SL}(2,\mathbb{R})$ such that

$$\gamma(\overline{\mathbb{R}}\setminus I_2^\circ)=I_1\quad\text{where}\quad I_1:=[1,2],\quad I_2:=[-1,0].$$

Note that these γ are all hyperbolic, i.e. $|\operatorname{tr}\gamma|>2$, which implies that γ has two fixed points on $\mathbb{R}$, one attractive and one repulsive. Find these fixed points. Show that any point in I_1° is the attractive point of some γ and similarly for repulsive points and I_2°.

Solution: We need

$$\gamma(-1)=2,\quad \gamma(0)=1.$$

Writing

$$\gamma=\begin{pmatrix}a & b\\ c & d\end{pmatrix},\quad ad-bc=1$$

we get the equations

$$\frac{b-a}{d-c}=2,\quad \frac{b}{d}=1.$$

Writing out in terms of a,b, we get

$$c=\frac{a+b}{2},\quad d=b,$$

and using the equation $ad-bc=1$ we get

$$(a-b)b=2.$$

So it makes sense to parametrize by $b\neq 0$, obtaining

$$\gamma=\begin{pmatrix}b+\frac{2}{b} & b\\ b+\frac{1}{b} & b\end{pmatrix},\quad \gamma(x)=1+\frac{x}{(b^2+1)x+b^2}.$$

The fixed point equation is $\gamma(x)=x$, which can be written as the quadratic equation

$$cx^2+(d-a)x-b=0$$

which has solutions

$$x_\pm = \frac{a-d\pm\sqrt{(a+d)^2-4}}{2c} = \frac{1\pm\sqrt{b^4+b^2+1}}{b^2+1}.$$

To see which one is attractive and which one is repulsive, compute

$$\gamma'(x_\pm) = \frac{1}{(cx_\pm+d)^2} \quad \text{where} \quad cx_\pm + d = \frac{a+b\pm\sqrt{(a+d)^2-4}}{2}.$$

We see that $\gamma'(x_+) < 1 < \gamma'(x_-)$, so x_+ is the attractive point and x_- is the repulsive one. From the mapping properties of γ, or by direct computation, we see that $x_+ \in I_1$ and $x_- \in I_2$. Moreover, as $b \to 0$ we have

$$x_+ \to 2, \quad x_- \to 0$$

and as $b \to \infty$ we have

$$x_+ \to 1, \quad x_- \to -1$$

which gives the last statement.

2. Let $\Gamma \subset \mathrm{SL}(2,\mathbb{R})$ be a Schottky group, with generators $\gamma_1, \dots, \gamma_r$. Show that it is a free group with these generators, i.e. for any word $\mathbf{a} \in \mathcal{W}$, if $\gamma_{\mathbf{a}} = I$ then $\mathbf{a} = \emptyset$.

Solution: Assume that $\mathbf{a} = a_1 \dots a_n$ is a nonempty word. Since ∞ is contained in the complement of $I_{\overline{a_n}}$, we have $\gamma_{a_n}(\infty) \in I_{a_n}$. Since $a_n \neq \overline{a_{n-1}}$, $\gamma_{a_n}(\infty)$ is in the complement of $I_{\overline{a_{n-1}}}$, thus $\gamma_{a_{n-1}a_n}(\infty) \in I_{a_{n-1}}$. Repeating this argument, we get $\gamma_{\mathbf{a}}(\infty) \in I_{a_1}$. In particular, $\gamma_{\mathbf{a}}(\infty) \neq \infty$, so $\gamma_{\mathbf{a}}$ cannot be the identity.

3. This exercise explains why elements of Schottky groups have bounded distortion.

(a) We first discuss the way that a general element $\gamma \in \mathrm{SL}(2,\mathbb{R})$ can map an interval to another interval. Assume that $I, J \subset \mathbb{R}$ are intervals such that $\gamma(I) = J$. Define the *distortion factor of* γ *on* I by

$$\alpha(\gamma, I) := \log \frac{\gamma^{-1}(\infty) - x_1}{\gamma^{-1}(\infty) - x_0} \in \mathbb{R} \quad \text{where} \quad I = [x_0, x_1].$$

(If $\gamma^{-1}(\infty) = \infty$, that is γ is an affine map, then we put $\alpha(\gamma, I) := 0$.) Show that γ can be factorized as

$$\gamma = \gamma_J\, \gamma_{\alpha(\gamma,I)}\, \gamma_I^{-1}, \quad \gamma_\alpha := \begin{pmatrix} e^{\alpha/2} & 0 \\ e^{\alpha/2} - e^{-\alpha/2} & e^{-\alpha/2} \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$$

where $\gamma_I, \gamma_J \in \mathrm{SL}(2,\mathbb{R})$ are the affine maps such that $\gamma_I([0,1]) = I$, $\gamma_J([0,1]) = J$.

