Fractal uncertainty principle and quantum chaos

Semyon Dyatlov

October 4, 2018

Overview

- This talk presents two recent results in quantum chaos
- Central ingredient: fractal uncertainty principle (FUP)

No function can be localized
in both position and frequency
near a fractal set

- Using tools from
  - Microlocal analysis (classical/quantum correspondence)
  - Hyperbolic dynamics (classical chaos)
  - Fractal geometry
  - Harmonic analysis

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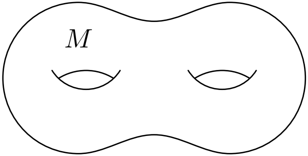
First result: lower bound on mass

- (M, g) compact hyperbolic surface
- Geodesic flow on M: a standard model of classical chaos (perturbations diverge exponentially from the original geodesic)
- Eigenfunctions of the Laplacian $-\Delta_g$ studied by quantum chaos



$$(-\Delta_g - \lambda^2)u = 0, \quad \|u\|_{L^2} = 1$$

Theorem 1 [D–Jin '18, using D–Zahl '16 and Bourgain–D '18]

Let $\Omega \subset M$ be a nonempty open set. Then there exists c depending on M, Ω but not on λ such that

$$\|u\|_{L^2(\Omega)} \geq c > 0$$

For bounded λ this follows from unique continuation principle

The new result is in the high frequency limit $\lambda \to \infty$

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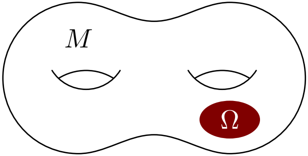

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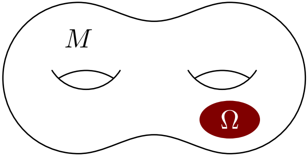

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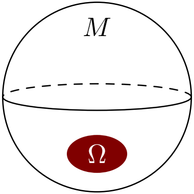

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The chaotic nature of geodesic flow is important

For example, Theorem 1 is false if M is the round sphere

Theorem 1

Let M be a hyperbolic surface, $\Omega \subset M$ nonempty open set. Then $\exists c_\Omega > 0$:

$$(-\Delta_g - \lambda^2)u = 0 \quad \Longrightarrow \quad \|u\|_{L^2(\Omega)} \geq c_\Omega \|u\|_{L^2(M)}$$

Application to control theory:

Theorem 2 [Jin '17]

Fix $T > 0$ and nonempty open $\Omega \subset M$. Then there exists $C = C(T, \Omega)$:

$$\|f\|^2_{L^2(M)} \leq C \int_0^T \int_\Omega |e^{it\Delta_g} f(x)|^2 \, dxdt \quad \text{for all } f \in L^2(M)$$

Control by any nonempty open set previously known only for flat tori: Haraux '89, Jaffard '90

Work in progress

- Datchev–Jin: an estimate on c_Ω in terms of Ω (using Jin–Zhang '17)
- D–Jin–Nonnenmacher: Theorems 1 and 2 for surfaces of variable negative curvature

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Weak limits of eigenfunctions

Original motivation: study high frequency sequences of eigenfunctions

$$(-\Delta_g - \lambda_j^2)u_j = 0, \quad \|u_j\|_{L^2} = 1, \quad \lambda_j \to \infty$$

in terms of weak limit: probability measure μ on M such that $u_j \to \mu$ i.e.

$$\int_M a(x)|u_j(x)|^2 \, d\,\mathrm{vol}_g(x) \to \int_M a \, d\mu \quad \text{for all} \quad a \in C^\infty(M)$$

Theorem 1 $\Rightarrow$ for hyperbolic surfaces, $\operatorname{supp}\mu = M$: 'no whitespace'

