Minicourse on fractal uncertainty principle
Lecture 1: Schottky groups and spectral gaps

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March 22–25, 2021

Preview of results

Theorem 1 [Spectral gap]

Let M be a convex co-compact hyperbolic surface. Let $\mathscr{L}_M$ be the set of lengths of primitive closed geodesics on M with multiplicity. Define the Selberg zeta function as the holomorphic extension to $s \in \mathbb{C}$ of

$$Z_M(s) = \prod_{\ell \in \mathscr{L}_M} \prod_{k=0}^{\infty} (1 - e^{-(s+k)\ell}), \quad \operatorname{Re} s \gg 1.$$

Then $\exists \beta > 0$: Z has only finitely many zeroes in $\{\operatorname{Re} s \geq \frac{1}{2} - \beta\}$.

Theorem 2 [Fractal Uncertainty Principle / FUP]

Assume that $X, Y \subset \mathbb{R}$ are ν-porous on scales $\geq h$, i.e. for every interval $I \subset \mathbb{R}$ with $h \leq |I| \leq 1$, $I \setminus X$ and $I \setminus Y$ contain intervals of length $\nu|I|$. Here $\nu > 0$ is fixed, $h \to 0$. Then $\exists \beta = \beta(\nu) > 0, C: \ \forall h > 0, f \in L^2(\mathbb{R})$

$$f \in L^2(\mathbb{R}), \quad \operatorname{supp} \hat{f} \subset h^{-1} Y \quad \Longrightarrow \quad \|\mathbb{1}_X f\|_{L^2(\mathbb{R})} \leq C h^{\beta} \|f\|_{L^2(\mathbb{R})}.$$

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Plan of the minicourse

- Lecture 1: Schottky groups, convex co-compact surfaces, transfer operators, the spectral gap problem, overview of history
- Lecture 2: Statement of FUP, known results on FUP, and a proof of FUP in the model case of Cantor sets following D–Jin '16 [arXiv:1608.02238]
- Lecture 3–4: How FUP implies a spectral gap following D–Zworski '20 [arXiv:1710.05430]
- + two tutorials led by Malo Jézéquel

Möbius maps

- $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ identified with $\mathbb{S}^1$
- The group $\mathrm{SL}(2,\mathbb{R})$ acts on $\overline{\mathbb{R}}$ by Möbius maps:

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}) \quad \Longrightarrow \quad \gamma(x) = \frac{ax+b}{cx+d}$$

- For any closed intervals $I_1, I_2 \subset \mathbb{R}$ with $I_1 \cap I_2 = \emptyset$ there exists

$$\gamma \in \mathrm{SL}(2,\mathbb{R}) \quad \text{such that} \quad \gamma(\overline{\mathbb{R}} \setminus I_2^\circ) = I_1$$

Note that $\gamma^{-1}(\overline{\mathbb{R}} \setminus I_1^\circ) = I_2$

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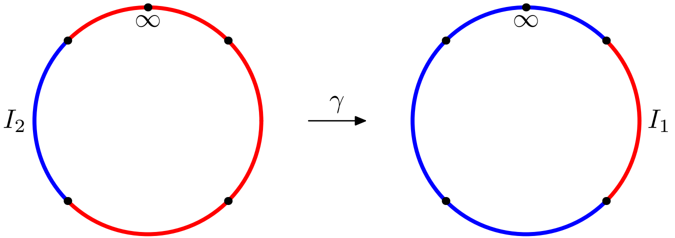

Schottky subgroups of $\mathrm{SL}(2,\mathbb{R})$

To define a Schottky group:

- Fix $r \geq 1$ and put $\mathcal{A} := \{1,\dots,2r\}$; $\bar{a} := (a+r) \bmod 2r$, $a \in \mathcal{A}$
- Fix $2r$ nonintersecting intervals $I_a \subset \mathbb{R}$, $a \in \mathcal{A}$
- Fix the generators $\gamma_a \in \mathrm{SL}(2,\mathbb{R})$, $a \in \mathcal{A}$:

$$\gamma_a(\overline{\mathbb{R}} \setminus I_{\bar{a}}^\circ) = I_a, \quad \gamma_{\bar{a}} = \gamma_a^{-1}$$

