LECTURE NOTES ON QUANTUM CHAOS
COURSE 18.158, MIT, APRIL 2016, VERSION 2

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Abstract. We give an overview of the quantum ergodicity result.

1. Quantum ergodicity in the physical space

1.1. **Concentration of eigenfunctions.** First, let us consider the case when $M \subset \mathbb{R}^2$ is a bounded domain with piecewise C^∞ boundary and we take the operator

$$\Delta = -\partial_{x_1}^2 - \partial_{x_2}^2.$$

We study the Dirichlet eigenvalues $\lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$ (with multiplicities taken into account) and the corresponding L^2 normalized eigenfunctions $u_j \in H_0^1(M)$, so

$$\Delta u_j = \lambda_j^2 u_j, \quad u_j|_{\partial M} = 0, \quad \|u_j\|_{L^2(M)} = 1. \tag{1.1}$$

We are interested in the following question regarding the high energy limit:

Question 1.1. *How do u_j concentrate as $j \to \infty$?*

In general, u_j become rapidly oscillating at high energies, so we have to study their concentration in some rough sense. A natural way to do that is to take the weak limits of the measures $|u_j(x)|^2\,dx$ along subsequences:

Definition 1.2. *Let u_{j_k} be a subsequence of (u_j) and μ a probability measure on M. We say that $u_{j_k} \to \mu$ weakly, if*

$$\int_M a(x)|u_{j_k}(x)|^2\,dx \to \int_M a(x)\,d\mu(x) \quad \text{for all } a \in C_0^\infty(M) \tag{1.2}$$

We say that u_{j_k} ***equidistributes in*** *M, if it converges weakly to the volume measure:*

$$u_{j_k} \to \frac{dx}{\mathrm{Vol}(M)}. \tag{1.3}$$

Remarks. 1. By a standard density argument, once (1.2) holds for all $a \in C_0^\infty(M)$, it holds for all $a \in C(\overline{M})$.

2. By a diagonal argument (see [Zw, Theorem 5.2]), there always exists a subsequence of u_j converging to some measure.

As a basic example, consider the square

$$M=[0,1]^2.$$

The Dirichlet eigenfunctions have the form

$$u_{j\ell}(x_1,x_2)=2\sin(j\pi x_1)\sin(\ell\pi x_2),\quad j,\ell\in\mathbb{N};\quad \lambda_{j\ell}=\pi\sqrt{j^2+\ell^2}.$$

Exercise 1.3. *Show that as $j\to\infty$,*

$$u_{jj}\to dx_1dx_2;\quad u_{1j}\to 2\sin^2(\pi x_1)\,dx_1dx_2,$$

that is the sequence u_{jj} equidistributes in M but the sequence u_{1j} does not.

More generally, we will consider the case when (M,g) is a compact Riemannian manifold with piecewise smooth boundary and replace Δ by the Laplace–Beltrami operator Δ_g which can be defined using the identity

$$\int_M \langle du,dv\rangle_g\,d\operatorname{Vol}_g=\int_M(\Delta_g u)v\,d\operatorname{Vol}_g,\quad u,v\in C_0^\infty(M).$$

The generalization of the measure (1.3) to this case is given by the Riemannian volume measure

$$\frac{d\operatorname{Vol}_g}{\operatorname{Vol}_g(M)}.$$

Exercise 1.4. *Let $M=\mathbb{S}^2$ be the two-dimensional sphere embedded into $\mathbb{R}^3$.*

(a) *Using the expression for Laplacian on $\mathbb{R}^3$ in spherical coordinates, show that each homogeneous harmonic polynomial v on $\mathbb{R}^3$ of degree m, the restriction $u:=v|_{\mathbb{S}^2}$ is an eigenfunction of $\Delta_{\mathbb{S}^2}$ with eigenvalue $m(m+1)$. (In fact, with a bit more work one can see that all eigenfunctions of $\Delta_{\mathbb{S}^2}$ are obtained in this way.)*

(b) *Using the coordinates (x_1,x_2,x_3) in $\mathbb{R}^n$, define for each $m\in\mathbb{N}_0$,*

$$v_m^\pm=(x_1\pm ix_2)^m,\quad u_m^\pm:=c_m v_m^\pm|_{\mathbb{S}^2}.$$

where the constant c_m is chosen so that $\|u_m^\pm\|_{L^2(\mathbb{S}^2)}=1$. Show that as $m\to\infty$, $u_m^\pm$ converge weakly to a probability measure on $\mathbb{S}^2$ which is supported on the equator $\{x_1^2+x_2^2=1,\ x_3=0\}$.

We see that the limit of u_{j_k} may depend on the choice of the sequence. It turns out that the limits in fact also depend in an essential way on the dynamics of a natural flow on (M,g), and quantum chaos studies in particular how the dynamical properties of (M,g) influence the behavior of eigenstates.

1.2. **Chaotic dynamics and ergodicity.** For (M, g) a Riemannian manifold, define the unit cotangent bundle

$$S^*M = \{(x, \xi) \in T^*M : |\xi|_g = 1\}.$$

When M is a domain in $\mathbb{R}^2$, we can write

$$S^*M = \{(x, \xi) \in M \times \mathbb{R}^2 : |\xi| = 1\}$$

and parametrize this space by (x_1, x_2, θ) where $\xi = (\cos\theta, \sin\theta)$. (The unit tangent and cotangent bundles can be identified with each other using the metric g, and it will become apparent later why it is much more convenient for us to use the cotangent bundle here.)

We consider the geodesic billiard ball flow on M,

$$\varphi_t : S^*M \to S^*M, \quad t \in \mathbb{R}.$$

For M a domain in $\mathbb{R}^2$, every trajectory of φ_t follows a straight line with velocity vector ξ until it hits the boundary, when it bounces off by the law of reflection. For (M, g) a Riemannian manifold, straight lines are replaced by geodesics induced by the metric g.

If M has a boundary, then the resulting map is not continuous and it is defined everywhere except a measure zero set in $\mathbb{R} \times S^*M$, corresponding to trajectories that either hit non-smooth parts of the boundary or become tangent to the boundary. We will ignore these issues in our note and send the reader to [ZeZw] for a detailed explanation of how they can be handled.