A (much) stronger property is equidistribution: $\mu = d\,\mathrm{vol}_g$

- Quantum ergodicity: most eigenfunctions equidistribute if the geodesic flow is chaotic: Shnirelman '74, Zelditch '87, Colin de Verdière '85 … Zelditch–Zworski '96
- QUE conjecture: all eigenfunctions equidistribute for strongly chaotic systems. Only proved in arithmetic situations: Lindenstrauss '06
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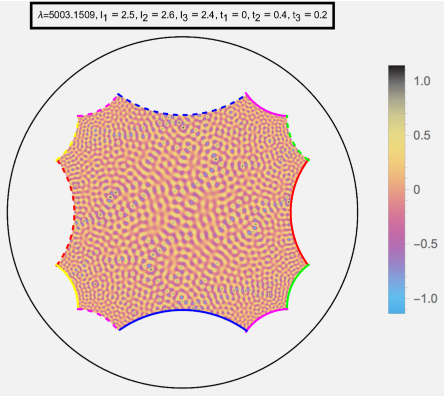
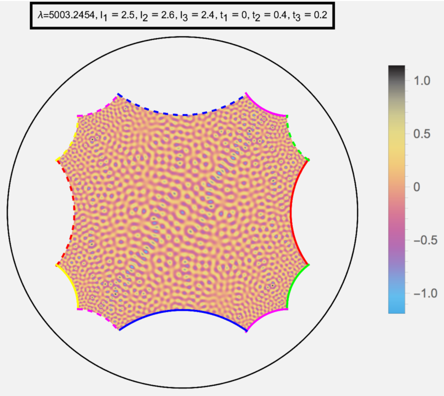
Pictures of eigenfunctions (courtesy of Alex Strohmaier)

Hyperbolic surfaces, using Strohmaier–Uski '12





- No whitespace (Theorem 1)
- Equidistribution conjectured by QUE

Pictures of eigenfunctions (courtesy of Alex Barnett)

One can also study Dirichlet eigenfunctions on a domain with boundary
The geodesic flow is replaced by the billiard ball flow

Completely integrable
Whitespace in the center (easy)

Mildly chaotic
Whitespace on the sides (conjectured)
Lack of equidistribution [Hassell '10]

Strongly chaotic
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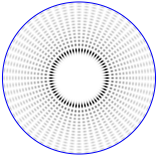

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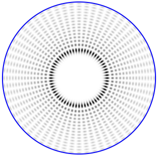

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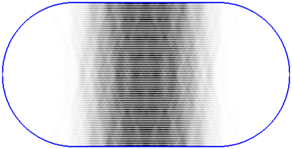

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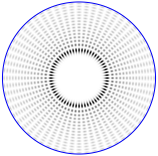

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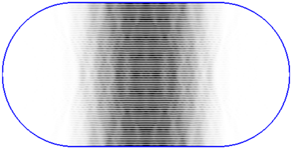

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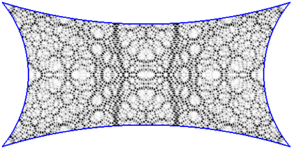

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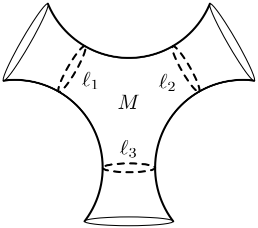
Second result: spectral gaps for noncompact surfaces

(M, g) convex co-compact hyperbolic surface

Pictures of resonances (by David Borthwick and Tobias Weich)



Resonances: zeroes of the Selberg zeta function

$$Z_M(s) = \prod_{\ell \in \mathcal{L}_M} \prod_{k=0}^{\infty} \left(1 - e^{-(s+k)\ell}\right)$$

$\mathcal{L}_M = \{$lengths of primitive closed geodesics$\}$

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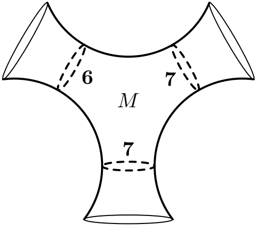

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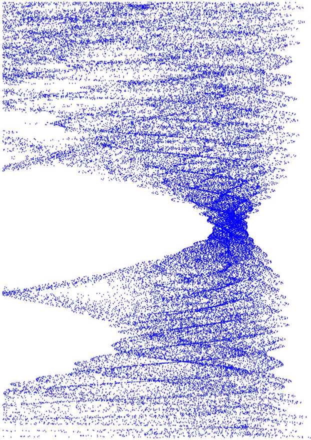

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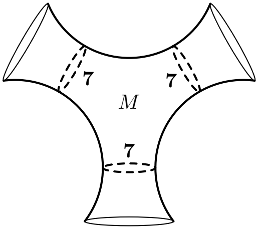

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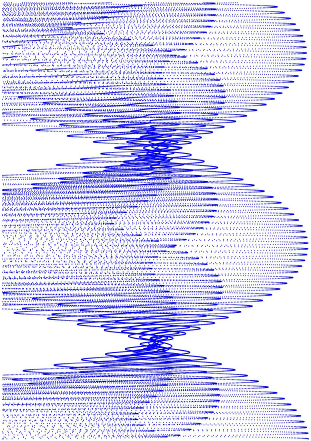