- Define the Schottky group $\Gamma \subset \mathrm{SL}(2,\mathbb{R})$ generated by $\gamma_1,\dots,\gamma_r$
- Define the sets of admissible words

$$\mathcal{W}^n := \{a_1 \dots a_n \in \mathcal{A}^n \mid \forall j : a_{j+1} \neq \overline{a_j}\}, \quad \mathcal{W} := \bigcup_{n \geq 0} \mathcal{W}^n$$

- We have $\Gamma = \{\gamma_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{W}\}$ where

$$\gamma_{a_1 \dots a_n} = \gamma_{a_1} \cdots \gamma_{a_n} \in \Gamma$$

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The limit set

- If $a, b \in \mathcal{A}$, $\bar{a} \neq b$, then $\gamma_a(I_b) \Subset \gamma_a(\overline{\mathbb{R}} \setminus I^\circ_{\bar{a}}) = I_a$
- For $\mathbf{a} \in \mathcal{W}^\circ := \mathcal{W} \setminus \{\emptyset\}$ define the closed interval

$$I_{\mathbf{a}} = \gamma_{\mathbf{a}'}(I_{a_n}) \quad \text{where} \quad \mathbf{a} = a_1 \dots a_n, \quad \mathbf{a}' := a_1 \dots a_{n-1}$$

- These intervals form a tree: $I_{\mathbf{a}} \subset I_{\mathbf{a}'}$ and

$$\mathbf{a}, \mathbf{b} \in \mathcal{W}^n, \quad \mathbf{a} \neq \mathbf{b} \quad \Longrightarrow \quad I_{\mathbf{a}} \cap I_{\mathbf{b}} = \emptyset$$

- They are also exponentially small: $\exists \theta > 0 : \max_{\mathbf{a} \in \mathcal{W}^n} |I_{\mathbf{a}}| = \mathcal{O}(e^{-\theta n})$
- The limit set of Γ is given by a Cantor-like procedure:

$$\Lambda_\Gamma = \bigcap_n \bigsqcup_{\mathbf{a} \in \mathcal{W}^n} I_{\mathbf{a}}$$

and is a compact subset of $\mathbb{R}$

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A minimal example: 2 initial intervals

- Assume $r = 1$, i.e. there are only 2 initial intervals I_1, I_2 and

$$\gamma_1(\overline{\mathbb{R}} \setminus I_2^\circ) = I_1, \quad \gamma_2(\overline{\mathbb{R}} \setminus I_1^\circ) = I_2, \quad \gamma_2 = \gamma_1^{-1}$$

- Then $\mathcal{W}^n$ consists of only 2 words, $1 \dots 1$ and $2 \dots 2$
- The limit set consists of only 2 points: $\Lambda_\Gamma = \{x_1, x_2\}$, where $x_1 \in I_1$, $x_2 \in I_2$ are the fixed points of γ_1 (and thus of γ_2)
- x_1 is attractive ($\gamma_1'(x_1) < 1$) and x_2 is repulsive ($\gamma_1'(x_2) > 1$)

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A more interesting example

- If $r \geq 2$, then $\mathcal{W}^n$ has $2r(2r-1)^{n-1}$ words
- The limit set Λ_Γ has fractal structure
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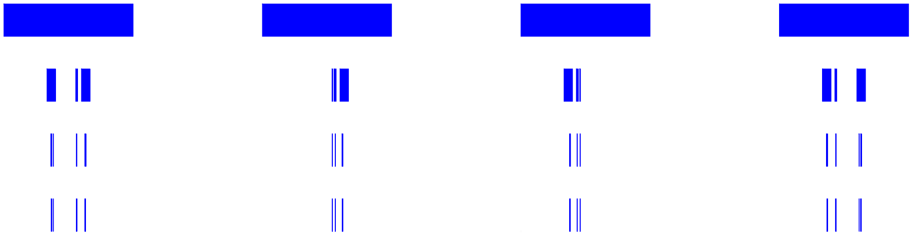

