# Minicourse on fractal uncertainty principle
# Lecture 2: from control of eigenfunctions to FUP

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# Review: general setup

- $(M, g)$ compact hyperbolic surface (curvature $\equiv -1$)
- We are given $a \in C_c^\infty(T^*M)$ such that $a|_{S^*M} \not\equiv 0$
- Goal (Theorem 1'): prove that for all $h \ll 1$ and $u \in C^\infty(M)$

$$(-h^2\Delta_g - 1)u = 0 \quad \Longrightarrow \quad \|u\| \leq C\|\operatorname{Op}_h(a)u\|$$

(all norms are $L^2$ or operator norm $L^2 \to L^2$)

- Take two functions $a_1, a_2 \in C_c^\infty(T^*M \setminus 0; [0,1])$ such that

$$a_1 + a_2 = 1 \text{ near } S^*M, \quad \operatorname{supp} a_1 \subset \{a \neq 0\}, \quad S^*M \setminus \operatorname{supp} a_j \neq \emptyset$$

The operators $A_j := \operatorname{Op}_h(a_j)$ satisfy $\|A_j\| \leq 1 + \mathcal{O}(h)$ and

$$\|A_1(j)u\| \leq C\|\operatorname{Op}_h(a)u\| + \mathcal{O}(h^\infty)\|u\|$$

where $A(j) := U(-t)AU(t)$ and $U(t) = \exp(-it\sqrt{-\Delta_g})$

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# Review: proof under GCC

- For a word $\mathbf{w} = w_0 \dots w_{N-1} \in \mathcal{W}(N)$, define

$$A_{\mathbf{w}} := A_{w_{N-1}}(N-1) \cdots A_{w_1}(1) A_{w_0}(0), \quad a_{\mathbf{w}} := \prod_{j=0}^{N-1} (a_{w_j} \circ \varphi_j)$$

- $A_{\mathbf{w}} = \operatorname{Op}_h(a_{\mathbf{w}}) + \mathcal{O}_N(h)$ and $u = \sum_{\mathbf{w} \in \mathcal{W}(N)} A_{\mathbf{w}} u + \mathcal{O}(h^\infty)\|u\|$
- Previously we gave the proof under the geometric control condition: there exists $N$ such that $a_{2\dots2} = 0$ where $2\dots2 \in \mathcal{W}(N)$
- To do that we split $u = A_{\mathcal{X}} u + A_{\mathcal{Y}} u$ where $A_{\mathcal{X}} = A_{2\dots2} = \mathcal{O}(h)$ and $\|A_{\mathcal{Y}} u\| \leq CN \|\operatorname{Op}_h(a)u\| + \mathcal{O}(h^\infty)\|u\|$
- Without GCC, we have $\sup_{S^*M} |a_{2\dots2}| = 1$ and thus $\|A_{\mathcal{X}}\| = 1 + \mathcal{O}(h)$
- Key fact for Theorem 1' without GCC: for $N \approx 2\log(1/h)$, we do not have $A_{2\dots2} = \operatorname{Op}_h(a_{2\dots2}) + \dots$ and in fact $\|A_{2\dots2}\| \ll 1$

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# Scheme of the proof of Theorem 1'

## Key estimate

Let $N := 2\lfloor \log(1/h) \rfloor$. Then there exist $\beta > 0, C$ such that for all $h$

$$\|A_{\mathbf{w}}\| \leq Ch^{\beta} \quad \text{for all} \quad \mathbf{w} \in \mathcal{W}(N)$$

- Why 2? Related to expansion rate of the geodesic flow, more below
- Can write $u = \sum_{\mathbf{w}\in\mathcal{W}(N)} A_{\mathbf{w}} u = A_{\mathcal{X}} u + A_{\mathcal{Y}} u, \quad A_{\mathcal{X}} := A_{2\dots2}$
- By the key estimate, $\|A_{\mathcal{X}} u\| \leq Ch^{\beta}\|u\| \ll \|u\|$
- Can estimate $A_{\mathcal{Y}} u$ as before:

$$\|A_{\mathcal{Y}} u\| \leq 2\sum_{j=0}^{N-1} \|A_1(j)u\| \leq C\log(1/h)\|\operatorname{Op}_h(a)u\| + \mathcal{O}(h^{\infty})\|u\|$$

