Minicourse on fractal uncertainty principle
Lecture 1: overview and geometric control

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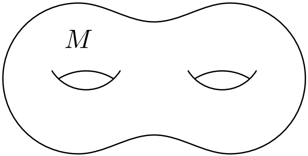
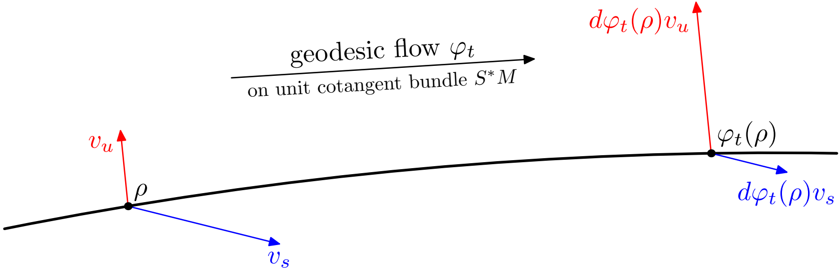
May 31, 2021

Lower bound on mass

- (M, g) compact negatively curved surface
- Geodesic flow on M: a standard model of classical chaos (perturbations diverge exponentially from the original geodesic)



More precisely, we have the stable/unstable decomposition:



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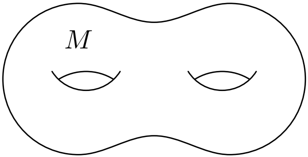
- Eigenfunctions of the Laplacian $-\Delta_g$ studied by quantum chaos



$$(-\Delta_g - \lambda^2)u = 0, \quad \|u\|_{L^2} = 1$$

Theorem 1

Let $\Omega \subset M$ be a nonempty open set. Then there exists c depending on M, Ω but not on λ such that

$$\|u\|_{L^2(\Omega)} \geq c > 0$$

Constant curvature: D–Jin '18, using D–Zahl '16 and Bourgain–D '18

Variable curvature: D–Jin–Nonnenmacher '19, using Bourgain–D '18

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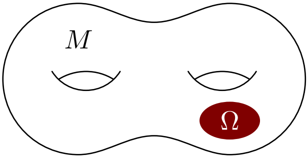

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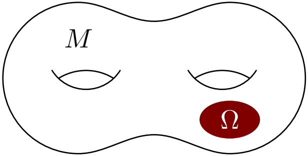

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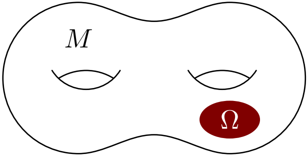

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For bounded λ this follows from unique continuation principle

The new result is in the high frequency limit $\lambda \to \infty$

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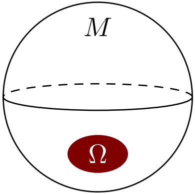

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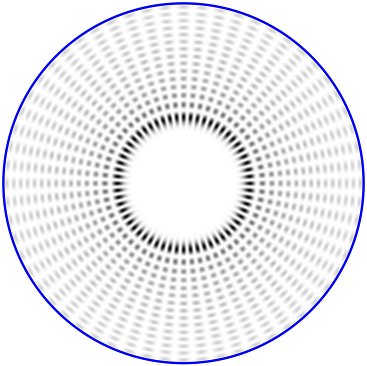
The chaotic nature of geodesic flow is important

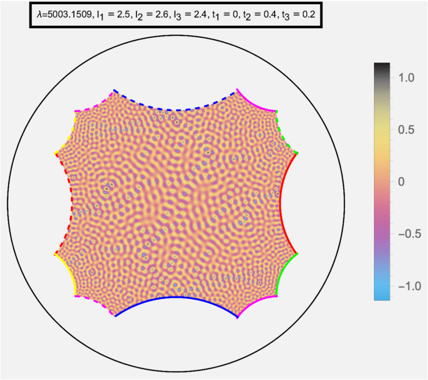
For example, Theorem 1 is false if M is the round sphere

An illustration

Picture on the right courtesy of Alex Strohmaier, using Strohmaier–Uski '12



Disk (Dirichlet b.c.)
Whitespace in the middle



Hyperbolic surface
No whitespace

Application to control theory [Jin '18]

Fix $T > 0$ and nonempty open $\Omega \subset M$. Then there exists $C = C(T, \Omega)$:

$$\|f\|^2_{L^2(M)} \leq C \int_0^T \int_{\Omega} |e^{it\Delta_g} f(x)|^2 \, dxdt \quad \text{for all } f \in L^2(M)$$

Control by any nonempty open set previously known only for flat tori: Haraux '89, Jaffard '90, Burq–Zworski '04, Anantharaman–Macia '14...

