Minicourse on fractal uncertainty principle
Lecture 2: Fractal Uncertainty Principle

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An uncertainty principle

- Unitary semiclassical Fourier transform $\mathcal{F}_h : L^2(\mathbb{R}) \to L^2(\mathbb{R})$

$$\mathcal{F}_h f(x) = (2\pi h)^{-\frac{1}{2}}\, \widehat{f}\Big(\frac{x}{h}\Big) = (2\pi h)^{-\frac{1}{2}} \int_{\mathbb{R}} e^{-\frac{i}{h}xy} f(y)\, dy$$

  Here $0 < h \ll 1$ is called the semiclassical parameter

- For $X \subset \mathbb{R}$, denote by $\mathbb{1}_X : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ the multiplication operator by the indicator function of X
- We say that two h-dependent sets $X = X(h)$, $Y = Y(h) \subset \mathbb{R}$ satisfy the uncertainty principle with exponent β if

$$\|\mathbb{1}_X\, \mathcal{F}_h\, \mathbb{1}_Y\|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} = \mathcal{O}(h^\beta) \quad \text{as} \quad h \to 0$$

- This is equivalent to the following estimate for all $f \in L^2(\mathbb{R})$:

$$\operatorname{supp} \widehat{f} \subset h^{-1} Y \quad \Longrightarrow \quad \|\mathbb{1}_X f\|_{L^2(\mathbb{R})} \le C h^\beta \|f\|_{L^2(\mathbb{R})}$$

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Basic uncertainty principles

- Looking for

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- Trivial bound: $\beta = 0$ as $\| \mathbb{1}_X \mathcal{F}_h \mathbb{1}_Y \|_{L^2 \to L^2} \leq 1$
- Volume bound: if $|X|, |Y| = \mathcal{O}(h^{1-\delta})$ then get $\beta = \frac{1}{2} - \delta$:

$$\| \mathbb{1}_X \mathcal{F}_h \mathbb{1}_Y \|_{L^2 \to L^2} \leq \| \mathbb{1}_X \|_{L^\infty \to L^2} \|\mathcal{F}_h\|_{L^1 \to L^\infty} \| \mathbb{1}_Y \|_{L^2 \to L^1}$$
$$\leq \sqrt{\frac{|X| \cdot |Y|}{2\pi h}} = \mathcal{O}(h^{\frac{1}{2} - \delta})$$

- Cannot be improved if we only know the volume, e.g.

$$X = Y = [-\sqrt{h}, \sqrt{h}] \quad \Longrightarrow \quad \text{cannot get} \quad \beta > 0$$

So we need to know more about the structure of X and Y

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- Cannot be improved if we only know the volume, e.g.

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So we need to know more about the structure of X and Y

Ahlfors–David regular sets

Definition

We call a set $X \subset \mathbb{R}$ δ-regular up to scale h with constant C if there exists a finite measure μ on X such that

$$C^{-1}|I|^\delta \le \mu(I) \le C|I|^\delta$$

for every interval $I \subset \mathbb{R}$ with $h \le |I| \le 1$ whose center lies in X.

- Example: the mid-third Cantor set is $\log_3 2$-regular up to scale 0
- The limit set Λ_Γ of a Schottky group is δ-regular up to scale 0, taking μ = Patterson–Sullivan measure
- If X is δ-regular up to scale 0, then its h-neighborhood $X(h) = X + [-h, h]$ is δ-regular up to scale h
- Relation to porous sets mentioned in Lecture 1: X is porous $\iff$ $X \subset \widetilde{X}$ for some δ-regular $\widetilde{X}$ with $\delta < 1$

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Fractal uncertainty principle for the Fourier transform

Theorem

Assume that $X, Y \subset [0,1]$ are δ-regular with constant C_R up to scale h where $0 < \delta < 1$. Then there exist $\beta = \beta(\delta, C_R) > \max(0, \frac{1}{2} - \delta)$ and $C = C(\delta, C_R)$ such that

$$\| \mathbb{1}_X \, \mathcal{F}_h \, \mathbb{1}_Y \|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} \le C h^\beta.$$

- $\beta > 0$ proved by Bourgain–D '18 using methods of harmonic analysis. Jin–Zhang '20 got for some universal constant K

$$\beta = \exp\Big[- \exp\big(K(C_R \delta^{-1}(1-\delta)^{-1})^{K(1-\delta)^{-2}}\big)\Big]$$

