STATISTICAL PROPERTIES OF EIGENFUNCTIONS

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Abstract. This is a translation of the paper "Статистические свойства собственных функций" which appeared in the Proceedings of the All-USSR School in Differential Equations with Infinite Number of Independent Variables and in Dynamical Systems with Infinitely Many Degrees of Freedom, Dilijan, Armenia, May 21–June 3, 1973; published by the Armenian Academy of Sciences, Erevan, 1974. Translated from the Russian original by Semyon Dyatlov.

1. Introduction

1.1. Let M be a compact smooth manifold and L be an elliptic differential operator on M; if M has a boundary, let us impose for instance Dirichlet boundary conditions. What is the structure of eigenfunctions of L?

We restrict ourselves to the case when $L = \Delta$ is the Laplace operator of a Riemannian metric on M. If $\dim M = n = 1$ then there is only one Laplace operator and its eigenfunctions (up to a change of variables) have the form $e^{\pm i\lambda x}$.

A natural generalization of this to the higher dimensional case is the "quasianalytic" asymptotic, looking for a high frequency eigenfunction with eigenvalue $-\lambda^2$ in the (local) form

$$\sum_{j=1}^{m} g_j(x) e^{i\lambda f_j(x)}.$$

See [1, p.69].

We look for the amplitude $g_j(x)$ and phase $f_j(x)$ in the form of asymptotic series

$$\begin{aligned} g_j(x) &\sim g_j^{(0)}(x) + g_j^{(1)}(x)/\lambda + \ldots; \\ f_j(x) &\sim f_j^{(0)}(x) + f_j^{(1)}(x)/\lambda + \ldots \quad (\lambda \to \infty). \end{aligned} \tag{1}$$

Substituting these asymptotic series into the equation $\Delta u + \lambda^2 u = 0$ we obtain a sequence of equations on the functions $f_j^{(k)}$ and $g_j^{(k)}$, starting from the following equation on $f_j^{(0)}(x)$:

$$|\operatorname{grad} f_j^{(0)}(x)| \equiv 1. \tag{2}$$

When considering this equation globally one has to keep in mind that $\lambda f_j^{(0)}(x)$ plays the role of phase, so when going around a loop it may increase by a multiple of 2π. Moreover, the number of leaves m of the function $f^{(0)}(x)$ is different at different points of M and these leaves are glued to each other along certain $n-1$ dimensional submanifolds (those submanifolds, known as caustics, are boundaries of envelopes of characteristics of the equation (2)). In particular, in some region in M (known as the shadow region) the function $f^{(0)}$ is not defined at all.

When crossing a caustic from one leaf to the other, the phase has a shift which is a multiple of $\frac{\pi}{2}$ (the Arnold–Maslov index [2]). If M has a boundary, then pairs of leaves are also glued to each other there as well, and there is a phase shift when crossing from one leaf to the other, depending on the boundary conditions used.

Typically there exists no global solution of the equation (2) with prescribed multi-valued behavior. This is certainly true for compact Riemannian manifolds of negative curvature. More 'real-life' examples are given by Laplace operators on domains with concave boundary (for instance the interior of a hypocycloid, or a square minus a disk). In these cases the eigenfunctions, which undoubtedly exist at arbitrarily large frequencies, certainly do not have the "quasiclassical" form. *So what do they look like?*

1.2. The above question is not well-defined since we did not specify what information we want to know about an eigenfunction. For instance, one could interpret this question literally. Let the manifold M be very large (the entire universe). Assume that it is excited at a wavelength in the visible light range. Then our eyes, observing at some point of M, will see something on the 'celestial sphere'.

What our eyes see is to some extent the answer to the question 'what does an eigenfunction look like?'.

In the 'quasiclassical' case we will see on the celestial sphere several separate 'stars'. If we move on the manifold, then the relative position of these 'stars' and their brightness will vary. In particular, when crossing a caustic we will see that two 'stars' merge into one (at which point their brightness increases by a scale) and then disappear (or the same events in reverse).