A natural probability measure on S^*M is the *Liouville measure*, defined for a general Riemannian manifold by

$$d\mu_L = \frac{d\operatorname{Vol}_g(x)\, d\mu_{\mathbb{S}^{n-1}}(\xi)}{\operatorname{Vol}_g(M) \cdot \operatorname{Vol}(\mathbb{S}^{n-1})}, \quad n = \dim M,$$

where $\mu_{\mathbb{S}^{n-1}}$ is the standard surface measure on the sphere, transported to a measure on each fiber of S^*M. For M a domain in $\mathbb{R}^2$, in coordinates (x_1, x_2, θ) we have

$$d\mu_L = \frac{dx_1 dx_2 d\theta}{2\pi \operatorname{Vol}(M)}.$$

The measure μ_L is invariant under the flow:

$$\mu_L(\varphi_t(U)) = \mu_L(U), \quad U \subset S^*M, \quad t \in \mathbb{R}.$$

We now introduce the notion of *ergodicity* for the flow φ_t, which is a rather weak way of saying that φ_t is a chaotic flow:

Definition 1.5. *We say that φ_t is ergodic with respect to μ_L, if for each flow invariant set*

$$U \subset S^*M; \quad \varphi_t(U) = U, \quad t \in \mathbb{R},$$

we have either $\mu_L(U) = 0$ or $\mu_L(U) = 1$.

One important consequence of ergodicity is the following statement about *ergodic averages*

$$\langle a\rangle_T := \frac{1}{T}\int_0^T a\circ\varphi_t\,dt,\quad T>0,\quad a\in L^1(S^*M;\mu_L). \tag{1.4}$$

Theorem 1 (L^2 ergodic theorem)**.** *Assume that φ_t is ergodic with respect to μ_L. Then for each $a\in L^2(S^*M;\mu_L)$,*

$$\langle a\rangle_T \to \int_{S^*M} a\,d\mu_L \quad \textit{in } L^2(S^*M;\mu_L).$$

Proof. We will only sketch the proof, sending the reader to [Zw, Theorem 15.1] for an alternative proof, and we restrict ourselves to the case when M has no boundary. Consider the vector field X on S^*M generating the flow, so that

$$\varphi_t = \exp(tX).$$

This vector field gives rise to a first order differential operator, still denotes X. Since μ_L is a φ_t-invariant measure, we have $\mathcal{L}_X\mu_L = 0$ and thus $-iX$ is an unbounded self-adjoint operator on $L^2(S^*M)$.

Let dE_X be the spectral measure of $-iX$, which is an operator-valued measure on $\mathbb{R}$ which is constructed via the spectral theorem for unbounded self-adjoint operators. Then for $a\in L^2(S^*M;\mu_L)$,

$$a\circ\varphi_t = \exp(tX)a = \int_{\mathbb{R}} e^{it\lambda}\,dE_X(\lambda)a.$$

Therefore, for $T>0$

$$\langle a\rangle_T = \int_{\mathbb{R}}\bigg(\frac{1}{T}\int_0^T e^{it\lambda}\,dt\bigg)dE_X(\lambda)a = \int_{\mathbb{R}}\frac{e^{iT\lambda}-1}{iT\lambda}dE_X(\lambda)a.$$

Now the function $\frac{e^{iT\lambda}-1}{iT\lambda}$ is bounded uniformly in T,λ, and it has the pointwise in λ limit

$$\frac{e^{iT\lambda}-1}{iT\lambda}\to \mathbb{1}_{\{0\}}(\lambda) = \begin{cases}1, & \lambda=0;\\ 0, & \lambda\neq 0,\end{cases}\quad \text{as } T\to\infty.$$

Since integral over the spectral measure is a strongly continuous function of the interval, one can see from here that

$$\langle a\rangle_T \to \int_{\{0\}} dE_X(\lambda)a \quad \text{in } L^2(S^*M,\mu_L). \tag{1.5}$$

The right-hand side is the orthogonal projection of a onto the space $V_0\subset L^2(S^*M,\mu_L)$ of functions satisfying the equation $Xf=0$. However, for each such f we have $f\circ\varphi_t=\varphi_t$ and thus the sublevel sets $\{f\le c\}$ are invariant under the flow (modulo a measure zero set which can be removed). By ergodicity, V_0 must then consist of constant functions. Then the right-hand side of (1.5) is the integral of a with respect to μ_L, finishing the proof. □

Exercise 1.6. *Show that neither* $[0,1]^2$ *nor* $\mathbb{S}^2$ *have ergodic* φ_t. *(Hint: on* $\mathbb{S}^2$*, the angular momentum with respect to any axis gives a conserved quantity. Any sublevel set of this function will be invariant under the flow.)*

There are many important examples of ergodic systems, including

- Sinai billiards;
- Bunimovich stadiums;
- Riemannian manifolds (M, g) without boundary which have negative sectional curvature, in particular closed negatively curved surfaces.

1.3. **Statement of quantum ergodicity.**

The following theorem (together with its generalizations is Theorems 4, 8 below) is the main result to be proved in this course:

Theorem 2 (Quantum ergodicity in the physical space)**.** *Assume that* φ_t *is ergodic with respect to* μ_L. *Then there exists a density 1 subsequence* λ_{j_k}, *that is*

$$\frac{\#\{k \mid \lambda_{j_k} \leq R\}}{\#\{j \mid \lambda_j \leq R\}} \to 1 \quad \text{as } R \to \infty,$$

such that u_{j_k} *equidistributes in* M*:*

$$u_{j_k} \to \frac{d\,\mathrm{Vol}_g}{\mathrm{Vol}_g(M)}.$$

This theorem was stated by Shnirelman [Sh] and proved by Zelditch [Ze] and Colin de Verdère [CdV]. The case of the domains with boundary was established by Zelditch–Zworski [ZeZw]. See [Zw, Theorem 15.5] for a detailed proof in the boundaryless case. (All of the results mentioned above prove the more general Theorems 4,8.)

We see that Theorem 2 uses information about the cotangent bundle on M to derive a statement on the manifold M itself. It turns out that to prove it, we should generalize the statement of equidistribution to T^*M, which we call the *phase space*.

2. Phase space concentration and proof of quantum ergodicity

We will henceforth assume that M has no boundary, referring the reader to [ZeZw] for the boundary case.

