EXERCISES FOR THE MINICOURSE ON FRACTAL UNCERTAINTY PRINCIPLE

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Abstract. These are companion exercises to the minicourse given at the Spring School on Transfer Operators, organized by the Bernoulli Center, Lausanne, in March 2021.

1. Describe all the elements $\gamma \in \mathrm{SL}(2,\mathbb{R})$ such that

$$\gamma(\overline{\mathbb{R}} \setminus I_2^\circ) = I_1 \quad \text{where} \quad I_1 := [1,2], \quad I_2 := [-1,0].$$

Note that these γ are all hyperbolic, i.e. $|\operatorname{tr}\gamma| > 2$, which implies that γ has two fixed points on $\mathbb{R}$, one attractive and one repulsive. Find these fixed points. Show that any point in I_1° is the attractive point of some γ and similarly for repulsive points and I_2°.

2. Let $\Gamma \subset \mathrm{SL}(2,\mathbb{R})$ be a Schottky group, with generators $\gamma_1, \dots, \gamma_r$. Show that it is a free group with these generators, i.e. for any word $\mathbf{a} \in \mathcal{W}$, if $\gamma_{\mathbf{a}} = I$ then $\mathbf{a} = \emptyset$.

3. This exercise explains why elements of Schottky groups have bounded distortion.

(a) We first discuss the way that a general element $\gamma \in \mathrm{SL}(2,\mathbb{R})$ can map an interval to another interval. Assume that $I, J \subset \mathbb{R}$ are intervals such that $\gamma(I) = J$. Define the *distortion factor of γ on I* by

$$\alpha(\gamma, I) := \log \frac{\gamma^{-1}(\infty) - x_1}{\gamma^{-1}(\infty) - x_0} \in \mathbb{R} \quad \text{where} \quad I = [x_0, x_1].$$

(If $\gamma^{-1}(\infty) = \infty$, that is γ is an affine map, then we put $\alpha(\gamma, I) := 0$.) Show that γ can be factorized as

$$\gamma = \gamma_J\, \gamma_{\alpha(\gamma,I)}\, \gamma_I^{-1}, \quad \gamma_\alpha := \begin{pmatrix} e^{\alpha/2} & 0 \\ e^{\alpha/2} - e^{-\alpha/2} & e^{-\alpha/2} \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$$

where $\gamma_I, \gamma_J \in \mathrm{SL}(2,\mathbb{R})$ are the affine maps such that $\gamma_I([0,1]) = I$, $\gamma_J([0,1]) = J$.

(b) Show that for each R there exists C such that in the notation of part (a)

$$|\alpha(\gamma, I)| \le R \quad \Longrightarrow \quad C^{-1}\frac{|J|}{|I|} \le \gamma'(x) \le C\frac{|J|}{|I|} \quad \text{for all} \quad x \in I.$$

(c) Let Γ be a Schottky group generated by $\gamma_1, \dots, \gamma_r \in \mathrm{SL}(2,\mathbb{R})$. Show that there exists C_Γ such that for all nonempty $\mathbf{a} = a_1 \dots a_n \in \mathcal{W}$ we have

$$C_\Gamma^{-1}|I_{\mathbf{a}}| \le \gamma'_{\mathbf{a}'}(x) \le C_\Gamma |I_{\mathbf{a}}| \quad \text{for all} \quad x \in I_{a_n}.$$

That is, the derivatives of the map $\gamma_{\mathbf{a}'}$ are of comparable size at different points of I_{a_n}.

(d) Using the following special case of Γ-equivariance of the Patterson–Sullivan measure μ:

$$\mu(I_{\mathbf{a}})=\int_{I_{a_n}}(\gamma'_{\mathbf{a}'}(x))^\delta\,d\mu(x)$$

and the fact that $\mu(I_a)>0$ for every $a\in\mathcal{A}$, show that for some constant C_Γ depending only on Γ

$$C_\Gamma^{-1}|I_{\mathbf{a}}|^\delta\leq\mu(I_{\mathbf{a}})\leq C_\Gamma|I_{\mathbf{a}}|^\delta.$$

Using this, show that Λ_Γ is δ-regular up to scale 0 with some constant depending only on Γ.