If M has negative curvature, then we will see something opposite. Typically the entire celestial sphere will appear *uniformly bright*; moreover, the brightness does not depend on our position and direction. This is the result whose precise formulation and a sketch of the proof are given in the rest of this talk.

2. The result

2.1. Let Δ be the Laplace operator on M and put $\Lambda = \sqrt{-\Delta}$. We may define Sobolev spaces $H_s(M)$ as follows:

$$H_s = (\Lambda + 1)^{-s} L_2, \quad \|u\|_s = \|(\Lambda + 1)^s u\|_{L^2} = \|(\Lambda + 1)^s u\|_0. \tag{3}$$

In particular if $u = u_k$ is an eigenfunction of the operator Δ with eigenvalue $-\lambda_k^2$, then $\|u_k\|_s = (\lambda_k + 1)^s$, since from now on we assume that eigenfunctions are normalized: $\|u_k\|_0 = 1$.

Denote by T^*M the cotangent bundle of M, T_x^*M be the cotangent space at a point $x \in M$, S^*M be the unit cotangent bundle, and S_x^*M be the unit sphere in T_x^*M.

Let $A(x, \xi)$ $(x \in M, \xi \in T_x^*M)$ be a pseudodifferential symbol of order 0, i.e. A is homogeneous of order 0 in ξ and smooth when $\xi \neq 0$.

Let $\widehat{A}$ be a pseudodifferential operator with symbol $A(x, \xi)$. Our object of study is the asymptotic behavior of the expression $(\widehat{A} u_k, u_k)$ (where $(\bullet, \bullet)$ denotes the inner product of $L^2(M)$) as $k \to \infty$.

Note that the above inner product depends only on the symbol $A(x, \xi)$. Indeed, if $\widehat{A}_1$ and $\widehat{A}_2$ are two pseudodifferential operators with the symbol $A(x, \xi)$, then $\|(\widehat{A}_1 - \widehat{A}_2)u\|_0 \le C\|u\|_{-1}$; in particular, $\|(\widehat{A}_1 - \widehat{A}_2)u_k\|_0 \le C\lambda_k^{-1}$.

Thus we may fix a way to construct a pseudodifferential operator $\widehat{A}$ from its symbol (i.e. fix local coordinates on M, partition of unity etc.); the relation between the pseudodifferential operator $\widehat{A}$ and its symbol will be bijective and linear.

Next, the symbol $A(x, \xi)$ is determined (as it is homogeneous) by its restriction $A(x, \omega)$ to S^*M $(x \in M, \omega \in S_x^*M)$. Denote by $\langle A \rangle$ the average of $A(x, \omega)$ over S^*M:

$$\langle A \rangle = \frac{1}{V(M)\Omega_n} \int_{S^*M} A(x, \omega)\, dx d\omega;$$

here $V(M)$ is the volume of M and Ω_n is the area of the unit sphere in the n-dimensional Euclidean space.

We assume that the eigenvalues of Δ are simple. Then we introduce the probability measure $P_s(k)$ on the set $\mathbb{Z}_+$ of natural numbers depending on a parameter $s > 0$:

$$P_s(k) = \frac{e^{-\lambda_k^2 s}}{\sum_{j=1}^{\infty} e^{-\lambda_j^2 s}}. \tag{4}$$

A subsequence $\{u_{k_j}\}$ of the sequence $\{u_k\}$ is called a density one subsequence if

$$\lim_{s \to 0} \sum_{j=1}^{\infty} P_s(k_j) = 1, \tag{5}$$

this definition of the density of a sequence is somewhat more general than the usual one.