(b) Show that for each R there exists C such that in the notation of part (a)

$$|\alpha(\gamma, I)| \le R \quad \Longrightarrow \quad C^{-1}\frac{|J|}{|I|} \le \gamma'(x) \le C\frac{|J|}{|I|} \quad \text{for all} \quad x \in I.$$

(c) Let Γ be a Schottky group generated by $\gamma_1, \dots, \gamma_r \in \mathrm{SL}(2, \mathbb{R})$. Show that there exists C_Γ such that for all nonempty $\mathbf{a} = a_1 \dots a_n \in \mathcal{W}$ we have

$$C_\Gamma^{-1}|I_{\mathbf{a}}| \leq \gamma'_{\mathbf{a}'}(x) \leq C_\Gamma |I_{\mathbf{a}}| \quad \text{for all} \quad x \in I_{a_n}.$$

That is, the derivatives of the map $\gamma_{\mathbf{a}'}$ are of comparable size at different points of I_{a_n}.

(d) Using the following special case of Γ-equivariance of the Patterson–Sullivan measure μ:

$$\mu(I_{\mathbf{a}}) = \int_{I_{a_n}} (\gamma'_{\mathbf{a}'}(x))^\delta \, d\mu(x)$$

and the fact that $\mu(I_a) > 0$ for every $a \in \mathcal{A}$, show that for some constant C_Γ depending only on Γ

$$C_\Gamma^{-1}|I_{\mathbf{a}}|^\delta \leq \mu(I_{\mathbf{a}}) \leq C_\Gamma |I_{\mathbf{a}}|^\delta.$$

Using this, show that Λ_Γ is δ-regular up to scale 0 with some constant depending only on Γ.

Solution: See §2 in arXiv:1704.02909.

4. This exercise explains why the transfer operator is of trace class on $\mathcal{H}(D)$. (See for instance Dyatlov–Zworski, *Mathematical Theory of Scattering Resonances,* Appendix B.4, for an introduction to trace class operators.) We consider the following simpler setting: $D \subset \mathbb{C}$ is the unit disk, $\mathcal{H}(D)$ is the space of holomorphic functions in $L^2(D)$ (it is a closed subspace of L^2 and thus a Hilbert space), and we consider the operator

$$L : \mathcal{H}(D) \to \mathcal{H}(D), \quad Lf(z) = f(z/2).$$

Show that L is trace class using one or both of the following methods:

(a) the fact that $\{z^k\}_{k \in \mathbb{N}_0}$ is an orthogonal basis in $\mathcal{H}(D)$;

Solution: We have $L(z^k) = 2^{-k} z^k$, so L is self-adjoint on $\mathcal{H}(D)$ and has eigenvalues 2^{-k}, $k \in \mathbb{N}_0$. The series $\sum_{k=0}^\infty 2^{-k}$ converges, so L is trace class.

(b) the Cauchy integral formula, where $\gamma \subset D$ is a contour surrounding the disk $\{|z| \leq \frac{1}{2}\}$

$$Lf(z) = \frac{1}{2\pi i} \oint_\gamma L_w f(z) \, dw, \quad L_w f(z) = \frac{f(w)}{w - z/2},$$

together with the fact that each L_w is a rank 1 operator. (This solution easily adapts to the transfer operators that we study, where the key fact is that $\gamma_a(D_b) \Subset D_a$ when $a \neq \bar{b}$.)

Solution: Each L_w is a rank 1 operator, in fact $L_w = u_w \otimes \delta_w$ where $\delta_w : \mathcal{H}(D) \to \mathbb{C}$ is the delta function at w, $\delta_w(f) = f(w)$, and $u_w(z) = \frac{1}{w - z/2} \in \mathcal{H}(D)$. Thus in particular L_w is trace class. Since both δ_w and u_w depend continuously on w (the first one as a functional on $\mathcal{H}(D)$ with operator norm, the second one as an element of $\mathcal{H}(D)$), L_w depends continuously on w in the Banach space of trace class operators on $\mathcal{H}(D)$. So the integral above converges in that Banach space, which shows that L is trace class.

5. Assume that Γ is a Schottky group generated by just two intervals I_1, I_2. (The corresponding convex co-compact hyperbolic surface is a hyperbolic cylinder.) Let $x_1 \in I_1, x_2 \in I_2$ be the fixed points of γ_1 (and thus of $\gamma_2 = \gamma_1^{-1}$). Let $\mathcal{L}_s : \mathcal{H}(D) \to \mathcal{H}(D)$ be the transfer operator where $D = D_1 \sqcup D_2 \subset \mathbb{C}$.