4. This exercise explains why the transfer operator is of trace class on $\mathcal{H}(D)$. (See for instance Dyatlov–Zworski, *Mathematical Theory of Scattering Resonances,* Appendix B.4, for an introduction to trace class operators.) We consider the following simpler setting: $D\subset\mathbb{C}$ is the unit disk, $\mathcal{H}(D)$ is the space of holomorphic functions in $L^2(D)$ (it is a closed subspace of L^2 and thus a Hilbert space), and we consider the operator

$$L:\mathcal{H}(D)\to\mathcal{H}(D),\quad Lf(z)=f(z/2).$$

Show that L is trace class using one or both of the following methods:

(a) the fact that $\{z^k\}_{k\in\mathbb{N}_0}$ is an orthogonal basis in $\mathcal{H}(D)$;

(b) the Cauchy integral formula, where $\gamma\subset D$ is a contour surrounding the disk $\{|z|\leq\frac{1}{2}\}$

$$Lf(z)=\frac{1}{2\pi i}\oint_\gamma L_wf(z)\,dw,\quad L_wf(z)=\frac{f(w)}{w-z/2},$$

together with the fact that each L_w is a rank 1 operator. (This solution easily adapts to the transfer operators that we study, where the key fact is that $\gamma_a(D_b)\Subset D_a$ when $a\neq\bar{b}$.)

5. Assume that Γ is a Schottky group generated by just two intervals I_1,I_2. (The corresponding convex co-compact hyperbolic surface is a hyperbolic cylinder.) Let $x_1\in I_1,x_2\in I_2$ be the fixed points of γ_1 (and thus of $\gamma_2=\gamma_1^{-1}$). Let $\mathcal{L}_s:\mathcal{H}(D)\to\mathcal{H}(D)$ be the transfer operator where $D=D_1\sqcup D_2\subset\mathbb{C}$.

Show that the resonances (i.e. the values $s\in\mathbb{C}$ for which the equation $\mathcal{L}_su=u$ has a nonzero solution $u\in\mathcal{H}(D)$) are given by

$$s=-j+\frac{2\pi i}{\ell}k,\quad j\in\mathbb{N}_0,\quad k\in\mathbb{Z},\quad \ell:=-\log\gamma_1'(x_1)=-\log\gamma_2'(x_2)>0.$$

(In fact, ℓ is the length of the closed geodesic on the cylinder $\Gamma\backslash\mathbb{H}^2$.)

Hint: if $\mathcal{L}_su=u$, then let j be the vanishing order of u at x_1 and expand the equation at $z=x_1$.

6. Show the following version of the 'Patterson–Sullivan' gap: if $\operatorname{Re} s > \delta$ then the equation $\mathcal{L}_s u = u$ has no nonzero solution $u \in \mathcal{H}(D)$. To do this, show that a sufficiently large power $\mathcal{L}_s^n$ is a contracting operator on $C(I)$ with the supremum norm, by writing out $\mathcal{L}_s^n$ as a sum over words in $\mathcal{W}^n$ and using the results of Exercise 3.

7. Fix $\delta \in [0,1]$ and define the h-dependent intervals

$$X = Y = [-h^{1-\delta}, h^{1-\delta}].$$

Show that there exists a constant $c > 0$ such that

$$\| \mathbb{1}_X \mathcal{F}_h \mathbb{1}_Y \|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} \geq c h^{\max(0, \frac{1}{2} - \delta)}.$$

(Hint: apply this operator to a dilated cutoff function supported in Y.)

8. Let $Z \subset \mathcal{W}$ be a partition, i.e. a finite set of nonempty words such that

$$\Lambda_\Gamma = \bigsqcup_{\mathbf{a} \in Z} (\Lambda_\Gamma \cap I_{\mathbf{a}}).$$

Let $\overline{Z} := \{\overline{\mathbf{a}} \mid \mathbf{a} \in Z\}$ where $\overline{a_1 \dots a_n} := \overline{a_n} \dots \overline{a_1}$. Define the transfer operator $\mathcal{L}_{\overline{Z},s}$ by

$$\mathcal{L}_{\overline{Z},s} f(z) = \sum_{\mathbf{a} \in \overline{Z},\ \mathbf{a} \rightsquigarrow b} (\gamma_{\mathbf{a}'}(z))^s f(\gamma_{\mathbf{a}'}(z)), \quad z \in D_b$$

where for $\mathbf{a} = a_1 \dots a_n$ we put $\mathbf{a}' := a_1 \dots a_{n-1}$ and say $\mathbf{a} \rightsquigarrow b$ if $a_n = b$. Assume that $u \in \mathcal{H}(D)$ satisfies $\mathcal{L}_s u = u$. Show that $\mathcal{L}_{\overline{Z},s} u = u$.