We may now formulate the main result:

Theorem 1. *Let B be a bounded set in $C^\infty(S^*M)$. Then there exists a density one subsequence $\{u_{k_j}\}$ of $\{u_k\}$ such that $\lim_{j\to\infty}(\widehat{A}u_{k_j}, u_{k_j}) = \langle A\rangle$ for each $A(x,\omega) \in B$ (here $\widehat{A}$ is the pseudodifferential operator constructed from the symbol $A(x,\omega)$).*

An equivalent formulation is the following: let $\mathcal{F}_s(A;t) = P_s\{k \mid (\widehat{A}u_k, u_k) < t\}$ be the distribution function of the 'random variable' $(\widehat{A}u_k, u_k)$ with respect to the measure P_s. Then

$$\lim_{s\to 0}\mathcal{F}_s(A;t) = \begin{cases} 0, & t < \langle A\rangle, \\ 1, & t > \langle A\rangle. \end{cases} \tag{6}$$

The relation of this theorem with the explanation in the introduction is specified below.

2.2. The proof of the theorem consists of four steps. Most of the argument applies to any manifold; the fact that M is negatively curved is only used in one place.

For brevity, henceforth we denote a point $(x,\omega) \in S^*M$ by the letter z.

Lemma 2.1. *There exists a sequence of probability measures $\mu_k(dz)$ on S^*M such that*

$$\lim_{k\to\infty}\left[(\widehat{A}u_k, u_k) - \int_{S^*M} A(z)\mu_k(dz)\right] = 0 \tag{7}$$

uniformly for $A(z)$ in any bounded set $B \subset C^\infty(S^*M)$.

Proof. To prove this, consider the expression $(\widehat{A}u_k, u_k)$ for a fixed k. It is linear in $A(z)$, so we may regard it as the value at $A(z)$ of some generalized function $U_k(z)$:

$$(\widehat{A}u_k, u_k) = \langle U_k(z), A(z)\rangle \tag{8}$$

(here $\langle \bullet, \bullet\rangle$ denote the inner product on S^*M rather than on M). In fact for each finite k the function $U_k(z)$ will be smooth but this is not important for us.

Let $A(z)$ be a real-valued function. Then by the Gårding inequality [4]

$$\begin{aligned} \operatorname{Re}(\widehat{A}u_k, u_k) &\ge \inf_z A(z)\|u_k\|_0^2 - C\|u_k\|_{-\frac{1}{2}}^2, \\ |\operatorname{Im}(\widehat{A}u_k, u_k)| &\le C\|u_k\|_{-\frac{1}{2}}^2. \end{aligned} \tag{9}$$

Moreover by construction

$$(\widehat{I}u_k, u_k) = 1$$

(where $\widehat{I}$ is the identity operator with symbol $I(z) \equiv 1$). The constant C is bounded when $A(z) \in B$. Therefore

$$\begin{aligned} \operatorname{Re}\langle U_k(z), A(z)\rangle &\geq \inf_z A(z) - c\lambda_k^{-1}, \\ |\operatorname{Im}\langle U_k(z), A(z)\rangle| &\leq c\lambda_k^{-1}, \\ \langle U_k(z), 1\rangle &= 1. \end{aligned} \tag{10}$$

If not for the term $C\lambda_k^{-1}$ on the right-hand sides then these would imply that $U_k(z)$ is a probability measure.

However, for large k the offending term becomes small and we may turn the generalized function $U_k(z)$ into a measure $\mu_k(dz)$, slightly changing its value on functions $A(z) \in B$. □

2.3. It is not difficult to understand the meaning of the measure $\mu_k(dz)$. Consider a solution to the wave equation of the form $u_k(x)e^{i\lambda_k t}$ where λ_k is very large. Then $\mu_k(dz) = \mu_k(dx \wedge d\omega)$ is (up to a factor) the *energy flux* through the volume element dx on M inside the solid angle $d\omega$. This interpretation lets us understand an important property of the measure $\mu_k(dz)$: its invariance under the geodesic flow.

The geodesic flow is a one-parameter group G_t of transformations of the manifold S^*M which preserve the volume dz. If $z = (x, \omega) \in S^*M$ then $G_t(z)$ is constructed as follows: take the geodesic of length t through the point $x \in M$ in the direction dual to $\omega \in S_x^*M$. If x' is its endpoint and $\omega' \in S_{x'}^*M$ is the unit cotangent vector dual to the unit tangent vector to the geodesic at the point x', then we put $z' = (x', \omega') = G_t(z)$.