Show that the resonances (i.e. the values $s \in \mathbb{C}$ for which the equation $\mathcal{L}_s u = u$ has a nonzero solution $u \in \mathcal{H}(D)$) are given by

$$s = -j + \frac{2\pi i}{\ell} k, \quad j \in \mathbb{N}_0, \quad k \in \mathbb{Z}, \quad \ell := -\log \gamma_1'(x_1) = -\log \gamma_2'(x_2) > 0.$$

(In fact, ℓ is the length of the closed geodesic on the cylinder $\Gamma \backslash \mathbb{H}^2$.)

Hint: if $\mathcal{L}_s u = u$, then let j be the vanishing order of u at x_1 and expand the equation at $z = x_1$.

Solution: First of all, putting $x := x_1$, $y := x_2$ in the identity $|\gamma_1(x) - \gamma_1(y)|^2 = \gamma_1'(x)\gamma_1'(y)|x-y|^2$ we get $\gamma_1'(x_1)\gamma_1'(x_2) = 1$. Thus the definition of ℓ makes sense.

We have for $u \in \mathcal{H}(D)$

$$\mathcal{L}_s u(z) = \begin{cases} (\gamma_1'(z))^s u(\gamma_1(z)), & z \in D_1; \\ (\gamma_2'(z))^s u(\gamma_2(z)), & z \in D_2. \end{cases}$$

The disks D_1, D_2 do not interact so we can consider u separately on these two. Let us focus on D_1.

Assume that $\mathcal{L}_s u = u$ for some $s \in \mathbb{C}$ and $u \in \mathcal{H}(D_1) \setminus \{0\}$. Let $j \in \mathbb{N}_0$ be the vanishing order of u at $z = x_1$. Multiplying u by a constant we may assume that

$$u(z) = (z - x_1)^j + \mathcal{O}(|z - x_1|^{j+1}) \quad \text{as} \quad z \to x_1.$$

Expanding the identity $u(z) = \mathcal{L}_s u(z)$ at $z = x_1$ and using that

$$\gamma_1(z) - x_1 = e^{-\ell}(z - x_1) + \mathcal{O}(|z - x_1|^2)$$

we get

$$(z - x_1)^j + \mathcal{O}(|z - x_1|^{j+1}) = e^{-\ell(s+j)}(z - x_1)^j + \mathcal{O}(|z - x_1|^{j+1})$$

which implies that $e^{-\ell(s+j)} = 1$ and thus

$$s = -j + \frac{2\pi i}{\ell} k \quad \text{for some} \quad k \in \mathbb{Z}. \tag{0.1}$$

Now, assume that s has the form (0.1) for some $j \in \mathbb{N}_0$, $k \in \mathbb{Z}$. We construct a nonzero $u \in \mathcal{H}(D)$ such that $\mathcal{L}_s u = u$. Let us write

$$\gamma_1'(z) = e^{-\varphi(z)}, \quad \gamma_1(z) - x_1 = (z - x_1)e^{-\psi(z)}, \quad z \in D_1$$

where φ, ψ are holomorphic and bounded on D_1 and $\varphi(x_1) = \psi(x_1) = \ell$. We look for u in the form

$$u(z) = (z - x_1)^j e^{v(z)}$$

where v is some bounded holomorphic function on D_1. Then $\mathcal{L}_s u = u$ is equivalent to the following equation for v:

$$e^{v(z)} = e^{-s\varphi(z) - j\psi(z) + v(\gamma_1(z))}, \quad z \in D_1.$$

To satisfy the latter it suffices to construct v such that

$$v(z) = v(\gamma_1(z)) + \theta(z), \quad z \in D_1 \tag{0.2}$$

where $\theta(z) := -s\varphi(z) - j\psi(z) + 2\pi i k$ is holomorphic and bounded on D_1 and $\theta(x_1) = 0$. Now, to solve (0.2) we put

$$v(z) := \sum_{n=0}^{\infty} \theta(\gamma_1^n(z)), \quad z \in D_1$$

where the terms of the series are holomorphic in D_1 and the series converges uniformly in D_1 since $\gamma_1^n(z) \to x_1$ exponentially fast as $n \to \infty$.

6. Show the following version of the 'Patterson–Sullivan' gap: if $\operatorname{Re} s > \delta$ then the equation $\mathcal{L}_s u = u$ has no nonzero solution $u \in \mathcal{H}(D)$. To do this, show that a sufficiently large power $\mathcal{L}_s^n$ is a contracting operator on $C(I)$ with the supremum norm, by writing out $\mathcal{L}_s^n$ as a sum over words in $\mathcal{W}^n$ and using the results of Exercise 3.