- $\beta > \frac{1}{2} - \delta$ proved by D–Jin '18, inspired by Dolgopyat's method. Get

$$\beta = \tfrac{1}{2} - \delta + (5C_R)^{-160\delta^{-1}(1-\delta)^{-1}}$$

- See also D–Zahl '16, Cladek–Tao '20 which use additive combinatorics

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Hyperbolic FUP

For applications to hyperbolic surfaces, we replace the phase xy in $\mathcal{F}_h$ by $2\log|x-y|$ and introduce a cutoff $\chi \in C_c^\infty(\mathbb{R}^2)$, $\operatorname{supp}\chi \cap \{x=y\} = \emptyset$:

$$\mathcal{B}_{\chi,h} f(x) = (2\pi h)^{-\frac{1}{2}} \int_{\mathbb{R}} |x-y|^{-\frac{2i}{h}} \chi(x,y) f(y)\, dy$$

For $\chi \equiv 1$, $\mathcal{B}$ is equivariant under all $\gamma \in \mathrm{SL}(2,\mathbb{R})$:

$$(\mathcal{B}_{1,h} f) \circ \gamma = (\gamma')^{-\frac{i}{h}} \mathcal{B}_{1,h}\big((\gamma')^{1-\frac{i}{h}}(f \circ \gamma)\big)$$

For $\mathcal{B}_{\chi,h}$ we have the same FUP:

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FUP and spectral gaps

- Let $\Gamma \subset SL(2,\mathbb{R})$ be a Schottky group, $M = \Gamma\backslash\mathbb{H}^2$
- $\Lambda_\Gamma \subset \mathbb{R}$ the limit set, $\Lambda_\Gamma(h) := \Lambda_\Gamma + [-h, h]$ its h-neighborhood

Theorem [D–Zahl '16, D–Zworski '17], explained in Lecture 3–4

Assume that the sets $X = Y = \Lambda_\Gamma(h)$ satisfy hyperbolic FUP:

$$\forall\chi \quad \| \mathbb{1}_{\Lambda_\Gamma(h)} \mathcal{B}_{\chi,h} \mathbb{1}_{\Lambda_\Gamma(h)} \|_{L^2\to L^2} \le Ch^\beta, \quad C = C(\chi)$$

Then for any $\alpha > \frac{1}{2} - \beta$, M has finitely many resonances with $\operatorname{Re} s \ge \alpha$.

- Trivial bound $\beta = 0 \implies$ 'Lax–Phillips' gap $\operatorname{Re} s > \frac{1}{2}+$
- Volume bound $\beta = \frac{1}{2} - \delta \implies$ 'Patterson–Sullivan' gap $\operatorname{Re} s > \delta+$
- FUP on previous slide $\implies$ gap with some $\alpha = \alpha(\Gamma) < \min(\frac{1}{2}, \delta)$
- Specialized FUP for Λ_Γ [Bourgain–D '17] $\implies \alpha = \alpha(\delta) < \delta$

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Discrete Cantor sets

We now present a proof of FUP in the special setting of Cantor sets. This is much simpler than the general case but keeps some key features. We follow D–Jin '17, with the exposition from [arXiv:1903.02599]

- Discrete unitary Fourier transform $\mathcal{F}_N : \mathbb{C}^N \to \mathbb{C}^N$

$$\mathcal{F}_N u(j) = \frac{1}{\sqrt{N}} \sum_{\ell=0}^{N-1} e^{-\frac{2\pi i j \ell}{N}} u(\ell)$$

- Fix $M \geq 3$, $\mathscr{A} \subset \{0, \dots, M-1\}$. Put $N := M^k$, $k \gg 1$ and define

$$\mathcal{C}_k := \{a_0 + a_1 M + \cdots + a_{k-1} M^{k-1} \mid a_0, \dots, a_{k-1} \in \mathscr{A}\}$$

- Example: if $M = 3$, $\mathscr{A} = \{0, 2\}$, then $\mathcal{C}_k \subset \{0, \dots, N-1\}$, $N = 3^k$, is the discrete mid-3rd Cantor set $\{0, 2, 6, 8, 18, 20, 24, 26, \dots\}$
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Uncertainty principle for discrete Cantor sets

Theorem

Assume that $0 < \delta < 1$, i.e. $1 < |\mathscr{A}| < M$. Then there exists $\beta = \beta(M, \mathscr{A}) > \max(0, \frac{1}{2} - \delta)$ such that as $N = M^k \to \infty$,

$$\| \mathbb{1}_{\mathcal{C}_k} \mathcal{F}_N \mathbb{1}_{\mathcal{C}_k} \|_{\mathbb{C}^N \to \mathbb{C}^N} = \mathcal{O}(N^{-\beta}).$$