Connection to hyperbolic surfaces

- Möbius transformations act on $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ by the same formula
- In particular they act by isometries on the hyperbolic upper half-plane

$$\mathbb{H}^2 = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}, \quad g = \frac{|dz|^2}{(\operatorname{Im} z)^2}$$

- The quotient $\Gamma\backslash\mathbb{H}^2$ is a convex co-compact hyperbolic surface
- Our generators γ_a, $a \in \mathcal{A}$, satisfy

$$\gamma_a(\overline{\mathbb{C}} \setminus D^\circ_{\bar{a}}) = D_a$$

  where $D_a \subset \mathbb{C}$ is the disk centered on $\mathbb{R}$ such that $D_a \cap \mathbb{R} = I_a$
- Can define the tree of disks $D_{\mathbf{a}} := \gamma_{\mathbf{a}'}(D_{a_n})$, $\mathbf{a} \in \mathcal{W}^\circ$, with $D_{\mathbf{a}} \cap \mathbb{R} = I_{\mathbf{a}}$

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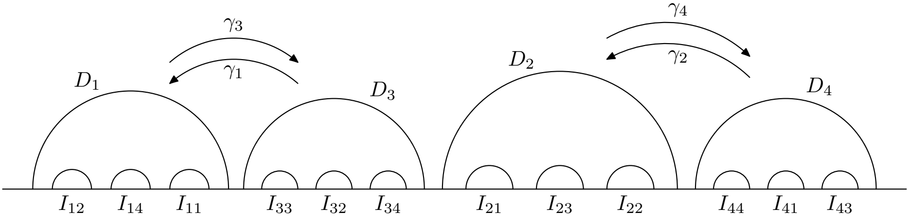
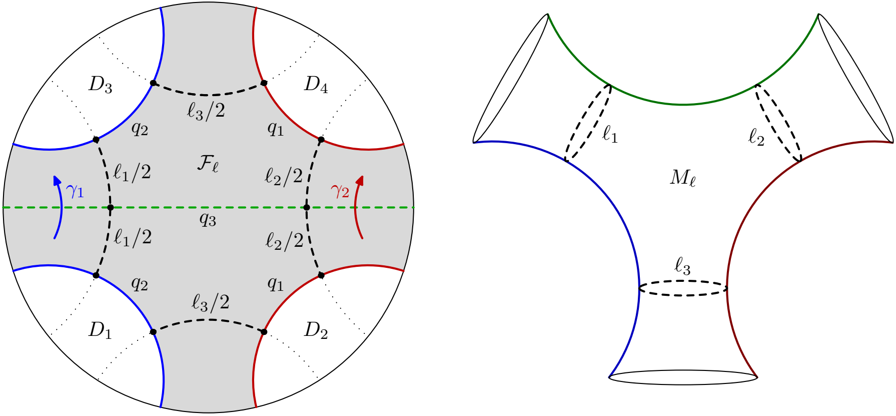
A picture of the tree of half-disks (4 initial intervals)



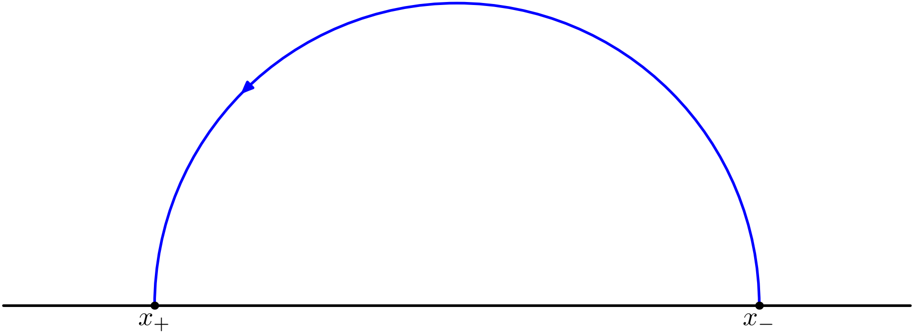
A picture of a convex co-compact hyperbolic surface

Here is how to glue a three-funnel surface from the fundamental domain $\mathbb{H}^2 \setminus \bigsqcup_{a=1}^4 D_a$ in the Poincaré disk model (old picture, replace γ_j by γ_j^{-1})