2.1. **Semiclassical quantization.**

Assume that $a \in C_0^\infty(T^*M)$. Semiclassical quantization associates to a, which is called *symbol* or *classical observable*, an operator

$$\mathrm{Op}_h(a) : L^2(M) \to L^2(M)$$

which is called a *semiclassical pseudodifferential operator* or *quantum observable*. This procedure depends on a parameter $h > 0$, called the semiclassical parameter, and we will be interested in the limit $h \to 0$. Originally h referred to (a dimensionless version of) Planck constant; in general it is the wavelength at which we want to study our eigenfunctions.

We will not give a definition of $\operatorname{Op}_h(a)$ here but will instead send the reader to [Zw, Chapters 4 and 14], and will give some explanations regarding semiclassical quantization later in the course. We remark that the procedure is independent of the choice of coordinates on M only modulo an $\mathcal{O}(h)$ remainder in the symbol, but the defined class of operators is geometrically invariant.

In fact we may define $\operatorname{Op}_h(a)$ for a in a more general class $S^m(T^*M)$, $m \in \mathbb{R}$, given by the conditions

$$a \in S^m(T^*M) \iff |\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha\beta}(1+|\xi|)^{m-|\beta|}.$$

The resulting operator acts on Sobolev spaces

$$\operatorname{Op}_h(a) : H^s(M) \to H^{s-m}(M), \quad s \in \mathbb{R}.$$

We note that if $a(x,\xi)$ is a polynomial in ξ,

$$a(x,\xi) = \sum_{|\gamma| \leq m} a_\gamma(x)\xi^\gamma, \quad a_\gamma(x) \in C^\infty(M),$$

then Op_h is a differential operator; on $\mathbb{R}^n$, the standard quantization procedure gives

$$\operatorname{Op}_h(a) = \sum_{|\gamma| \leq m} a_\gamma(x)(hD_x)^\gamma, \quad D_x = \frac{1}{i}\partial_x. \tag{2.1}$$

In particular, if $a(x,\xi) = a(x)$, then we get a multiplication operator

$$\operatorname{Op}_h(a)u(x) = a(x)u(x),$$

and on $\mathbb{R}^n$,

$$\operatorname{Op}_h(\xi_j) = hD_{x_j} = \frac{h}{i}\partial_{x_j}.$$

Also, if X is a vector field on M, then we have

$$\frac{h}{i}X = \operatorname{Op}_h(p_X) + \mathcal{O}(h), \quad p_X(x,\xi) = \langle \xi, X(x) \rangle.$$

This explains why our symbols are functions on the cotangent bundle rather than the tangent bundle – a vector field naturally gives a linear function on the fibers of the cotangent bundle.

We list below some fundamental properties of the quantization operation. We leave the remainders ambiguous, but they will have appropriate mapping properties in Sobolev spaces.

Theorem 3. *For $a \in S^m(T^*M), b \in S^k(T^*M)$, we have*

$$\mathrm{Op}_h(a)^* = \mathrm{Op}_h(\bar{a}) + \mathcal{O}(h), \tag{2.2}$$

$$\mathrm{Op}_h(a)\,\mathrm{Op}_h(b) = \mathrm{Op}_h(ab) + \mathcal{O}(h), \tag{2.3}$$

$$[\mathrm{Op}_h(a), \mathrm{Op}_h(b)] = \frac{h}{i}\,\mathrm{Op}_h(\{a,b\}) + \mathcal{O}(h^2), \tag{2.4}$$

where $\{a,b\}$ is the Poisson bracket, given in coordinates by

$$\{a,b\} = \sum_j (\partial_{\xi_j} a \partial_{x_j} b - \partial_{x_j} a \partial_{\xi_j} b).$$

Moreover, for $a \in S^0(T^*M)$ the operator norm of $\mathrm{Op}_h(a)$ on L^2 can be estimated as follows [Zw, Theorem 5.1]: for some constant C independent of a, h,

$$\limsup_{h\to 0} \|\,\mathrm{Op}_h(a)\|_{L^2\to L^2} \leq C\|a\|_{L^\infty(T^*M)}. \tag{2.5}$$

2.2. **Quantum ergodicity in phase space.** We now generalize Theorem 2 to a statement about quantum observables

$$\langle \mathrm{Op}_h(a) u_j, u_j\rangle_{L^2}, \quad a \in S^0(T^*M).$$

For that we need to pick the value of h and it will be convenient to put

$$h_j = \frac{1}{\lambda_j}.$$

We then define

$$V_j(a) := \langle \mathrm{Op}_{h_j}(a) u_j, u_j\rangle_{L^2(M)}, \quad a \in S^0(T^*M).$$

Definition 2.1 (Weak limits in phase space)**.** *Let u_{j_k} be a subsequence of u_j and μ be a measure on T^*M. We say that $u_{j_k} \to \mu$ in the sense of semiclassical measures, if*

$$V_{j_k}(a) \to \int_{T^*M} a\, d\mu \quad \textit{for all } a \in S^0(T^*M).$$

We say that u_{j_k} ***equidistribute in phase space*** *if they converge to the Liouville measure:*

$$u_{j_k} \to \mu_L.$$

Remark. There is always a subsequence converging to some measure, and all resulting measures are supported on the unit cosphere bundle S^*M and invariant under the flow φ_t – see [Zw, Chapter 5] and (2.8), (2.10) below.

Theorem 4 (Quantum ergodicity in phase space)**.** *Assume φ_t is ergodic with respect to μ_L. Then there exists a density 1 subsequence u_{j_k} such that u_{j_k} equidistribute in the phase space.*

Theorem 2 follows from here by taking a to be a function of x, so that

$$V_j(a(x)) = \int_M a(x)|u_j(x)|^2\,dx,$$

and using the fact that the pushforward of μ_L to M is the volume measure:

$$\int_{S^*M} a(x)\,d\mu_L = \frac{1}{\mathrm{Vol}_g(M)}\int_M a(x)\,d\,\mathrm{Vol}_g\,.$$

In the rest of this section, we prove Theorem 4, following several steps.