- Putting together, get $\|u\| \leq C\log(1/h)\|\operatorname{Op}_h(a)u\|$ for $h \ll 1$
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# Long time propagation

By Egorov's Theorem + composition property, for $N$ independent of $h$

$$A_{\mathbf{w}} = \mathrm{Op}_h(a_{\mathbf{w}}) + \mathcal{O}(h) \quad \text{for all} \quad \mathbf{w} \in \mathcal{W}(N)$$

Can this work when $N \to \infty$ as $h \to 0$?

- The proof of Egorov's Theorem uses basic semiclassical calculus. So the real question is: ¿Can we still quantize $a_{\mathbf{w}}$?
- The problem with $a_{\mathbf{w}} = \prod_{j=0}^{N-1}(a_{w_j} \circ \varphi_j)$ is that the derivatives of $a_{w_j} \circ \varphi_j$ are large when $j \gg 1$. How large?
- The geodesic flow $\varphi_t : S^*M \to S^*M$ of a hyperbolic surface has the flow/unstable/stable decomposition $T(S^*M) = E_0 \oplus E_u \oplus E_s$:

$$|d\varphi_t(x,\xi)v| = \begin{cases} |v|, & v \in E_0(x,\xi) \\ e^t|v|, & v \in E_u(x,\xi) \\ e^{-t}|v|, & v \in E_s(x,\xi) \end{cases}$$

So $\sup|\partial^\alpha(a_{w_j} \circ \varphi_j)| \le C_\alpha e^{N|\alpha|}$

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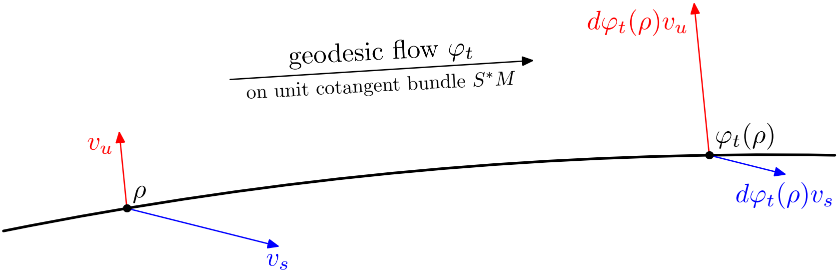
# Picture of the unstable/stable decomposition




## Remarks

- We often ignore the flow direction $E_0$ because there is no expansion or contraction in it
- We also often restrict to $S^*M$, where $u$ lives microlocally, and ignore the dilation direction $\xi \cdot \partial_\xi$
- So the effective dynamics (on a Poincaré section in $S^*M$, transversal to the flow) is similar to 2-dimensional hyperbolic maps (e.g. cat map)

# Pushing the limits of quantization: isotropic symbols

Let us look at the standard quantization on $\mathbb{R}^n$:

$$\mathrm{Op}_h(a)f(x) = (2\pi h)^{-n} \int_{\mathbb{R}^{2n}} e^{\frac{i}{h}\langle x-y,\xi\rangle} a(x,\xi) f(y)\, dy d\xi$$

Composition formula: $\mathrm{Op}_h(a)\,\mathrm{Op}_h(b) = \mathrm{Op}_h(a\#b)$ where

$$a\#b \sim \sum_{k=0}^{\infty} (-ih)^k \sum_{|\alpha|=k} \frac{1}{\alpha!} \partial_\xi^\alpha a \cdot \partial_x^\alpha b \quad \text{as} \quad h \to 0$$