Datchev–Jin WIP: an estimate on $C(T, \Omega)$, using Jin–Zhang '17

Application to damped wave equation [Jin '17, D–Jin–Nonnenmacher '19]

Assume that $b \in C^\infty(M)$, $b \geq 0$, $b \not\equiv 0$. Then $\exists \nu > 0$: every solution

$$v \in C^\infty([0, \infty) \times M), \quad (\partial_t^2 + b(x)\partial_t - \Delta_g)v(t, x) = 0$$

to the damped wave equation has exponentially decaying energy:

$$\int_M |\partial_t v|^2 + |\nabla_x v|^2 \, dx = \mathcal{O}(e^{-\nu t}) \quad \text{as} \quad t \to \infty$$

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Microlocal analysis

Localization in position and frequency using semiclassical quantization

$$a(x,\xi)\in C^\infty(T^*M)\ \mapsto\ \operatorname{Op}_h(a)=a\Big(x,\frac{h}{i}\partial_x\Big):C^\infty(M)\to C^\infty(M)$$

Properties of quantization in the semiclassical limit $h\to 0$

- $\operatorname{Op}_h(a)\operatorname{Op}_h(b)=\operatorname{Op}_h(ab)+\mathcal{O}(h)$
- $\operatorname{Op}_h(a)^*=\operatorname{Op}_h(\overline{a})+\mathcal{O}(h)$
- $[\operatorname{Op}_h(a),\operatorname{Op}_h(b)]=-ih\operatorname{Op}_h(\{a,b\})+\mathcal{O}(h^2)$
- $\sup|a|<\infty\quad\Longrightarrow\quad\|\operatorname{Op}_h(a)\|_{L^2\to L^2}=\mathcal{O}(1)$
- $\operatorname{supp} b\subset\{a\neq 0\}\quad\Longrightarrow\quad\|\operatorname{Op}_h(b)u\|\leq C\|\operatorname{Op}_h(a)u\|+\mathcal{O}(h^\infty)\|u\|$

Rescale $\quad(-\Delta_g-\lambda^2)u=0,\quad\lambda\to\infty$

to obtain $\quad(-h^2\Delta_g-1)u=0,\quad h=\lambda^{-1}\to 0$

where $\quad -h^2\Delta_g-1=\operatorname{Op}_h(p^2-1),\quad p(x,\xi)=|\xi|_g$

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$$\text{Rescale}\quad (-\Delta_g-\lambda^2)u=0,\quad \lambda\to\infty$$
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$$\text{where}\quad -h^2\Delta_g-1=\mathrm{Op}_h(p^2-1),\quad p(x,\xi)=|\xi|_g$$

Microlocal version of Theorem 1

Define the cosphere bundle $S^*M := \{(x,\xi) \in T^*M \colon |\xi|_g = 1\}$

Theorem 1′

Let $a \in C_c^\infty(T^*M)$ satisfy $a|_{S^*M} \not\equiv 0$. Then for $h \ll 1$ and all $u \in L^2(M)$

$$\|u\| \leq C\|\operatorname{Op}_h(a)u\| + \frac{C\log(1/h)}{h}\|(-h^2\Delta_g - 1)u\|$$

where the constant C depends only on M, a, but not on h, u

Remarks

- If $(-h^2\Delta_g - 1)u = 0$ then we get $\|\operatorname{Op}_h(a)u\| \geq c\|u\|$ for some $c > 0$
- Implies Theorem 1: $a = a(x) \quad \Longrightarrow \quad \operatorname{Op}_h(a)u = au$
- Sharp: $a|_{S^*M} \equiv 0$, $(-h^2\Delta_g - 1)u = 0 \quad \Longrightarrow \quad \|\operatorname{Op}_h(a)u\| \leq Ch\|u\|$
- Cannot work for $\mathcal{O}(h/\log(1/h))$ quasimodes: Brooks '15, Eswarathasan–Nonnenmacher '17, Eswarathasan–Silberman '17