Solution: Put $\alpha := \operatorname{Re} s > \delta$. Take large n. Then for any $f \in C(I)$ we have

$$\mathcal{L}_s^n f(x) = \sum_{\substack{\mathbf{a} \in \mathcal{W}^n \\ \mathbf{a} \to b}} (\gamma'_{\mathbf{a}}(x))^s f(\gamma_{\mathbf{a}}(x)), \quad x \in I_b$$

where $\mathbf{a} \to b$ means that $a_n \neq \bar{b}$ where $\mathbf{a} = a_1 \dots a_n$.

By Exercise 3(c) we have $|(\gamma'_{\mathbf{a}}(x))^s| = |\gamma'_{\mathbf{a}}(x)|^\alpha \leq C|I_{\mathbf{a}}|^\alpha$ for $x \in I_b$, $\mathbf{a} \to b$. Here C is a constant independent of n. Therefore

$$\sup_I |\mathcal{L}_s^n f| \leq r_n \sup_I |f|, \quad r_n := C \sum_{\mathbf{a} \in \mathcal{W}^n} |I_{\mathbf{a}}|^\alpha.$$

Now by Exercise 3(d) we have

$$\sum_{\mathbf{a} \in \mathcal{W}^n} |I_{\mathbf{a}}|^\delta \leq C \sum_{\mathbf{a} \in \mathcal{W}^n} \mu(I_{\mathbf{a}}) \leq C.$$

Since $\alpha > \delta$ and $\max_{\mathbf{a} \in \mathcal{W}^n} |I_{\mathbf{a}}| \to 0$ as $n \to \infty$, we get $r_n \to 0$ as $n \to \infty$. Thus for n large enough, $\mathcal{L}_s^n$ is a contraction on $C(I)$ with the uniform norm. If $u \in \mathcal{H}(D)$ and $\mathcal{L}_s u = u$, then it is easy to see that $f := u|_I \in C(I)$ and $\mathcal{L}_s^n f = f$, which implies that $u|_I = 0$ and thus (by analytic continuation for instance) $u = 0$.

7. Fix $\delta \in [0,1]$ and define the h-dependent intervals

$$X = Y = [-h^{1-\delta}, h^{1-\delta}].$$

Show that there exists a constant $c > 0$ such that

$$\| \mathbb{1}_X \mathcal{F}_h \mathbb{1}_Y \|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} \geq c h^{\max(0, \frac{1}{2} - \delta)}.$$

(Hint: apply this operator to a dilated cutoff function supported in Y.)

Solution: Fix $\chi \in C_c^\infty((-1,1))$ such that $\|\chi\|_{L^2} = 1$ and $\widehat{\chi}(0) \neq 0$ and define

$$u(y;h) = h^{\frac{\delta-1}{2}} \chi(h^{\delta-1} y), \quad \|u\|_{L^2} = 1, \quad \operatorname{supp} u \subset Y.$$

Then

$$\mathcal{F}_h u(x) = \frac{h^{-\delta/2}}{\sqrt{2\pi}} \widehat{\chi}(h^{-\delta} x),$$

so we compute

$$\| \mathbb{1}_X \mathcal{F}_h \mathbb{1}_Y u \|_{L^2(\mathbb{R})} = \frac{1}{\sqrt{2\pi}} \|\widehat{\chi}\|_{L^2([-h^{1-2\delta}, h^{1-2\delta}])} \geq c h^{\max(0, \frac{1}{2} - \delta)}.$$

8. Let $Z \subset \mathcal{W}$ be a partition, i.e. a finite set of nonempty words such that

$$\Lambda_\Gamma = \bigsqcup_{\mathbf{a} \in Z} (\Lambda_\Gamma \cap I_{\mathbf{a}}).$$

Let $\overline{Z} := \{\overline{\mathbf{a}} \mid \mathbf{a} \in Z\}$ where $\overline{a_1 \dots a_n} := \overline{a_n} \dots \overline{a_1}$. Define the transfer operator $\mathcal{L}_{\overline{Z},s}$ by

$$\mathcal{L}_{\overline{Z},s} f(z) = \sum_{\mathbf{a} \in \overline{Z},\ \mathbf{a} \rightsquigarrow b} (\gamma_{\mathbf{a}'}(z))^s f(\gamma_{\mathbf{a}'}(z)), \quad z \in D_b$$

where for $\mathbf{a} = a_1 \dots a_n$ we put $\mathbf{a}' := a_1 \dots a_{n-1}$ and say $\mathbf{a} \rightsquigarrow b$ if $a_n = b$. Assume that $u \in \mathcal{H}(D)$ satisfies $\mathcal{L}_s u = u$. Show that $\mathcal{L}_{\overline{Z},s} u = u$.

Solution: See Lemma 2.4 in arXiv:1704.02909.