2.3. **Step 1: using the eigenfunction equation.** We first rewrite the eigenfunction equation

$$\Delta_g u_j = \lambda_j^2 u_j$$

in the form

$$\mathrm{Op}_{h_j}(p)u_j = 0 \tag{2.6}$$

where the symbol

$$p(x,\xi) = p_0(x,\xi) + \mathcal{O}(h), \quad p_0(x,\xi) = \frac{|\xi|_g^2 - 1}{2} \tag{2.7}$$

is chosen so that

$$P = \mathrm{Op}_h(p) = \frac{h^2\Delta_g - 1}{2}.$$

Define the Hamiltonian vector field

$$H_{p_0} = \sum_j \partial_{\xi_j}p\cdot\partial_{x_j} - \partial_{x_j}p\cdot\partial_{\xi_j},$$

and note that

$$H_{p_0}a = \{p_0, a\}, \quad a \in C^\infty(T^*M).$$

Define the flow

$$\varphi_t = \exp(tH_{p_0}),$$

then (explaining the choice of $\frac{1}{2}$ in the definition of p_0) the restriction of φ_t to S^*M is the geodesic flow.

A key tool in the proof is the Schrödinger propagator

$$U(t) = U(t;h) = \exp\Big(-\frac{itP}{h}\Big) : L^2(M) \to L^2(M).$$

It quantizes the flow φ_t as made precise by the following

Theorem 5 (Egorov's Theorem)**.** *For $a \in C_0^\infty(T^*M)$, we have*

$$U(-t)\,\mathrm{Op}_h(a)U(t) = \mathrm{Op}_h(a\circ\varphi_t) + \mathcal{O}(h)_{L^2(M)\to L^2(M)}.$$

Proof. We only sketch the proof, see [Zw, Theorem 15.2] for details. It is enough to prove that, denoting $a_t := a \circ \varphi_t$,

$$\partial_t\big(U(t)\operatorname{Op}_h(a_t)U(-t)\big) = \mathcal{O}(h)_{L^2(M)\to L^2(M)}$$

The left-hand side is

$$U(t)\Big(\operatorname{Op}_h(\partial_t a_t) - \frac{i}{h}\Big[P, \operatorname{Op}_h(a_t)\Big]\Big)U(-t)$$

By (2.4), this becomes

$$U(t)\operatorname{Op}_h(\partial_t a_t - \{p_0, a_t\})U(-t) + \mathcal{O}(h)$$

and it remains to use that $\partial_t a_t = \{p_0, a_t\}$. □

We then have the following

Lemma 2.2. *Assume that $a \in C_0^\infty(T^*M)$. Then for any $T > 0$,*

$$V_j(a) = V_j\big(\langle a\rangle_T\big) + \mathcal{O}_T(h_j)$$

where $\langle a\rangle_T$ is defined in (1.4) and the constant in the remainder depends on T.

Proof. We have for each t, $U(t;h_j)u_j = u_j$ and thus

$$\begin{aligned} V_j(a) &= \langle \operatorname{Op}_{h_j}(a)u_j, u_j\rangle_{L^2} = \langle U(-t;h_j)\operatorname{Op}_{h_j}(a)U(t;h_j)u_j, u_j\rangle_{L^2} \\ &= \langle \operatorname{Op}_{h_j}(a\circ\varphi_t)u_j, u_j\rangle_{L^2} + \mathcal{O}_t(h_j) = V_j(a\circ\varphi_t) + \mathcal{O}_t(h_j) \end{aligned} \tag{2.8}$$

and it remains to average both sides over $t \in [0,T]$. □

This statement uses the fact that u_j are eigenfunctions and features ergodic averages along the flow φ_t.

2.4. **Step 2: basic bounds.** We record here a few standard bounds on $V_j(a)$. First of all, by (2.5) we have for some global constant C and each $a \in S^0(T^*M)$,

$$\limsup_{j\to\infty} |V_j(a)| \le C\|a\|_{L^\infty(T^*M)}. \tag{2.9}$$

Moreover, if a vanishes on S^*M, then

$$\lim_{j\to\infty} V_j(a) = 0 \tag{2.10}$$

as follows immediately from

Lemma 2.3 (Elliptic bound)**.** *Assume $a \in S^0(T^*M)$ and $a|_{S^*M} = 0$. Then as $j \to \infty$,*

$$\|\operatorname{Op}_{h_j}(a)u_j\|_{L^2} = \mathcal{O}(h_j).$$

Proof. Since a vanishes on S^*M, we may write $a = bp_0 = bp + \mathcal{O}(h)$, where p, p_0 are defined in (2.7) and $b \in S^{-2}(T^*M)$. Then by (2.3),

$$\operatorname{Op}_{h_j}(a) = \operatorname{Op}_{h_j}(b)\operatorname{Op}_{h_j}(p) + \mathcal{O}(h_j)_{L^2\to L^2}.$$

Since $\operatorname{Op}_{h_j}(p)u_j = 0$ by (2.6), the proof is finished. □

2.5. **Step 3: bounding averages over eigenfunctions.** We know by Theorem 1 that for large T, the average $\langle a\rangle_T$ is close to the integral of a, but only in $L^2(T^*M)$. If we had an L^∞ estimate instead, then we could use (2.9) to control $V_j(a)$ for all j in the limit $j \to \infty$. However, ergodic averages typically do not converge in L^∞ (this can be seen for instance by considering a closed geodesic).

Therefore we will have to make the best out of the L^2 bound on a. It turns out that it produces a bound on $V_j(a)$ on average in j – see Lemma 2.4 below. The key statement is the following theorem, which we will try to prove later in the course (see §3.3):

Theorem 6 (Local Weyl Law). *Assume that $\chi \in C_0^\infty\big((0,\infty)\big)$ and $a \in S^0(T^*M)$. Then as $R \to \infty$,*

$$\sum_j \chi\Big(\frac{\lambda_j}{R}\Big)V_j(a) = \Big(\frac{R}{2\pi}\Big)^n\bigg(\int_{T^*M}\chi(|\xi|_g)\,a\Big(x,\frac{\xi}{|\xi|_g}\Big)\,dxd\xi + \mathcal{O}(R^{-1})\bigg).$$

Taking $a = 1$ in Theorem 6, we in particular get

$$\sum_j \chi\Big(\frac{\lambda_j}{R}\Big) = \Big(\frac{R}{2\pi}\Big)^n\Big(\int_{T^*M}\chi(|\xi|_g)\,dxd\xi + \mathcal{O}(R^{-1})\Big).$$

Approximating $\chi = \mathbb{1}_{[0,1]}$ by functions in $C_0^\infty\big((0,\infty)\big)$, this gives

Theorem 7 (Weyl Law). *We have as $R \to \infty$,*

$$\#\{j \mid \lambda_j \le R\} = \frac{\omega_n}{(2\pi)^n}\operatorname{Vol}_g(M)R^n + o(R^n)$$

where $\omega_n > 0$ is the volume of the unit ball in $\mathbb{R}^n$.