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Semiclassical measures

Take a high frequency sequence of Laplacian eigenfunctions

$$(-h_j^2\Delta_g - 1)u_j = 0, \quad \|u_j\|_{L^2(M)} = 1, \quad h_j \to 0$$

We say u_j converges weakly to a measure μ on T^*M if

$$\forall a \in C_c^\infty(T^*M): \quad \langle \mathrm{Op}_{h_j}(a)u_j, u_j\rangle_{L^2} \to \int_{T^*M} a\,d\mu \quad \text{as } j \to \infty$$

Call such limits μ semiclassical measures

Basic properties

- μ is a probability measure, $\operatorname{supp}\mu \subset S^*M$
- μ is invariant under the geodesic flow $\varphi_t : S^*M \to S^*M$
- Natural candidate: Liouville measure $\mu_L \sim d\,\mathrm{vol}$ (equidistribution)
- Natural enemy: delta measure δ_γ on a closed geodesic (scarring)

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Theorem 1′: $a|_{S^*M} \not\equiv 0 \implies \|\mathrm{Op}_{h_j}(a)u_j\|_{L^2} \geq c > 0$

Theorem 1″

Let μ be a semiclassical measure on M. Then $\operatorname{supp}\mu = S^*M$

Brief overview of history

- Quantum Ergodicity [Shnirelman '74, Zelditch '87, Colin de Verdière '85]: $\mu = \mu_L$ for density 1 sequence of eigenfunctions
- Quantum Unique Ergodicity conjecture [Rudnick–Sarnak '94]: $\mu = \mu_L$ for all eigenfunctions, that is μ_L is the only semiclassical measure. Proved in the arithmetic case [Lindenstrauss '06]

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Brief overview of history, continued

- Entropy bound [Anantharaman '08, A–Nonnenmacher '07]: $H_{\mathrm{KS}}(\mu) \geq \frac{1}{2}$, in particular $\mu \neq \delta_\gamma$. Here H_{KS} denotes Kolmogorov–Sinai entropy. Note $H_{\mathrm{KS}}(\mu_L) = 1$ and $H_{\mathrm{KS}}(\delta_\gamma) = 0$
- Theorem 1″: between QE and QUE and 'orthogonal' to entropy bound. There exist φ_t-invariant μ with $\operatorname{supp}\mu \neq S^*M$, $H_{\mathrm{KS}}(\mu) > \frac{1}{2}$

Proof of Theorem 1′: first steps

- In the remainder of these lectures, we will ‘prove’ Theorem 1′
- We focus on the case when $(-h^2\Delta_g - 1)u = 0$ and show

$$a|_{S^*M} \not\equiv 0 \quad \Longrightarrow \quad \|u\| \leq C_a \|\operatorname{Op}_h(a)u\| \quad \text{for} \quad h \ll 1$$

- We also assume that (M, g) is a hyperbolic surface ($K = -1$)

Partition of unity

- Take $a_1, a_2 \in C_c^\infty(T^*M \setminus 0; [0,1])$ such that

$$a_1 + a_2 = 1 \text{ near } S^*M, \quad \operatorname{supp} a_1 \subset \{a \neq 0\}, \quad S^*M \setminus \operatorname{supp} a_j \neq \emptyset$$

- Define $A_j := \operatorname{Op}_h(a_j)$. Then $A_1 + A_2 = I$ microlocally near S^*M
- $(-h^2\Delta_g - 1)u = 0 \quad \Longrightarrow \quad u = (A_1 + A_2)u + \mathcal{O}(h^\infty)\|u\|$
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Control and propagation