Limit set and geodesics

- Each geodesic on a surface $M = \Gamma\backslash\mathbb{H}^2$ lifts to a geodesic on $\mathbb{H}^2$, which is a half-circle with endpoints $x_-, x_+ \in \overline{\mathbb{R}}$



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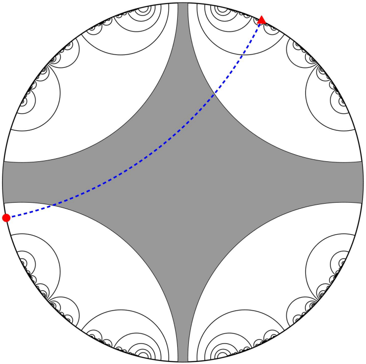
- The geodesic is trapped (i.e. stays in some compact set in M) if and only if $x_-, x_+ \in \Lambda_\Gamma$
- Here is a picture using the Poincaré disk model instead:



Patterson–Sullivan measure

- Denote by δ the Hausdorff dimension of the limit set Λ_Γ
- If $r = 1$ (2 initial intervals) then $\delta = 0$. If $r \geq 2$ then $0 < \delta < 1$
- Patterson–Sullivan measure: a probability measure μ on Λ_Γ such that

$$\int_{\Lambda_\Gamma} f(x)\,d\mu(x) = \int_{\Lambda_\Gamma} f(\gamma(x))(\gamma'(x))^\delta\,d\mu(x)$$

  for all $f \in C(\Lambda_\Gamma)$ and all $\gamma \in \Gamma$
- μ is δ-regular: for any interval I of size $|I| \leq 1$ centered at a point in Λ_Γ

$$\mu(I) \sim |I|^\delta$$

  where the constants in $\sim$ stay fixed as $|I| \to 0$

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Transfer operators

- For $s \in \mathbb{C}$ define the transfer operator $\mathcal{L}_s$ by

$$\mathcal{L}_s f(x) = \sum_{a\in\mathcal{A},\ a\neq\bar{b}} (\gamma_a'(x))^s f(\gamma_a(x)), \quad x \in I_b$$

- $\mathcal{L}_s$ maps $C(I) \to C(I)$ and also $\mathcal{H}(D) \to \mathcal{H}(D)$ where

$$I := \bigsqcup_{a\in\mathcal{A}} I_a \subset \mathbb{R}, \qquad D := \bigsqcup_{a\in\mathcal{A}} D_a \subset \mathbb{C}$$

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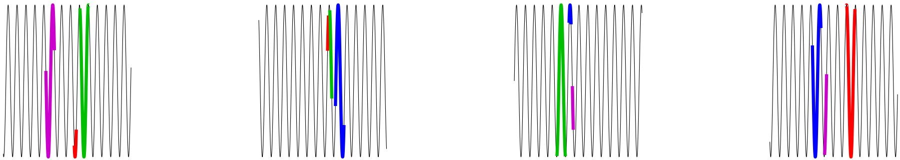
  and $\mathcal{H}(D)$ is the space of holomorphic functions in $L^2(D)$
- For $s = \delta$, the Patterson–Sullivan measure is in the kernel of $I - \mathcal{L}_s^*$:

$$\int_{\Lambda_\Gamma} f\, d\mu = \int_{\Lambda_\Gamma} (L_\delta f)\, d\mu \quad \text{for all} \quad f \in C(\Lambda_\Gamma)$$

Here is a picture for $\mathcal{L}_0 f(x) = \sum_{a \neq \bar{b}} f(\gamma_a(x))$, $x \in I_b$:



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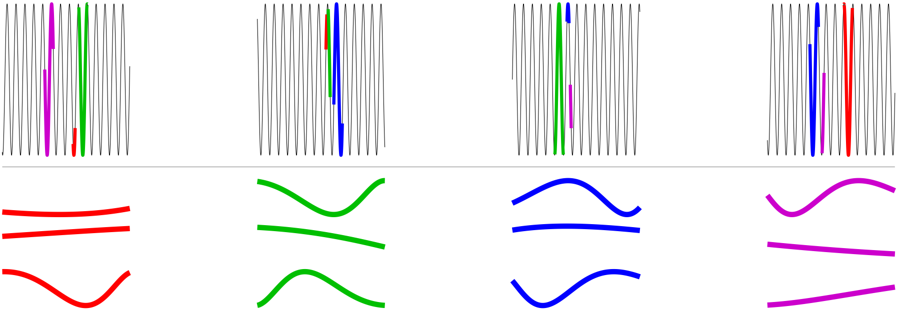