- Trivial bound $\beta = 0$: since $\mathcal{F}_N$ is unitary, $\| \mathbb{1}_{\mathcal{C}_k} \mathcal{F}_N \mathbb{1}_{\mathcal{C}_k} \|_{\mathbb{C}^N \to \mathbb{C}^N} \leq 1$
- Volume bound $\beta = \frac{1}{2} - \delta$: defining the Hilbert–Schmidt norm

$$\|A\|_{\mathrm{HS}}^2 = \sum_{j,k} |a_{jk}|^2 \quad \text{where} \quad A = (a_{jk})_{j,k=1}^N$$

we have

$$\| \mathbb{1}_{\mathcal{C}_k} \mathcal{F}_N \mathbb{1}_{\mathcal{C}_k} \|_{\mathbb{C}^N \to \mathbb{C}^N} \leq \| \mathbb{1}_{\mathcal{C}_k} \mathcal{F}_N \mathbb{1}_{\mathcal{C}_k} \|_{\mathrm{HS}} = N^{\delta - \frac{1}{2}}.$$

Uncertainty principle for discrete Cantor sets

Theorem

Assume that $0 < \delta < 1$, i.e. $1 < |\mathscr{A}| < M$. Then there exists $\beta = \beta(M, \mathscr{A}) > \max(0, \frac{1}{2} - \delta)$ such that as $N = M^k \to \infty$,

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Submultiplicativity

The proof of FUP for Cantor sets is greatly simplified by the

Submultiplicativity Lemma

Define $r_k := \| \mathbb{1}_{\mathcal{C}^k} \mathcal{F}_N \mathbb{1}_{\mathcal{C}_k} \|_{\mathbb{C}^N \to \mathbb{C}^N}$. Then $r_{k_1+k_2} \leq r_{k_1} \cdot r_{k_2}$ for all k_1, k_2.

To prove it, we employ the following decomposition also used in FFT:

- Write $k = k_1 + k_2, \quad N = M^k = N_1 \cdot N_2, \quad N_j := M^{k_j}$
- Identify $u \in \mathbb{C}^N$ with an $N_1 \times N_2$ matrix $U_{ab} = u(N_1 b + a)$
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An example of the 'Fast Fourier Transform' decomposition

Let's say $N = 4 = N_1 N_2$ where $N_1 = N_2 = 2$.

Take $u = (u_0, u_1, u_2, u_3) \in \mathbb{C}^4$. Follow the instructions on the last slide:

- Take $U = \begin{pmatrix} u_0 & u_2 \\ u_1 & u_3 \end{pmatrix}$, $\mathcal{F}_2$ each row to get $\frac{1}{\sqrt{2}} \begin{pmatrix} u_0 + u_2 & u_0 - u_2 \\ u_1 + u_3 & u_1 - u_3 \end{pmatrix}$
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- V gives the Fourier transform $\mathcal{F}_4 u$:

$$V = \begin{pmatrix} \mathcal{F}_4 u(0) & \mathcal{F}_4 u(1) \\ \mathcal{F}_4 u(2) & \mathcal{F}_4 u(3) \end{pmatrix}$$

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FUP with $\beta > 0$

- $r_{k_1+k_2} \le r_{k_1} \cdot r_{k_2}$ where $r_k := \| \mathbb{1}_{\mathcal{C}_k} \mathcal{F}_N \mathbb{1}_{\mathcal{C}_k} \|_{\mathbb{C}^N \to \mathbb{C}^N}$, $N = M^k$
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FUP with $\beta > \frac{1}{2} - \delta$ ('baby Dolgopyat')

- Similarly to the previous slide, enough to show that $\exists k : r_k < N^{\delta - \frac{1}{2}}$ where $r_k := \| \mathbb{1}_{\mathcal{C}_k} \mathcal{F}_N \mathbb{1}_{\mathcal{C}_k} \|_{\mathbb{C}^N \to \mathbb{C}^N}$, $N = M^k$
- We always have $r_k \leq \| \mathbb{1}_{\mathcal{C}_k} \mathcal{F}_N \mathbb{1}_{\mathcal{C}_k} \|_{\mathrm{HS}} = N^{\delta - \frac{1}{2}}$
- Assume $r_k = N^{\delta - \frac{1}{2}}$, then $\mathbb{1}_{\mathcal{C}_k} \mathcal{F}_N \mathbb{1}_{\mathcal{C}_k}$ has the same operator norm ($=$ max singular value σ_j) and H–S norm $\left(= \sqrt{\sigma_1^2 + \cdots + \sigma_N^2} \right)$
- This can only happen if $\mathbb{1}_{\mathcal{C}_k} \mathcal{F}_N \mathbb{1}_{\mathcal{C}_k}$ is a rank 1 matrix, i.e. each of its 2×2 minors is equal to 0. This gives

$$(j - j')(\ell - \ell') \in N\mathbb{Z} \quad \text{for all} \quad j, j', \ell, \ell' \in \mathcal{C}_k$$