- Use 2 derivatives for each power of $h$. This works if $a, b$ satisfy

$$\sup|\partial^\alpha a|, \sup|\partial^\alpha b| \le C_\alpha h^{-\rho|\alpha|} \quad \text{for some} \quad \rho < \tfrac{1}{2}$$

  with $k$-th term of the above expansion being $\mathcal{O}(h^{(1-2\rho)k})$
- The derivatives of $a_{w_j} \circ \varphi_j$ grow like $e^{N|\alpha|}$. So it appears that $A_{\mathbf{w}} = \mathrm{Op}_h(a_{\mathbf{w}}) + \dots$ until the Ehrenfest time: $N = \frac{1}{2}\log(1/h)$

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# Pushing the limits of quantization: anisotropic symbols

Can we quantize symbols which are rougher in $x$ but smoother in $\xi$?

$$a\#b \sim \sum_{k=0}^{\infty}(-ih)^k \sum_{|\alpha|=k} \frac{1}{\alpha!}\partial_\xi^\alpha a \cdot \partial_x^\alpha b \quad \text{as} \quad h \to 0$$

Can afford to lose $h^{-\rho}$, $\rho < 1$, differentiating in $x$,
if we lose nothing when differentiating in $\xi$

That is, we can take $a, b$ in the class $S_{L_0,\rho}$ defined by the inequalities

$$\sup |\partial_x^\alpha \partial_\xi^\beta a| \leq C_{\alpha\beta} h^{-\rho|\alpha|}$$

Or we could take $a, b$ in the class $S_{L_1,\rho}$ defined by losing in $\xi$ but not in $x$:

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But if $a \in S_{L_1,\rho}$ and $b \in S_{L_0,\rho}$, then the expansion diverges when $\rho > \frac{1}{2}$!

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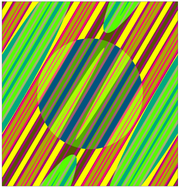
# Pushing the limits of quantization: hyperbolic surfaces

The derivatives of $a_{w_j} \circ \varphi_j$ are only large in the unstable direction.



Using this, we get $A_{\mathbf{w}} = \mathrm{Op}_h(a_{\mathbf{w}}) + \mathcal{O}(h^{1-\rho})$ for times $N \leq \rho \log(1/h)$ for any $\rho < 1$

Here $a_{\mathbf{w}} \in S_{L_s,\rho}(T^*M)$, putting $L_s := E_0 \oplus E_s$:

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$$\sup |X_1 \dots X_k Y_1 \dots Y_\ell a| \leq C h^{-\rho\ell}$$

for all vector fields $X_1, \dots, X_k, Y_1, \dots, Y_\ell$ s.t. $X_1, \dots, X_k$ are tangent to $L$

For $a \in S_{L,\rho}(T^*M)$, can define $\mathrm{Op}_h(a)$ by using a Fourier Integral Operator to conjugate to the case $L = L_0 = \mathrm{span}(\partial_\xi)$

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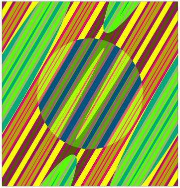

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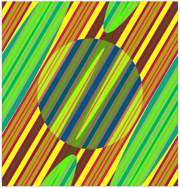

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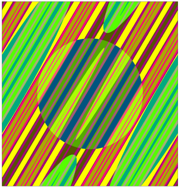

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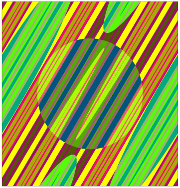

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# Proof of the key estimate: splitting in the middle

Key estimate: $\|A_{\mathbf{w}}\| \leq Ch^{\beta}$ for $\mathbf{w} \in \mathcal{W}(2N_1)$, $N_1 = \lfloor \rho \log(1/h) \rfloor$, $\rho < 1$

Write $A_{\mathbf{w}} = A_{w_{2N_1-1}}(2N_1 - 1) \cdots A_{w_0}(0)$ as $A_{\mathbf{w}} = U(-N_1) A_- A_+ U(N_1)$

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and $L_s = E_0 \oplus E_s$, $\quad L_u = E_0 \oplus E_u$. Reformulate the key estimate as

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But $\mathrm{Op}_h(a_-)$ and $\mathrm{Op}_h(a_+)$ do not lie in the same calculus!