Note also that the integral on the right-hand side in Theorem 6 is zero if a vanishes on S^*M; this is in line with Lemma 2.3.

For the proof of quantum ergodicity, we use the following corollary of Theorem 6:

Lemma 2.4 (Variance bound). *We have for each $a \in S^0(T^*M)$, as $R \to \infty$*

$$R^{-n}\sum_{\lambda_j\in[R,2R]}|V_j(a)|^2 \le C\int_{S^*M}|a|^2\,d\mu_L + \mathcal{O}(R^{-1})$$

where the constant C depends on M, but not on a or R.

Proof. Take nonnegative $\chi \in C_0^\infty\big((0,\infty)\big)$ with $\chi = 1$ on $[1,2]$, then it is enough to estimate

$$R^{-n}\sum_j \chi\Big(\frac{\lambda_j}{R}\Big)\|\operatorname{Op}_{h_j}(a)u_j\|_{L^2}^2 = R^{-n}\sum_j \chi\Big(\frac{\lambda_j}{R}\Big)\langle \operatorname{Op}_{h_j}(a)^*\operatorname{Op}_{h_j}(a)u_j, u_j\rangle_{L^2}$$

and the right-hand side is bounded by Theorem 6 using that by (2.2) and (2.3)

$$\operatorname{Op}_{h_j}(a)^*\operatorname{Op}_{h_j}(a) = \operatorname{Op}_{h_j}\big(|a|^2\big) + \mathcal{O}(h_j) = \operatorname{Op}_{h_j}\big(|a|^2\big) + \mathcal{O}(R^{-1}). \quad \square$$

2.6. **Step 4: integrated quantum ergodicity.** We can now prove the following integrated (or, strictly speaking, summed) form of Theorem 4:

Theorem 8 (Integrated quantum ergodicity)**.** *Assume that $a \in S^0(T^*M)$ and*

$$L_a = \int_{S^*M} a\, d\mu_L. \tag{2.11}$$

Then as $R \to \infty$,

$$R^{-n}\sum_{\lambda_j\in[R,2R]} |V_j(a) - L_a|^2 \to 0.$$

Proof. By subtracting L_a from a and using that $\operatorname{Op}_h(1)$ is the identity operator, we reduce to the case $L_a = 0$:

$$\int_{S^*M} a\, d\mu_L = 0.$$

Moreover, by (2.10) we may assume that $a \in C_0^\infty(T^*M)$.

Take some $T > 0$. By Lemma 2.2 and then Lemma 2.4, we have

$$\begin{aligned} R^{-n}\sum_{\lambda_j\in[R,2R]} |V_j(a)|^2 &\le R^{-n}\sum_{\lambda_j\in[R,2R]} \big|V_j\big(\langle a\rangle_T\big)\big|^2 + \mathcal{O}_T(R^{-1}) \\ &\le C\|\langle a\rangle_T\|^2_{L^2(S^*M,\mu_L)} + \mathcal{O}_T(R^{-1}) \end{aligned}$$

where the constant C is independent of T and R. Taking the limit as $R \to \infty$, we have

$$\limsup_{R\to\infty} R^{-n}\sum_{\lambda_j\in[R,2R]} |V_j(a)|^2 \le C\|\langle a\rangle_T\|^2_{L^2(S^*M,\mu_L)}.$$

The left-hand side does not depend on T, and the right-hand side converges to 0 by Theorem 1. $\square$

2.7. **Step 5: end of the proof.** It remains to derive Theorem 4 from Theorem 8, that is to extract a density 1 sequence of eigenfunctions which equidistributes in phase space. For that we use Chebyshev inequality and a diagonal argument on dyadic pieces of the spectrum.

More precisely, for $r \in \mathbb{N}$ let

$$N_r := \#\{j \mid \lambda_j \in [2^r, 2^{r+1})\},$$

then $N_r \sim 2^{nr}$ as $r \to \infty$ by the Weyl law (Theorem 7). Take a sequence

$$a_s \in C_0^\infty(T^*M), \quad s = 1, 2, \dots$$

which is dense in $C_0^\infty(T^*M)$ with respect to the uniform norm. Put $L_s = \int_{S^*M} a_s \, d\mu_L$ and

$$\varepsilon_{\ell,r} := \max_{s\leq \ell} \Big(\frac{1}{N_r} \sum_{\lambda_j \in [2^r, 2^{r+1})} |V_j(a_s) - L_s|^2 \Big).$$

Then $\varepsilon_{\ell,r} \to 0$ as $r \to \infty$ for each ℓ by Theorem 8. We pick $r(\ell)$ such that $r(\ell+1) > r(\ell)$ and

$$\varepsilon_{\ell,r} < 100^{-\ell} \quad \text{for } r \geq r(\ell).$$

Define the disjoint collection of sets $J_\ell \subset \mathbb{N}$ as follows:

$$j \in J_\ell \iff \lambda_j \in [2^{r(\ell)}, 2^{r(\ell+1)}) \text{ and } \max_{s\leq \ell} |V_j(a_s) - L_s| < 2^{-\ell}.$$

By Chebyshev inequality, for $r(\ell) \leq r < r(\ell+1)$,

$$\#\big(\{j \mid \lambda_j \in [2^r, 2^{r+1})\} \setminus J_\ell\big) \ \leq \ \frac{\ell \varepsilon_{\ell,r}}{2^{-2\ell}} N_r \ \leq \ 2^{-\ell} N_r,$$

therefore

$$1 - \frac{\#(J_\ell)}{\#\big(\{j \mid \lambda_j \in [2^{r(\ell)}, 2^{r(\ell+1)})\}\big)} \ \leq 2^{-\ell}.$$

It follows from here and the Weyl law that the sequence

$$j_k, \quad \{j_k\} = \bigcup_\ell J_\ell$$

is a density one subsequence in $\mathbb{N}$.