- This cannot happen already when $k = 2$ (and $|\mathscr{A}| > 1$): just take two different $a, b \in \mathscr{A}$ and put

$$j = \ell = Ma + a, \quad j' = \ell' = Ma + b$$

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- This can only happen if $\mathbb{1}_{\mathcal{C}_k} \mathcal{F}_N \mathbb{1}_{\mathcal{C}_k}$ is a rank 1 matrix, i.e. each of its 2×2 minors is equal to 0. This gives

$$(j - j')(\ell - \ell') \in N\mathbb{Z} \quad \text{for all} \quad j, j', \ell, \ell' \in \mathcal{C}_k$$

- This cannot happen already when $k = 2$ (and $|\mathscr{A}| > 1$): just take two different $a, b \in \mathscr{A}$ and put

$$j = \ell = Ma + a, \quad j' = \ell' = Ma + b$$

FUP with $\beta > \frac{1}{2} - \delta$ ('baby Dolgopyat')

- Similarly to the previous slide, enough to show that $\exists k : r_k < N^{\delta - \frac{1}{2}}$ where $r_k := \| \mathbb{1}_{\mathcal{C}_k} \mathcal{F}_N \mathbb{1}_{\mathcal{C}_k} \|_{\mathbb{C}^N \to \mathbb{C}^N}$, $N = M^k$
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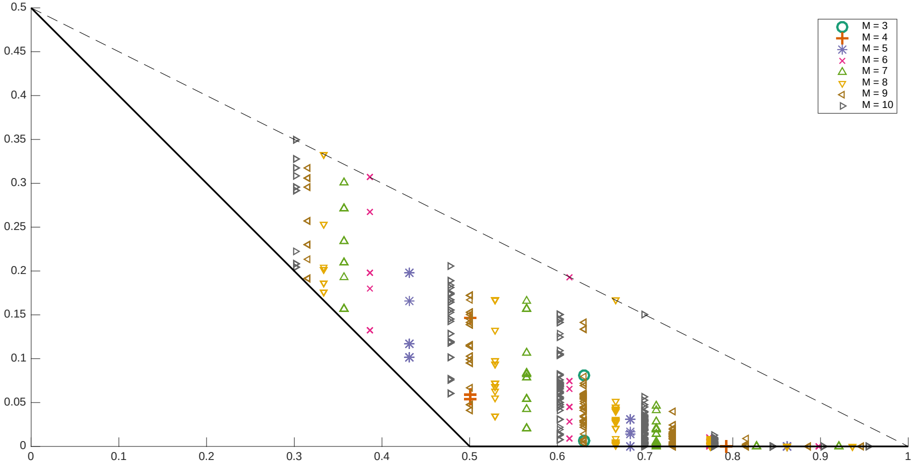
A picture of FUP exponents for all alphabets with $M \leq 10$



Horizontal axis: δ, vertical axis: β, solid line: $\beta = \max(0, \frac{1}{2} - \delta)$, dashed line: $\beta = \frac{1-\delta}{2}$ (corresponding to the gap conjectured by Jakobson–Naud)

A higher dimensional FUP?

- Open problem: get FUP with $\beta > 0$ on $\mathbb{R}^n$, $n > 1$. Let's take $n = 2$
- $\mathcal{F}_h f(x) = (2\pi h)^{-1}\widehat{f}(\frac{x}{h})$ semiclassical Fourier transform
- Want $\| \mathbb{1}_X \mathcal{F}_h \mathbb{1}_Y \|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)} = \mathcal{O}(h^\beta)$ where $X, Y \subset \mathbb{R}^2$ are δ-regular up to scale h and $\delta < 2$
- This is false: take $\delta = 1$, $X = [0, h] \times [0, 1]$, $Y = [0, 1] \times [0, h]$
- Han–Schlag '20: FUP holds with $\beta > 0$ if one of X, Y is contained in the product of 2 fractal sets
- It could be that the hyperbolic FUP (with $e^{-\frac{i}{h}\langle x,y\rangle}$ replaced by $|x-y|^{-\frac{2i}{h}}$) still holds.
  Partial result by D–Zhang WIP, when one of X, Y is a curve

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Thank you for your attention!