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Assume for simplicity that $\mathbf{w} = 2\dots2$, then

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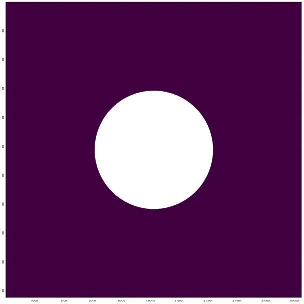
Using cat map for illustration:

$N_1 = 0$



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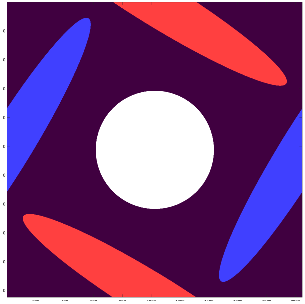

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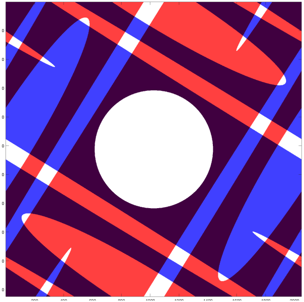

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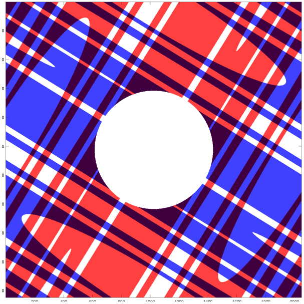

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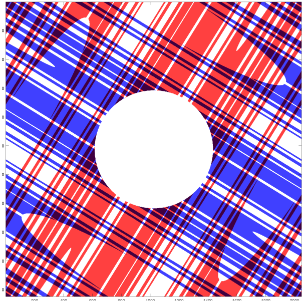

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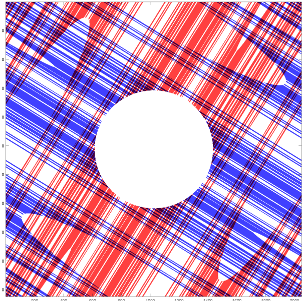
$V_-(N_1)$ is nice in the stable direction, porous up to scale $e^{-N_1} \sim h^\rho$ in the unstable direction

$V_+(N_1)$ is nice in the unstable direction, porous up to scale $h^\rho$ in the stable direction

**Want:** localizations to $V_-$, $V_+$ incompatible

$N_1 = 5$



# Main tool: fractal uncertainty principle (FUP)

**No function can be localized in both position and frequency near a fractal set**

**Definition**

Fix $\nu > 0$. A set $X \subset \mathbb{R}$ is $\nu$-porous up to scale $h$ if for each interval $I \subset R$ of length $h \leq |I| \leq 1$, there is an interval $J \subset I$, $|J| = \nu|I|$, $J \cap X = \emptyset$

Example: mid-third Cantor set $\mathcal{C} \subset [0,1]$ is $\frac{1}{6}$-porous on scales 0 to 1

**Theorem 2 [Bourgain–D '18]**

Assume that $X, Y \subset \mathbb{R}$ are $\nu$-porous up to scale $h$. Then $\exists \beta = \beta(\nu) > 0$:

$$\|\mathbb{1}_X(\tfrac{h}{i}\partial_x)\mathbb{1}_Y(x)\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} = \mathcal{O}(h^\beta) \quad \text{as } h \to 0$$

Note: enough to require porosity up to scales $h^\rho$ where $\rho > \frac{1}{2}$