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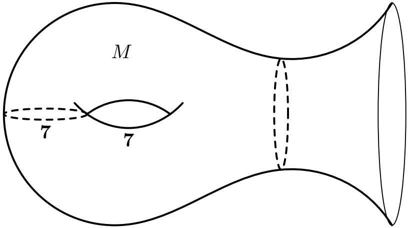

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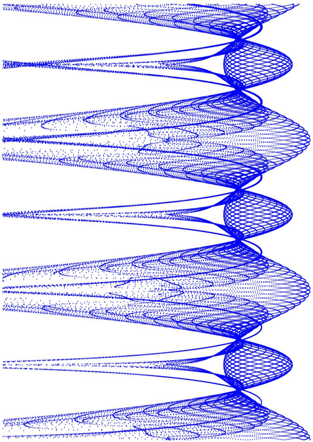

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Theorem 3 [D–Zahl '16, Bourgain–D '18, D–Zworski '18]

Let M be a convex co-compact hyperbolic surface. Then there exists an essential spectral gap of size $\beta = \beta(M) > 0$, namely M has only finitely many resonances s with $\operatorname{Re} s > \frac{1}{2} - \beta$

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- Gap for 'thin' open systems: Ikawa '88, Gaspard–Rice '89, Nonnenmacher–Zworski '09
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Main ingredient: fractal uncertainty principle (FUP)

Standard uncertainty principle for Fourier transform with 'Planck constant' $0 < h \ll 1$:

$$f \in L^2(\mathbb{R}), \quad \operatorname{supp} \hat{f} \subset [0,1] \quad \Longrightarrow \quad \|\mathbf{1}_{[0,h]} f\|_{L^2(\mathbb{R})} \le h^{1/2} \|f\|_{L^2(\mathbb{R})}$$

"Cannot concentrate in both position and frequency near one point"

Fractal uncertainty principle: if X, Y are h-neighborhoods of 'fractal sets' then for some $\beta > 0$

$$\operatorname{supp} \hat{f} \subset h^{-1} \cdot Y \quad \Longrightarrow \quad \|\mathbf{1}_X f\|_{L^2(\mathbb{R})} \le C h^{\beta} \|f\|_{L^2(\mathbb{R})}$$

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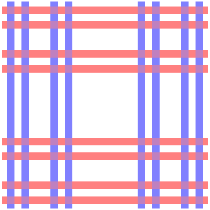

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Statement of the fractal uncertainty principle

Definition

Fix $\nu > 0$. A set $X \subset \mathbb{R}$ is ν-porous up to scale h if for each interval $I \subset \mathbb{R}$ of length $h \leq |I| \leq 1$, there exists an interval $J \subset I$, $|J| = \nu|I|$, $J \cap X = \emptyset$

Example: mid-third Cantor set $\mathcal{C} \subset [0,1]$ is $\frac{1}{6}$-porous up to scale 0

Theorem 4 [Bourgain–D '18]

Let X, Y be ν-porous up to scale $h \ll 1$. Then there exists $\beta = \beta(\nu) > 0$:

$$f \in L^2(\mathbb{R}), \quad \operatorname{supp} \hat{f} \subset h^{-1} \cdot Y \quad \Longrightarrow \quad \|\mathbb{1}_X f\|_{L^2(\mathbb{R})} \leq Ch^\beta \|f\|_{L^2(\mathbb{R})}$$

Recent progress: Jin–Zhang '17 (explicit $\beta(\nu)$),
Han–Schlag '18 (some higher dimensional cases)

Statement of the fractal uncertainty principle

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Fix $\nu > 0$. A set $X \subset \mathbb{R}$ is ν-porous up to scale h if for each interval $I \subset \mathbb{R}$ of length $h \leq |I| \leq 1$, there exists an interval $J \subset I$, $|J| = \nu|I|$, $J \cap X = \emptyset$

Example: mid-third Cantor set $\mathcal{C} \subset [0,1]$ is $\frac{1}{6}$-porous up to scale 0

Theorem 4 [Bourgain–D '18]