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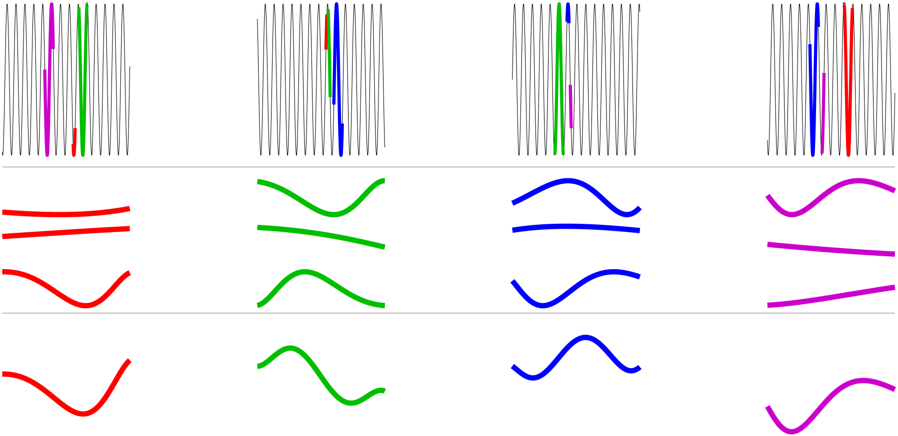

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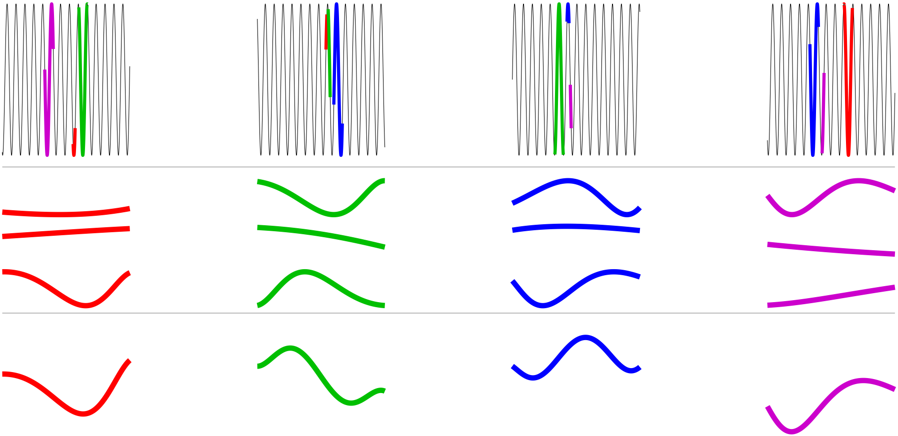

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Transfer operator and the Selberg zeta function

- $\Gamma \subset \mathrm{SL}(2,\mathbb{R})$ Schottky group
- $M = \Gamma \backslash \mathbb{H}^2$ convex co-compact hyperbolic surface
- Selberg zeta function: a product over the lengths of primitive closed geodesics on M

$$Z_M(s) = \prod_{\ell \in \mathscr{L}_M} \prod_{k=0}^{\infty} (1 - e^{-(s+k)\ell}), \quad \operatorname{Re} s \gg 1$$

- Transfer operator $\mathcal{L}_s : \mathcal{H}(D) \to \mathcal{H}(D)$, $s \in \mathbb{C}$, $D = \bigsqcup_{a \in \mathcal{A}} D_a \subset \mathbb{C}$:

$$\mathcal{L}_s f(z) = \sum_{a \in \mathcal{A},\ a \neq \bar{b}} (\gamma_a'(z))^s f(\gamma_a(z)), \quad z \in D_b$$

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which gives a way to continue $Z_M(s)$ to an entire function of s