On the other hand, we have for each s,

$$V_{j_k}(a_s) \to \int_{S^*M} a_s \, d\mu_L \quad \text{as } k \to \infty.$$

Using the bound (2.9) and the fact that $\{a_s\}$ is dense in C_0^∞ with the uniform norm, we see that

$$V_{j_k}(a) \to \int_{S^*M} a \, d\mu_L \quad \text{as } k \to \infty$$

for all $a \in C_0^\infty(T^*M)$. By (2.10) same is true for all $a \in S^0(T^*M)$, finishing the proof.

3. Overview of semiclassical quantization

We now briefly discuss how to define the quantization procedure Op_h, sending the reader to [Zw] for details.

3.1. **Quantization on $\mathbb{R}^n$.** We consider the following symbol classes on $T^*\mathbb{R}^n = \mathbb{R}^{2n}$,

$$S_h^m(T^*\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n), \quad m \in \mathbb{R},$$

defined as follows: $a(x,\xi;h) \in S^m(T^*\mathbb{R}^n)$ if for each multiiindices α, β there exists a constant $C_{\alpha\beta}$ such that for all x, ξ and small h,

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi;h)| \le C_{\alpha\beta}(1+|\xi|)^{m-|\beta|}.$$

Note that for $m \in \mathbb{N}_0$ this class includes polynomials of order m in ξ with coefficients bounded with all derivatives in x.

For $a \in S_h^m(T^*\mathbb{R}^n)$, we define the operator $\mathrm{Op}_h(a)$ on functions on $\mathbb{R}^n$ as follows:

$$\mathrm{Op}_h(a)f(x) = (2\pi h)^{-n} \int_{\mathbb{R}^{2n}} e^{\frac{i}{h}\langle x-y,\xi\rangle} a(x,\xi) f(y)\, dy d\xi. \tag{3.1}$$

The integral (3.1) does not always converge in the usual sense, so some explanations are in order. Assume first a is smooth and compactly supported, or more generally a lies in the Schwartz class $\mathscr{S}(T^*\mathbb{R}^n)$. If $f \in \mathscr{S}(\mathbb{R}^n)$, then integral in (3.1) converges absolutely and gives a Schwartz function.

When $a \in S^m(T^*\mathbb{R}^n)$, we see using the semiclassical Fourier transform

$$\mathcal{F}_h f(\xi) = (2\pi h)^{-n/2} \int_{\mathbb{R}^n} e^{-\frac{i}{h}\langle y,\xi\rangle} f(y)\, dy$$

that

$$\mathrm{Op}_h(a)f(x) = (2\pi h)^{-n/2} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x,\xi\rangle} a(x,\xi) \mathcal{F}_h f(\xi)\, d\xi \tag{3.2}$$

and since $\mathcal{F}_h f(\xi)$ is Schwartz, the integral still converges; integrating by parts in ξ, we see that it still gives a Schwartz function.

In fact, for any $a \in S^m(T^*\mathbb{R}^n)$, one can define $\mathrm{Op}_h(a)f$ for $f \in \mathscr{S}'(\mathbb{R}^n)$, where $\mathscr{S}'(\mathbb{R}^n)$, the dual to $\mathscr{S}(\mathbb{R}^n)$, is the space of tempered distributions. This can be seen either by duality or by treating (3.1) as an *oscillatory integral*, or by first considering the case of $a \in \mathscr{S}(T^*\mathbb{R}^n)$ and extending to general a by density. In either case, we obtain the quantization procedure on $\mathbb{R}^n$,

$$a \in S_h^m(T^*\mathbb{R}^n) \;\mapsto\; \mathrm{Op}_h(a) : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n),\; \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n).$$

Moreover, rapidly decaying symbols produce smoothing operators:

$$a \in \mathscr{S}(T^*\mathbb{R}^n) \implies \mathrm{Op}_h(a) : \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n).$$

Note that the mapping properties above are for any fixed h; we make no statement about the uniformity of norms as $h \to 0$ at this point.

Exercise 3.1. *Using* (3.2)*, show that when $a \in S^m(T^*\mathbb{R}^n)$ is polynomial in ξ, the operator* $\mathrm{Op}_h(a)$ *is the differential operator defined in* (2.1)*.*

In what follows, we will often ignore what happens as $x, \xi \to \infty$, so our proofs would immediately work for Schwartz symbols $a \in \mathscr{S}(T^*\mathbb{R}^n)$ and with more work can be extended to general symbols.

3.2. **Basic properties of quantization and stationary phase.** We now want to establish some properties of the quantization procedure Op_h. We start with the product formula (2.3). Assume that $a, b \in \mathscr{S}(T^*\mathbb{R}^n)$ uniformly in h. We would like to write

$$\mathrm{Op}_h(a)\,\mathrm{Op}_h(b) = \mathrm{Op}_h(c), \quad c(x,\xi;h) \in \mathscr{S}(T^*\mathbb{R}^n), \tag{3.3}$$

and understand the asymptotics of c as $h \to 0$.

We first find a formula for c using the following statement, known as *oscillatory testing*; see [Zw, Theorem 4.19]:

Lemma 3.2. *Assume that $a \in \mathscr{S}(T^*\mathbb{R}^n)$. Then for each fixed $h > 0$,*

1. We can recover the symbol a from the operator $A = \mathrm{Op}_h(a)$ as follows:

$$a(x,\xi) = e^{-\frac{i}{h}\langle x,\xi\rangle} A(e^{\frac{i}{h}\langle \bullet,\xi\rangle}). \tag{3.4}$$

2. If $A : \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ and the function $a \in \mathscr{S}(T^*\mathbb{R}^n)$ satisfies (3.4), then $A = \mathrm{Op}_h(a)$.

We now write out the symbol c from (3.3) as follows:

$$\begin{aligned} c(x,\xi) &= e^{-\frac{i}{h}\langle x,\xi\rangle}\,\mathrm{Op}_h(a)\,\mathrm{Op}_h(b)(e^{\frac{i}{h}\langle\bullet,\xi\rangle}) \\ &= e^{-\frac{i}{h}\langle x,\xi\rangle}\,\mathrm{Op}_h(a)(b(\bullet,\xi;h)e^{\frac{i}{h}\langle\bullet,\xi\rangle}) \\ &= (2\pi h)^{-n}\int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\eta-\xi\rangle} a(x,\eta;h)b(y,\xi;h)\,dyd\eta. \end{aligned} \tag{3.5}$$

To understand the behavior of c as $h \to 0$, we use the following (see [Zw, Theorem 3.16])

Theorem 9 (Method of stationary phase)**.** *Assume $U \subset \mathbb{R}^n$ is an open set and $\varphi \in C^\infty(U;\mathbb{R})$ has only one critical point $x_0 \in U$, that is $\nabla\varphi \neq 0$ on $U \setminus \{x_0\}$.*

Assume also that x_0 is a nondegenerate critical point, that is the Hessian $\nabla^2\varphi(x_0)$ gives a nondegenerate quadratic form. Denote by $\mathrm{sgn}(\nabla^2\varphi(x_0))$ the signature of this form (the number of positive eigenvalues minus the number of negative eigenvalues).