Lemma 2.2. *Let B be a bounded set in $C^\infty(S^*M)$, $T > 0$, and $\varepsilon > 0$. Then there exists k_0 such that for all $k > k_0$, $A(z) \in B$, $|t| < T$*

$$\left| \int A(G_t^{-1}(z))\mu_k(dz) - \int A(z)\mu_k(dz) \right| \leq \varepsilon. \tag{11}$$

It is known that every 'high frequency' solution $u(x, t)$ to the wave equation the energy is propagated along geodesics with unit speed. If $u(x, t) = u_k(x)e^{i\lambda_k t}$ then the energy flux is moreover stationary.

Therefore the measure $\mu_k(dz)$ is invariant under the geodesic flow with the same precision with which we can define the energy flux (i.e. the more precise the higher the frequency λ_k). Of course this argument does not replace a rigorous proof.

2.4. The following lemma shows that the measures $\mu_k(dz)$ are 'Lebesgue on average'.

Lemma 2.3. *Let B be a bounded set in $C^\infty(S^*M)$. Then*

$$\lim_{s \to 0} \sum_k P_k(s) \int_{S^*M} A(z)\mu_k(dz) = \langle A\rangle$$

uniformly in $A \in B$.

Proof. The proof uses a typical idea from spectral theory. Let $G(x, x', s)$ be the solution to the problem

$$\begin{cases} \frac{\partial}{\partial s} G(x, x', s) = \Delta_{x'} G(x, x', s) \\ G(x, x', s)|_{s=0} = \delta(x - x'); \end{cases} \tag{12}$$

$$G(x, x', s) = \sum_k u_k(x) u_k(x') e^{-\lambda_k^2 s}. \tag{13}$$

It follows that

$$\begin{aligned} &\sum_k P_s(k) \int_{S^*M} A(z)\mu_k(dz) \sim \sum_k P_s(k)(\widehat{A}u_k, u_k) \\ &\sim \frac{1}{V(M)} (2\pi s)^{\frac{n}{2}} \int_M \widehat{A}_{x'} G(x, x', s)|_{x=x'}\, dx. \quad (s \to 0) \end{aligned} \tag{14}$$

As $s \to 0$ the function $G(x, x', s)$ behaves as follows: for $x' \neq x$ it is exponentially small, while for x' close to x we have $G(x, x', s) \sim (2\pi s)^{-n/2} e^{-|x'-x|^2/(2s)}$.

This function has a sharp peak at $x' = x$ when $s = 0$. Thus we may asymptotically compute the result of applying to it the pseudodifferential operator $\widehat{A}$. We obtain

$$\widehat{A}_{x'} G(x, x', s)|_{x'=x} \sim (2\pi s)^{-\frac{n}{2}} \frac{1}{\Omega_n} \int_{S^*M} A(x, \omega)\, dx. \quad (s \to 0) \tag{15}$$

From here

$$\sum_k P_k(s) \int_{S^*M} A(z)\mu_k(dz) \sim \frac{1}{V(M)\Omega_n} \int_M dx \int_{S_x^*M} A(x, \omega)\, d\omega = \langle A \rangle \tag{16}$$

as needed. ∎

2.5. We now have everything needed for the proof of the theorem. More precisely:

(a) The manifold S^*M and the smooth flow G_t on it which preserves the Lebesgue measure dz;

(b) The sequence of measures $\mu_k(dz)$ which are 'almost invariant' under G_t (Lemma 2.2) and 'Lebesgue on average' (Lemma 2.3).

We will prove that for each function $A(z) \in C$ and each $\varepsilon > 0$ there exists a density one subsequence $\{k_j\}$ such that

$$\left| \int_{S^*M} A(z)\mu_{k_j}(dz) - \langle A \rangle \right| < \varepsilon. \tag{17}$$

This obviously implies the theorem.