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Resonances

We call $s \in \mathbb{C}$ a resonance of M if $Z_M(s) = 0$, i.e. $I - \mathcal{L}_s$ is not invertible

Ruelle–Perron–Frobenius/Patterson–Sullivan theory

- δ is a resonance: $\mathcal{L}_\delta^* \mu = \mu$ where μ is the Patterson–Sullivan measure
- No resonances with $\operatorname{Re} s > \delta$
- If $\delta > 0$, then δ is the only resonance on the line $\operatorname{Re} s = \delta$
- If $\delta = 0$, there is actually a lattice of resonances
  $s = -j + ick, \quad j \in \mathbb{N}_0, \quad k \in \mathbb{Z}, \quad c = c(\Gamma) > 0$

Lax–Phillips theory + Patterson–Perry '01

- There are only finitely many resonances with $\operatorname{Re} s \geq \frac{1}{2}$, and they all lie on the interval $[\frac{1}{2}, 1]$
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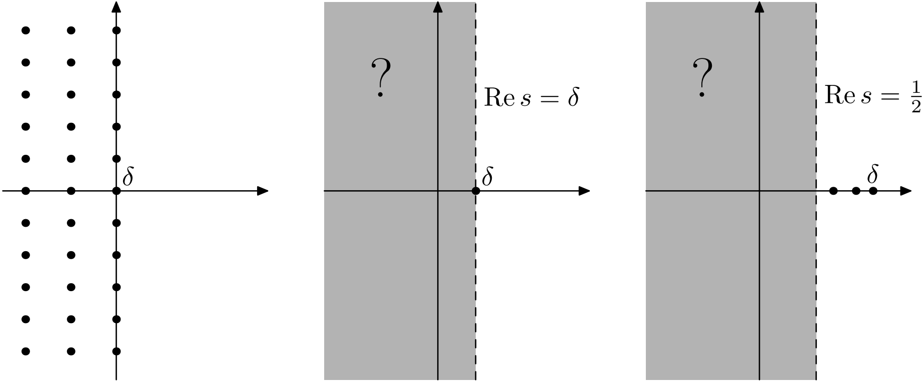
Schematic picture of resonances

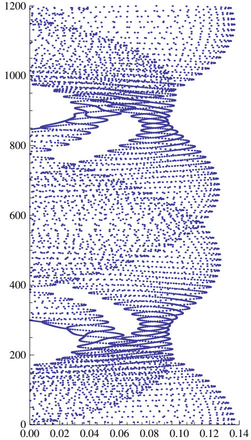


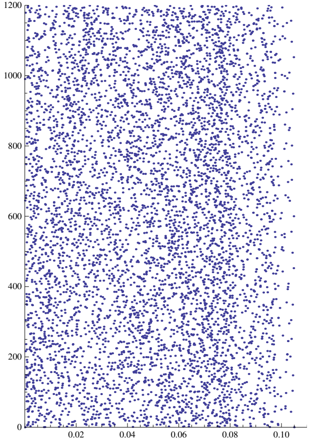
$\delta = 0$ $\quad$ $0 < \delta < \frac{1}{2}$ $\quad$ $\frac{1}{2} < \delta < 1$

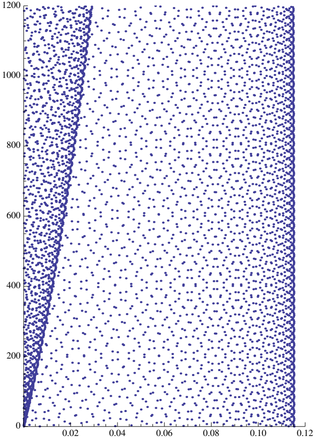
Numerical plots of resonances

Pictures by David Borthwick, see Borthwick '14, Borthwick–Weich '16

David Borthwick, *Spectral Theory of Infinite-Area Hyperbolic Surfaces,* Second Edition, Birkhäuser, 2016, Chapter 16







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- Both of these prove a fractal uncertainty principle (FUP). The fact that FUP implies a spectral gap is proved in D–Zahl '16. In this course we will give another proof, from D–Zworski '20

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