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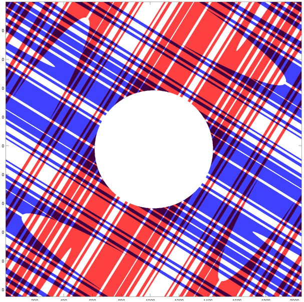
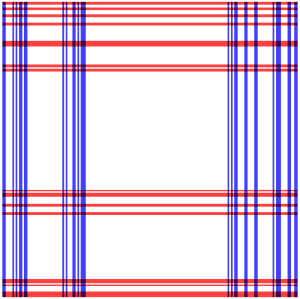
# From FUP to the key estimate

**Need:** $\|\mathrm{Op}_h(a_-)\mathrm{Op}_h(a_+)\| \leq Ch^\beta$,
supp $a_-$ porous in unstable direction,
supp $a_+$ porous in stable direction

**FUP:** $\|\mathrm{Op}_h(b_-)\mathrm{Op}_h(b_+)\| \leq Ch^\beta$,
$b_\pm \in C_c^\infty(\mathbb{R}^2)$, supp $b_- \subset \{\xi \in W_-\}$,
supp $b_+ \subset \{x \in W_+\}$, $W_\pm \subset \mathbb{R}$ porous

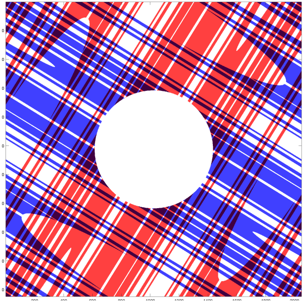




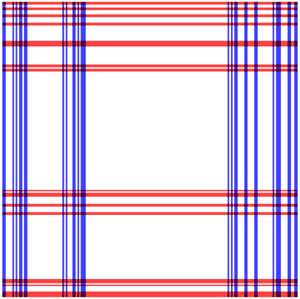
To pass from FUP to the key estimate, we can try to conjugate by a Fourier Integral operator to map $E_u \mapsto \mathbb{R}\partial_\xi$, $E_s \mapsto \mathbb{R}\partial_x$. Not quite possible but after some cutting and pasting can make it work...

# From FUP to the key estimate

**Need:** $\|\mathrm{Op}_h(a_-)\mathrm{Op}_h(a_+)\| \leq Ch^\beta$,
$\operatorname{supp} a_-$ porous in unstable direction,
$\operatorname{supp} a_+$ porous in stable direction



**FUP:** $\|\mathrm{Op}_h(b_-)\mathrm{Op}_h(b_+)\| \leq Ch^\beta$,
$b_\pm \in C_c^\infty(\mathbb{R}^2)$, $\operatorname{supp} b_- \subset \{\xi \in W_-\}$,
$\operatorname{supp} b_+ \subset \{x \in W_+\}$, $W_\pm \subset \mathbb{R}$ porous



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# Removing the log

So far we proved that $\|u\| \le C\log(1/h)\|\operatorname{Op}_h(a)u\|$ by writing

$$u = A_{\mathcal{X}}u + A_{\mathcal{Y}}u, \quad A_{\mathcal{X}} := A_{2\dots 2}$$

and estimating

$$\|A_{\mathcal{X}}u\| \le Ch^{\beta}\|u\|, \quad \|A_{\mathcal{Y}}u\| \le C\log(1/h)\|\operatorname{Op}_h(a)u\| + \mathcal{O}(h^{\infty})\|u\|$$

To get rid of the log prefactor, we will revise the decomposition $u = A_{\mathcal{X}}u + A_{\mathcal{Y}}u$ so that

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# Removing the log: uncontrolled words