Then for each $a \in C_0^\infty(U;\mathbb{C})$, we have as $h \to 0$

$$\int_U e^{\frac{i\varphi(x)}{h}} a(x)\,dx \sim (2\pi h)^{n/2} e^{\frac{i\varphi(x_0)}{h}} \sum_{j=0}^\infty h^j L_j(a)|_{x=x_0} \tag{3.6}$$

where each L_j is a φ-dependent linear differential operator of order $2j$. In particular

$$L_0(a)|_{x=x_0} = e^{\frac{i\pi}{4}\,\mathrm{sgn}\,\nabla^2\varphi(x_0)}\big|\det\nabla^2\varphi(x_0)\big|^{-1/2} a(x_0).$$

Proof. We only sketch a proof in a special case known as *quadratic stationary phase*:

$$n = 1, \quad \varphi(x) = \frac{x^2}{2}.$$

By Fubini's Theorem and a linear change of variables, one can pass from here to the case when φ is a nondegenerate quadratic form in higher dimensions. The general case can then be handled by the Morse Lemma, which gives a change of variables conjugating a general phase φ locally to a quadratic form.

We compute in terms of the standard (nonsemiclassical) Fourier transform $\hat{a}(\xi)$,

$$\int_{\mathbb{R}} e^{\frac{ix^2}{2h}} a(x)\, dx = e^{\frac{i\pi}{4}} \sqrt{\frac{h}{2\pi}} \int_{\mathbb{R}} e^{-\frac{ih\xi^2}{2}} \hat{a}(\xi)\, d\xi. \tag{3.7}$$

This follows from the more general statement true for any $z \in \mathbb{C}$, $\operatorname{Re} z \geq 0$, $z \neq 0$:

$$\int_{\mathbb{R}} e^{-\frac{zx^2}{2}} a(x)\, dx = \frac{1}{\sqrt{2\pi z}} \int_{\mathbb{R}} e^{-\frac{\xi^2}{2z}} \hat{a}(\xi)\, d\xi. \tag{3.8}$$

The statement (3.8) follows for $z > 0$ by direct calculation using the Fourier transform of the Gaussian and for all z by analytic continuation.

Now, taking the Taylor expansion of $e^{-ih\xi^2/2}$ as $h \to 0$ and using that $\hat{a} \in \mathscr{S}$, we get

$$\begin{aligned}\int_{\mathbb{R}} e^{\frac{ix^2}{2h}} a(x)\, dx \;&\sim\; e^{\frac{i\pi}{4}} \sqrt{\frac{h}{2\pi}} \sum_{j=0}^{\infty} \int_{\mathbb{R}} \frac{1}{j!} \Big(-\frac{ih\xi^2}{2} \Big)^j \hat{a}(\xi)\, d\xi \\ &\sim\; e^{\frac{i\pi}{4}} \sqrt{2\pi h} \sum_{j=0}^{\infty} \frac{1}{j!} \Big(\frac{ih}{2} \Big)^j \partial_x^{2j} a(0)\end{aligned}$$

finishing the proof. □

In the case (3.5) we integrate over y, η, thus the dimension is $2n$. The phase is given by

$$(y, \eta) \mapsto \langle x - y, \eta - \xi \rangle,$$

and the only critical point is $y = x, \eta = \xi$. The value of the phase at the critical point is equal to 0. The expansion (3.6) can be computed explicitly from quadratic stationary phase and yields

$$c(x, \xi) \sim a(x, \xi) b(x, \xi) + \frac{h}{i} \sum_{j=1}^{n} \partial_{\xi_j} a(x, \xi) \partial_{x_j} b(x, \xi) + \mathcal{O}(h^2),$$

explaining (2.3), (2.4).

3.3. **More on semiclassical quantization.** On a manifold M, we define the quantization $\operatorname{Op}_h(a)$ by covering M with a locally finite system of coordinate charts, splitting a into pieces using a partition of unity, quantizing it separately on each chart using (3.1), and adding the pieces back together. However, if we take different charts or the partition of unity, the resulting operator will change by an operator with symbol in $hS_h^{m-1}(T^*M)$.

Therefore, it is more convenient to consider the class of *semiclassical pseudodifferential operators*

$$\Psi^m(M) = \{\operatorname{Op}_h(a) \mid a \in S_h^m(T^*M)\}$$

which is independent of the choice of quantization, and the principal symbol map

$$\sigma_h : \Psi^m(M) \to S_h^m(T^*M)/hS_h^{m-1}(T^*M), \quad \sigma_h(\operatorname{Op}_h(a)) = a + hS_h^{m-1}(T^*M)$$

which is also independent of the quantization. We have the short exact sequence

$$0 \to h\Psi^{m-1}(M) \to \Psi^m(M) \xrightarrow{\sigma_h} S_h^m(T^*M)/hS_h^{m-1}(T^*M) \to 0.$$

The symbolic calculus makes it possible to construct more pseudodifferential operators by calculating their symbol term by term. For instance, we can find approximate inverses of operators with nonvanishing symbols:

Proposition 3.3. *Assume $a \in S_h^0(T^*M)$, $p \in S_h^m(T^*M)$, and $p \neq 0$ on $\operatorname{supp} a$. Then there exists $b \in S_h^{-m}(T^*M)$ such that*

$$\operatorname{Op}_h(a) = \operatorname{Op}_h(b)\operatorname{Op}_h(p) + \mathcal{O}(h^\infty).$$

Remark. This generalizes Lemma 2.3 in the following sense: if $(h^2\Delta_g - 1)u = 0$, then

$$\|\operatorname{Op}_h(a)u\|_{L^2} = \mathcal{O}(h^\infty)\|u\|_{L^2} \quad \text{for all } a \in S_h^0(T^*M),\ \operatorname{supp} a \cap \{|\xi|_g = 1\} = \emptyset.$$