Let X, Y be ν-porous up to scale $h \ll 1$. Then there exists $\beta = \beta(\nu) > 0$:

$$f \in L^2(\mathbb{R}), \quad \operatorname{supp} \hat{f} \subset h^{-1} \cdot Y \quad \Longrightarrow \quad \|\mathbf{1}_X f\|_{L^2(\mathbb{R})} \leq Ch^{\beta} \|f\|_{L^2(\mathbb{R})}$$

Recent progress: Jin–Zhang '17 (explicit $\beta(\nu)$), Han–Schlag '18 (some higher dimensional cases)

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- Write $X \subset \bigcap_j X_j$ where each $X_j \subset X_{j-1}$ has holes on scale $2^{-j} \ge h$
- Will show: for each j, $\|\mathbf{1}_{X_j} f\|_{L^2} \le (1-\epsilon)\|1_{X_{j-1}} f\|_{L^2}$
- This requires a lower bound on the mass of f on the 'holes' in $\mathbb{R} \setminus X_j$
- Such bounds exist if we know about decay of $\hat{f}$, e.g.

$$|\hat{f}(\xi)| \le C e^{-w(\xi)} \quad \text{where} \quad \int_{\mathbb{R}} \frac{w(\xi)}{1+\xi^2}\, d\xi = \infty$$

- To pass from $\operatorname{supp} \hat{f} \subset h^{-1} \cdot Y$ to Fourier decay bounds, take the convolution $f * g$, $\widehat{f * g} = \hat{f}\hat{g}$, where g is compactly supported and $\hat{g}$ has the right decay but only on $h^{-1} \cdot Y$
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Proof of Theorem 1 (lower bound on mass)

- Assume Theorem 1 fails, i.e. there exist $\lambda = \lambda_j \to \infty$, $u = u_j$ s.t.

$$(-\Delta_g - \lambda_j^2)u_j = 0, \quad \|u_j\|_{L^2(M)} = 1, \quad \|u_j\|_{L^2(\Omega)} \to 0$$

- Using semiclassical quantization $\mathrm{Op}_h(a) = a(x, \frac{h}{i}\partial_x)$, $h := \lambda^{-1} \ll 1$, $a \in C^\infty(T^*M)$, study localization of u in the phase space T^*M
- $(-\Delta_g - \lambda^2)u = 0 \implies$ phase space localization of u is invariant under the geodesic flow $\varphi_t : T^*M \to T^*M$
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- Thus u lives on the sets

$$\Gamma_\pm(T) := \{\rho \in T^*M \mid \varphi_{\mp t}(\rho) \notin U \text{ for all } t \in [0, T]\}, \quad T := \log(1/h)$$

- Can use microlocal analysis to make sense of localization on Γ_+, Γ_- but not on $\Gamma_+ \cap \Gamma_-$

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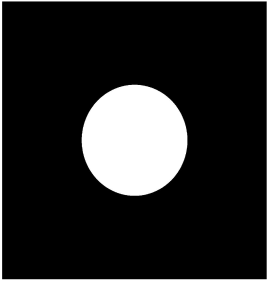
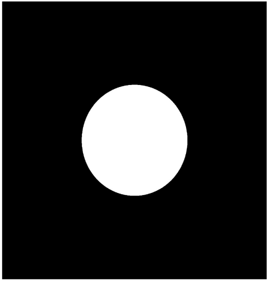
$\Gamma_{\pm}(T)$ smooth in stable/unstable directions, porous up to scale h in unstable/stable ones:



$\Gamma_{-}(T)$, $T = 0$



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FUP $\implies$ no function can be localized on both $\Gamma_{+}(T)$ and $\Gamma_{-}(T)$!

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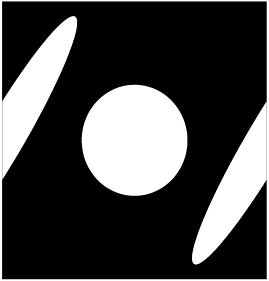
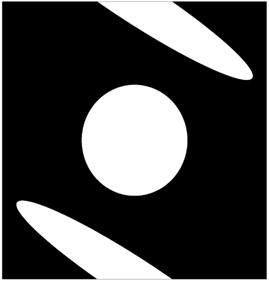


$\Gamma_-(T)$, $T = 1$



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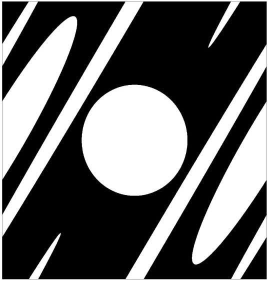
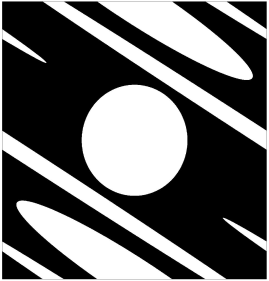