- Recall that we are dealing with words of length $2\lfloor \log(1/h) \rfloor$.
  Let's use instead the similar time $20N_0$ where $N_0 = \lfloor \frac{1}{10}\log(1/h) \rfloor$
- Define the set of controlled short logarithmic words

$$\mathcal{Z} := \{\mathbf{w} \in \mathcal{W}(N_0) \mid F(\mathbf{w}) \geq \alpha\}, \quad F(\mathbf{w}) := \frac{\#\{j \mid w_j = 1\}}{N_0}$$

  where $0 < \alpha \ll 1$ is chosen depending on $\beta$ from the key estimate
- Now write $\sum_{\mathbf{w} \in \mathcal{W}(20N_0)} A_{\mathbf{w}} = A_{\mathcal{X}} + A_{\mathcal{Y}}$ where, writing words in $\mathcal{W}(20N_0)$ as concatenations of 20 words in $\mathcal{W}(N_0)$

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- We have $\#(\mathcal{X}) \leq Ch^{100\alpha \log \alpha}$, so for $\alpha \ll_\beta 1$ the triangle inequality + the key estimate $\|A_{\mathbf{w}}\| \leq Ch^\beta$ give $\|A_{\mathcal{X}}\| \leq Ch^{\frac{\beta}{2}}$

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# Removing the log: controlled words I

It remains to bound $A_{\mathcal{Y}} u$ where

$$A_{\mathcal{Y}} := \sum_{\mathbf{w}\in\mathcal{Y}} A_{\mathbf{w}}, \quad \mathcal{Y} := \{\mathbf{w}^{(1)} \dots \mathbf{w}^{(20)} \mid \exists \ell : \mathbf{w}^{(\ell)} \in \mathcal{Z}\}$$

Similarly to the end of Lecture 1, since $u = A_{\mathcal{Z}} u + A_{\mathcal{Z}^\complement} u$, write

$$A_{\mathcal{Y}} u = \sum_{\ell=0}^{19} A_{\mathcal{Z}^\complement}(19N_0) \cdots A_{\mathcal{Z}^\complement}((\ell+1)N_0) A_{\mathcal{Z}}(\ell N_0) u$$

We can show that $\|A_{\mathcal{Z}^\complement}\| \le 1 + \mathcal{O}(h^{\frac{1}{10}})$, so

$$\|A_{\mathcal{Y}} u\| \le 2 \sum_{\ell=0}^{19} \|A_{\mathcal{Z}}(\ell N_0) u\| \le 40 \|A_{\mathcal{Z}} u\|$$

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Now it suffices to estimate $A_{\mathcal{Z}} u$ where $A_{\mathcal{Z}} := \sum_{\mathbf{w}\in\mathcal{Z}} A_{\mathbf{w}}$ and

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Because $N_0 = \lfloor \frac{1}{10} \log(1/h) \rfloor$ and $\frac{1}{10}$ is small, we have

$$A_{\mathcal{Z}} = \mathrm{Op}_h(a_{\mathcal{Z}}) + \mathcal{O}(h^{\frac{1}{10}}), \quad a_{\mathcal{Z}} := \sum_{\mathbf{w}\in\mathcal{Z}} a_{\mathbf{w}}.$$

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By the definition of $\mathcal{Z}$, we have $a_{\mathcal{Z}} \leq \alpha^{-1} a_F$. By sharp Gårding inequality

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We finally need to estimate $\|A_F u\|$ where

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Write $A_F = \frac{1}{N_0}\sum_{j=0}^{N_0-1} A_{F_j}$ where

$$F = \frac{1}{N_0}\sum_{j=0}^{N_0-1} F_j, \quad F_j(\mathbf{w}) := \begin{cases} 1, & w_j = 1 \\ 0, & w_j = 2 \end{cases}$$

Then (pretending that $A_1 + A_2 = I$) we have $A_{F_j} = A_1(j)$, so

$$\|A_F u\| \le \frac{1}{N_0}\sum_{j=0}^{N_0-1} \|A_1(j)u\| \le \|A_1 u\| \le C\|\operatorname{Op}_h(a)u\| + \mathcal{O}(h^\infty)\|u\|$$

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Thank you for your attention!