Sketch of proof. We first take

$$b_0 = \frac{a}{p} \in S_h^{-m}(T^*M).$$

Then by (2.3) we have for some $r_1 \in S_h^{-1}(T^*M)$,

$$\operatorname{Op}_h(a) = \operatorname{Op}_h(b_0)\operatorname{Op}_h(p) + h\operatorname{Op}_h(r_1) + \mathcal{O}(h^\infty).$$

Moreover, one can arrange so that $\operatorname{supp} r_1 \subset \{a \neq 0\}$. Then we repeat the procedure, putting

$$b_1 = \frac{r_1}{p} \in hS_h^{-m-1}(T^*M).$$

Arguing this way we construct some symbols $b_j \in h^j S_h^{-m-j}(T^*M)$ and it remains to take b such that

$$b \sim \sum_j b_j. \quad \square$$

We can also take functions of pseudodifferential operators, which we present in a special case. Namely we have

Theorem 10. *Assume (M,g) is a compact Riemannian manifold without boundary, and put $P = h^2\Delta_g = \operatorname{Op}_h(p) + \mathcal{O}(h)_{\Psi_h^1}$ where $p(x,\xi) = |\xi|_g^2$. Then for each $\chi \in C_0^\infty(\mathbb{R})$ and all N, we have*

$$\chi(P) \in \Psi_h^{-N}(M); \quad \sigma_h(\chi(P)) = \chi(p).$$

Sketch of proof. We write using the Fourier transform $\hat{\chi}$,

$$\chi(P) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{\chi}(t)e^{itP}\,dt. \tag{3.9}$$

For bounded t, we have

$$e^{itP} = \operatorname{Op}_h(p_t), \quad p_t = e^{itp} + \mathcal{O}(h), \tag{3.10}$$

as can be done solving the equation

$$\partial_t \operatorname{Op}_h(p_t) = iP \operatorname{Op}_h(p_t)$$

in symbolic calculus. When $\hat{\chi}$ is compactly supported, we get the desired formula. Otherwise we can write (3.10) up to $t \sim h^\varepsilon$ for some small ε, where the symbol p_t will have derivatives mildly growing in h, and use the integral (3.9) with the fact that $\hat{\chi}$ is Schwartz. □

From Theorem 10 we can derive the following version of local Weyl law of Theorem 6 (the original version, with h depending on u_j, can be proved using a rescaling and Lemma 2.3):

Theorem 11. *For $\chi \in C_0^\infty(\mathbb{R})$, $a \in S^0(T^*M)$, and λ_j, u_j defined in (1.1), we have as $h \to 0$*

$$\sum_j \chi(h^2\lambda_j)\langle \operatorname{Op}_h(a)u_j, u_j\rangle = (2\pi h)^{-n}\int_{T^*M} \chi(p(x,\xi))a(x,\xi)\,dxd\xi + \mathcal{O}(h^{1-n}).$$

Proof. Putting $P := -h^2\Delta_g$, the left-hand side is the trace

$$\operatorname{tr}\big(\chi(P)\operatorname{Op}_h(a)\big).$$

However, we know that $\chi(P)\operatorname{Op}_h(a) \in \Psi_h^{-N}(M)$ for all N, so we write $\chi(P)\operatorname{Op}_h(a) = \operatorname{Op}_h(b) + \mathcal{O}(h^\infty)$ where $b = \chi(p)a + \mathcal{O}(h)$ is rapidly decreasing in ξ.

It remains to use the following trace formula for pseudodifferential operators:

$$\operatorname{tr}\operatorname{Op}_h(b) = (2\pi h)^{-n}\int_{T^*M} b(x,\xi)\,dxd\xi + \mathcal{O}(h^{1-n})$$

which reduces to the case of quantization on $\mathbb{R}^n$ and there the trace can be computed by integrating the Schwartz kernel. □

3.4. **Another application of stationary phase: concentration of Lagrangian states.** Assume that $U \subset \mathbb{R}^n$ is open and we are given a phase function $\varphi \in C^\infty(U;\mathbb{R})$ and an amplitude $b \in C_0^\infty(U;\mathbb{C})$. We define the family of functions $u_h \in C_0^\infty(U;\mathbb{C})$, $h > 0$, by

$$u_h(x) = e^{i\varphi(x)/h} b(x).$$

We would like to understand the limits as $h \to 0$ of observables

$$\langle \mathrm{Op}_h(a) u_h, u_h \rangle_{L^2}, \quad a \in C_0^\infty(T^*\mathbb{R}^n),$$

specifically to write for some measure μ,

$$\langle \mathrm{Op}_h(a) u_h, u_h \rangle_{L^2} \to \int_{T^*\mathbb{R}^n} a\, d\mu. \tag{3.11}$$

This can be done by applying the method of stationary phase to

$$\mathrm{Op}_h(a) u_h(x) = (2\pi h)^{-n} \int_{\mathbb{R}^{2n}} e^{\frac{i}{h}(\langle x-y,\xi\rangle + \varphi(y))} a(x,\xi) b(y)\, dy d\xi.$$

The phase function is

$$(y,\xi) \mapsto \langle x-y,\xi\rangle + \varphi(y),$$

the stationary point is given by

$$x = y, \quad \xi = \nabla\varphi(x),$$

and the value of the phase at the stationary point is equal to $\varphi(x)$. Applying (3.6), we obtain

$$\mathrm{Op}_h(a) u_h(x) = e^{i\varphi(x)/h} a(x, \nabla\varphi(x)) b(x) + \mathcal{O}(h).$$

Therefore we have the limit (3.11) with μ given by

$$\int_{T^*M} a\, d\mu = \int_U a(x, \nabla\varphi(x)) |b(x)|^2\, dx.$$

In particular, μ lives on

$$\Lambda_\varphi = \{(x, \nabla\varphi(x)) \mid x \in U\}$$

which is a Lagrangian submanifold of $T^*\mathbb{R}^n$.

Exercise 3.4. *Find the semiclassical limits (in the sense of Definition 2.1) of the functions u_{1j} and u_{jj} from Exercise 1.3. (Ignore the boundary issues by testing these functions against operators supported strictly inside the square.)*

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