$\Gamma_-(T)$, $T = 2$



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$\Gamma_{-}(T)$, $T = 3$ $\qquad$ $\Gamma_{+}(T)$, $T = 3$

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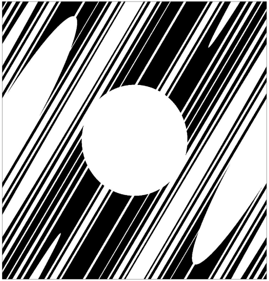
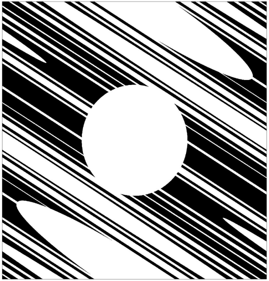


$\Gamma_-(T)$, $T = 4$



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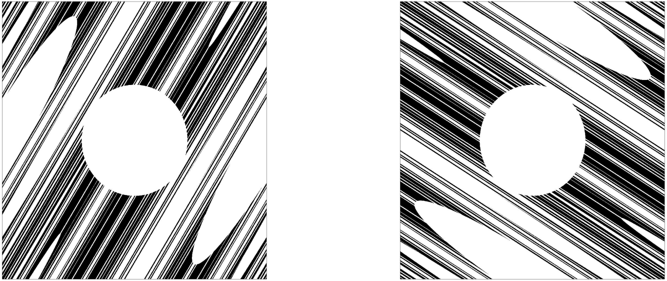

$\Gamma_-(T)$, $T = 5$ $\Gamma_+(T)$, $T = 5$

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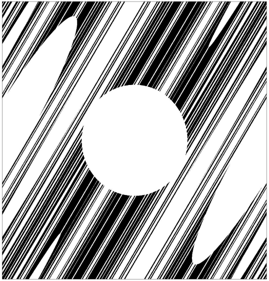

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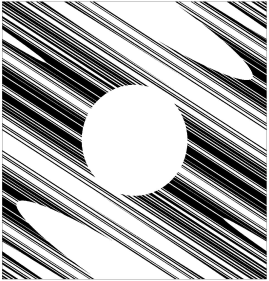

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Remarks on the proof of Theorem 1

- The porosity property for $\Gamma_\pm(T)$ is proved using that $U \neq \emptyset$ and unique ergodicity of horocycle flows. The condition $\varphi_{\mp t}(\rho) \notin U$ gives holes on scale e^{-t}
- To make sense of localization on $\Gamma_\pm(T)$, where $T = \log(1/h)$, we use the calculus developed in D–Zahl '16
- To use localization on $\Gamma_\pm(T)$ (foliated by stable/unstable leaves) together with FUP (corresponding to localization in position/frequency) we use a Fourier integral operator to straighten out the stable/unstable foliations
- These cannot be straightened out together: an additional linearization argument is needed
- We also remove the flow and dilation direction in T^*M: this is why one-dimensional FUP is used to get a result on surfaces
- For higher dimensional manifolds need higher-dimensional FUP: a big open problem

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Case of variable curvature

Theorem 5 [D–Jin–Nonnenmacher, in progress]

Let M be a surface of (variable) negative curvature and $\Omega \subset M$ a nonempty open set. Then there exists $c_\Omega > 0$ such that

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Two serious challenges compared to constant curvature:

- Non-constant expansion rates of $\varphi_t \quad \Longrightarrow \quad$ the propagation time to reach thickness h varies from point to point. Need propagation up to local Ehrenfest time
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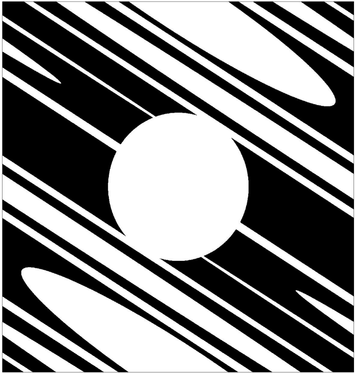
Variable curvature in pictures



Constant curvature

Variable curvature

(using perturbed Arnold cat map model for the figures)

Thank you for your attention!