By ellipticity, since $\operatorname{supp} a_1 \subset \{a \neq 0\}$,

$$\|A_1 u\| \leq C\|\operatorname{Op}_h(a)u\| + \mathcal{O}(h^\infty)\|u\|$$

Conjugate by the half-wave propagator: for $A : L^2(M) \to L^2(M)$ and $t \in \mathbb{R}$

$$A(t) := U(-t)AU(t), \quad U(t) := \exp(-it\sqrt{-\Delta_g})$$

Since $(-h^2\Delta_g - 1)u = 0$, we have $U(t)u = e^{-it/h}u$ and thus

$$\|A_1(t)u\| = \|A_1 u\| \leq C\|\operatorname{Op}_h(a)u\| + \mathcal{O}(h^\infty)\|u\|$$

Egorov's Theorem: if $a \in C_c^\infty(T^*M \setminus 0)$ and t is bounded then

$$U(-t)\operatorname{Op}_h(a)U(t) = \operatorname{Op}_h(a \circ \varphi_t) + \mathcal{O}(h)$$

where $\varphi_t = e^{tH_{|\xi|_g}} : T^*M \setminus 0 \to T^*M \setminus 0$ is the geodesic flow

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where $\varphi_t = e^{tH_{|\xi|_g}} : T^*M \setminus 0 \to T^*M \setminus 0$ is the geodesic flow

Control and propagation

By ellipticity, since $\operatorname{supp} a_1 \subset \{a \neq 0\}$,

$$\|A_1 u\| \leq C\|\operatorname{Op}_h(a)u\| + \mathcal{O}(h^\infty)\|u\|$$

Conjugate by the half-wave propagator: for $A : L^2(M) \to L^2(M)$ and $t \in \mathbb{R}$

$$A(t) := U(-t)AU(t), \quad U(t) := \exp(-it\sqrt{-\Delta_g})$$

Since $(-h^2\Delta_g - 1)u = 0$, we have $U(t)u = e^{-it/h}u$ and thus

$$\|A_1(t)u\| = \|A_1 u\| \leq C\|\operatorname{Op}_h(a)u\| + \mathcal{O}(h^\infty)\|u\|$$

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Sketch of the proof of Egorov's Theorem

- $a \in C_c^\infty(T^*M \setminus 0)$, $U(t) := \exp(-\frac{it}{h}P)$, $P := \sqrt{-h^2\Delta_g}$
- $\varphi_t := e^{tH_p}$ is the Hamiltonian flow of $p(x,\xi) = |\xi|_g$
- Microlocally on $T^*M \setminus 0$, $P = \mathrm{Op}_h(p) + \mathcal{O}(h)$
- Define $A_t := \mathrm{Op}_h(a \circ \varphi_t)$, then, since $\partial_t(a \circ \varphi_t) = \{p, a \circ \varphi_t\}$,

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So $U(t)A_tU(-t) = \mathrm{Op}_h(a) + \mathcal{O}(h)$, giving Egorov's Theorem:

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Words

- Recall: $A_1 = \mathrm{Op}_h(a_1), \quad A_2 = \mathrm{Op}_h(a_2), \quad A(t) := U(-t)AU(t)$
- Words: $\mathcal{W}(N) := \{\mathbf{w} = w_0 \dots w_{N-1} \mid w_0, \dots, w_{N-1} \in \{1, 2\}\}$
- For $\mathbf{w} \in \mathcal{W}(N)$, define

$$A_{\mathbf{w}} := A_{w_{N-1}}(N-1) \cdots A_{w_1}(1) A_{w_0}(0), \quad a_{\mathbf{w}} := \prod_{j=0}^{N-1} (a_{w_j} \circ \varphi_j)$$

Egorov's Theorem $\Longrightarrow$ $A_{\mathbf{w}} = \mathrm{Op}_h(a_{\mathbf{w}}) + \mathcal{O}_N(h)$

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$$u = \sum_{\mathbf{w} \in \mathcal{W}(N)} A_{\mathbf{w}} u$$

- Our proof will work by splitting this sum into 2 parts: controlled (words with enough A_1 in them) and uncontrolled, but small

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Words for the cat map (for illustration only)

$$a_{\mathbf{w}} = \prod_{j=0}^{N-1} (a_{w_j} \circ \varphi_j), \quad \mathbf{w} = w_0 \dots w_{N-1} \in \mathcal{W}(N)$$