For simplicity we assume that the measures μ_k are exactly invariant under G_t.

Proof of the theorem. By Birkhoff's Theorem there exists a full measure set $\mathfrak{M}$ on S^*M such that for each $z \in \mathfrak{M}$ there exists $\lim_{T\to\infty} \int_0^T A(G_t(z))\,dt$ – the time average of the function $A(z)$ which is a measurable function of z. Since the manifold M is negatively curved, the flow G_t is ergodic (this is the only place where we use that the curvature is negative).

Therefore, the time average is equal to $\langle A\rangle$ almost everywhere on $\mathfrak{M}$.

Consider the set

$$\mathfrak{M}_1(\tau,\varepsilon) = \left\{ z\colon \left| \frac{1}{T}\int_0^T A(G_t(z))\,dt - \langle A\rangle \right| < \varepsilon \ \forall T \geq \tau \right\}.$$

It is easy to see that $\forall \varepsilon > 0\ \forall \delta > 0\ \exists \tau > 0$

$$\operatorname{mes}(S^*M \setminus \mathfrak{M}_1(\tau,\varepsilon)) < \delta$$

where mes denotes Lebesgue measure.

The flow G_t is *smooth*. Thus there exists an open neighborhood $\widetilde{\mathfrak{M}}$ of the set $\mathfrak{M}_1(\tau,\varepsilon)$ with smooth boundary and such that

$$\forall z \in \widetilde{\mathfrak{M}} \quad \left| \frac{1}{\tau}\int_0^\tau A(G_t(z))\,dt - \langle A\rangle \right| < 2\varepsilon.$$

Let $\chi(z)$ be the characteristic function of $\widetilde{\mathfrak{M}}$.

At this point it is appropriate to use the language of probability. Each sequence $\{\alpha_k\}$ can be treated as a random variable with respect to the measure $P_s(k)$; then $\sum_k P_s(k)\alpha_k = M_s(\alpha)$ will be its expected value.

As proved before the measures μ_k are 'Lebesgue on average'. In particular this means that

$$\lim_{s\to 0} M_s\biggl(\int_{S^*M} (1-\chi(z))\mu_k(dz) \biggr) < \delta$$

so that for sufficiently small s

$$M_s\biggl(\int_{S^*M} (1-\chi(z))\mu_k(dz) \biggr) < \delta.$$

The random value in parentheses is nonnegative. Thus by the Chebyshev inequality

$$P_s\biggl\{ k\colon \int_{S^*M} (1-\chi(z))\,\mu_k(dz) < \sqrt{\delta} \biggr\} > 1 - \sqrt{\delta}. \tag{18}$$

Assume that the measure μ_k satisfies the condition $\int_{S^*M}(1-\chi(z))\mu_k(dz) < \sqrt{\delta}$. Then

$$\begin{aligned}\left|\int A(z)\mu_k(dz) - \langle A\rangle\right| &= \left|\int (1-\chi(z))A(z)\mu_k(dz) - \int (1-\chi(z))\langle A\rangle \mu_k(dz)\right.\\ &\quad \left. + \int \chi(z)A(z)\mu_k(dz) - \int \chi(z)\langle A\rangle\mu_k(dz)\right|\\ &\le \sqrt{\delta}\big(\max|A(z)| + \langle A\rangle\big)\\ &\quad + \int \chi(z)\mu_k(dz)\cdot\left|\frac{1}{\tau}\int_0^\tau A(G_t(z))\,dt - \langle A\rangle\right|\\ &\le 2\sqrt{\delta}\max|A(z)| + 2\varepsilon.\end{aligned}$$

We have thus shown that $\forall\varepsilon > 0\ \forall\delta > 0\ \exists s_0 > 0\ \forall s < s_0$

$$P_s\left\{k\colon \left|\int A(z)\mu_k(dz) - \langle A\rangle\right| < \varepsilon\right\} > 1-\delta.$$

Therefore $\lim_{s\to 0} P_s\{\dots\} = 1$. The theorem is proved. □

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