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Replace φ_j by the Arnold cat map

$$\varphi : \mathbb{T}^2 \to \mathbb{T}^2, \quad \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2,$$

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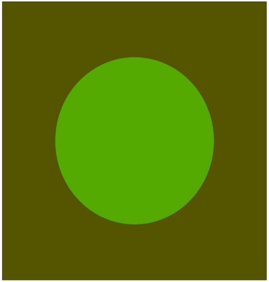

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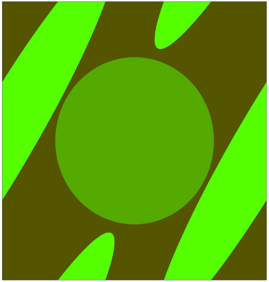

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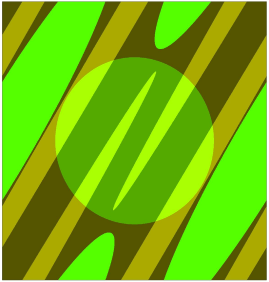

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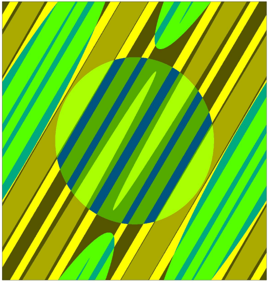

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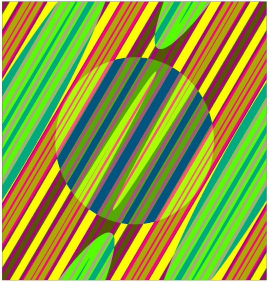

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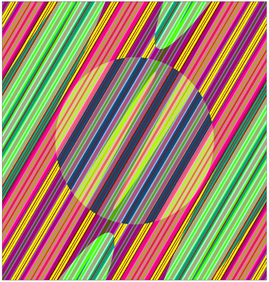

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We see structure due to the hyperbolicity of φ

Schwarz '21: Theorem 1' for quantum cat maps

Geometric control condition (GCC)

In the rest of today's lecture we show Theorem 1':

$$(-h^2\Delta_g - 1)u = 0 \quad \Longrightarrow \quad \|u\| \leq C\|\operatorname{Op}_h(a)u\|$$

in the simple case when $\{a \neq 0\}$ satisfies a geometric control condition:

$$\exists N > 0: \quad S^*M \subset \bigcup_{j=0}^{N-1} \varphi_{-j}(\{a \neq 0\})$$

(This proof actually works on any compact Riemannian manifold.)

Our partition $a_1 + a_2 = 1$ has $S^*M \setminus \operatorname{supp} a_2 \subset \operatorname{supp} a_1 \subset \{a \neq 0\}$.
But we can choose the partition so that these sets are close to each other, so $S^*M \setminus \operatorname{supp} a_2$ satisfies GCC. Then, taking $2\dots 2 \in \mathcal{W}(N)$,

$$\bigcap_{j=0}^{N-1} \varphi_{-j}(\operatorname{supp} a_2) = \emptyset \quad \Longrightarrow \quad a_{2\dots 2} = 0$$

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Controlled/small partition under GCC

We decompose (using that $u = A_1 u + A_2 u$)

$$u = \sum_{\mathbf{w} \in \mathcal{W}(N)} A_{\mathbf{w}} u = A_{\mathcal{X}} u + A_{\mathcal{Y}} u$$

- $A_{\mathcal{X}} := A_{2\dots 2} = A_2(N-1) \cdots A_2(1) A_2(0)$ is uncontrolled
- $A_{\mathcal{Y}} u = \sum_{j=0}^{N-1} A_2(N-1) \cdots A_2(j+1) A_1(j) u$ where j-th term corresponds to the words $\mathbf{w}$ such that $w_j = 1$, $w_{j+1} = \dots = w_{N-1} = 2$
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  Here we could take the constant 2 since $\|A_2\| \leq 1 + \mathcal{O}(h)$
- So $\|u\| \leq CN \| \operatorname{Op}_h(a) u\| + \mathcal{O}(h) \|u\|$ and we remove $\mathcal{O}(h)$ for $h \ll 1$ which gives Theorem 1'

Thank you for your attention!