Resonances for open quantum maps

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joint work with Long Jin (Purdue University)

September 12, 2016

- We study open quantum maps with underlying chaotic dynamics
- Much studied issue: existence of spectral gap
  (do waves decay exponentially?)
- Known under dynamical "pressure condition" $P(\frac{1}{2}) < 0$,
  but is the gap there when it is violated?
- The only known cases with gap and $P(\frac{1}{2}) > 0$:

  D–Zahl '16 hyperbolic surfaces "near" the critical pressure value

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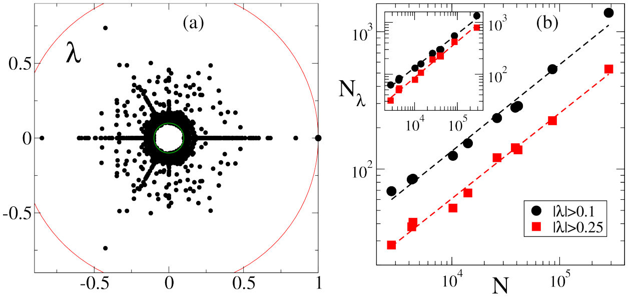
Overview of open quantum maps

- Resonances: complex characteristic frequencies of decaying waves in systems where energy is allowed to escape (e.g. obstacle scattering)
- Open quantum chaos studies the distribution of resonances, e.g. spectral gaps and fractal Weyl laws, with applications going as far as computer networks: Ermann–Frahm–Shepelyansky Rev.Mod.Phys.'15:



Eigenvalues for the Google Matrix of the Linux kernel and Weyl asymptotics

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- Open quantum maps: popular models in open quantum chaos
  See reviews by Nonnenmacher '11 (math), Novaes '13 (physics)
- Proposed experiments: Hannay–Keating–Ozorio de Almeida '94, Brun–Schack '99
- Attractive model for numerical experimentation:
  Schomerus–Tworzydło '04, Nonnenmacher–Zworski '05, '07, Keating et al. '06, Nonnenmacher–Rubin '07, Keating et al. '08, Novaes et al. '09, Carlo et al. '16 . . .

Open baker's maps

Open baker's maps $\varkappa = \varkappa_{M,\mathcal{A}}$ are determined by

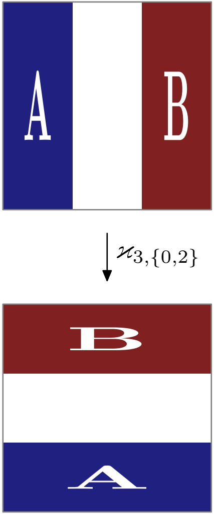
- an integer $M \geq 3$, the base
- a set $\mathcal{A} \subset \{0, \dots, M-1\}$, the alphabet
- we always assume $1 < |\mathcal{A}| < M$

$\varkappa$ is a canonical relation on $(0,1)_x \times (0,1)_\xi$:

$$\varkappa : (x, \xi) \mapsto \left(Mx - a, \frac{\xi + a}{M}\right)$$

$$\text{if} \quad x \in \left(\frac{a}{M}, \frac{a+1}{M}\right), \quad a \in \mathcal{A}$$

Basic model for a hyperbolic transformation with 'holes' through which one can escape



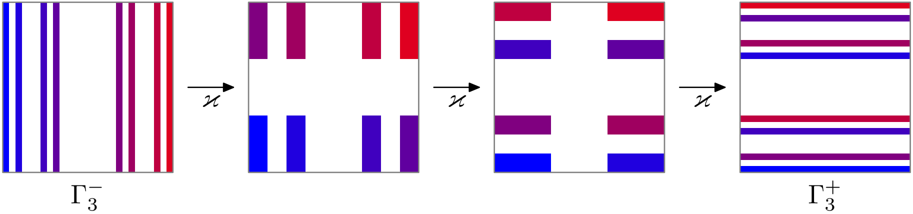
Cantor sets

For $k \in \mathbb{N}$, the domain and range of $\varkappa^k$ are

$$\Gamma_k^- := \text{Domain}(\varkappa^k) = \{(x,\xi) : \lfloor M^k \cdot x \rfloor \in \mathcal{C}_k\}$$
$$\Gamma_k^+ := \text{Range}(\varkappa^k) = \{(x,\xi) : \lfloor M^k \cdot \xi \rfloor \in \mathcal{C}_k\}$$

where $\mathcal{C}_k \subset \{0, \dots, M^k - 1\}$ is a discrete Cantor set:

$$\mathcal{C}_k = \mathcal{C}_k(M, \mathcal{A}) = \Big\{ \sum_{r=0}^{k-1} a_r M^r : a_0, \dots, a_{k-1} \in \mathcal{A} \Big\}$$



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The limiting Cantor set

$$\mathcal{C}_\infty := \bigcap_k \bigcup_{c \in \mathcal{C}_k} \Big[\frac{c}{M^k}, \frac{c+1}{M^k} \Big] \ \subset \ [0,1]$$

has Hausdorff dimension

$$\delta := \frac{\log |\mathcal{A}|}{\log M} \in (0,1)$$

Topological pressure: $P(s) = \delta - s, \ \ s \in \mathbb{R}$

Discrete microlocal analysis

Let $\ell^2_N := \ell^2(\mathbb{Z}_N)$, $\mathbb{Z}_N = \{0, \dots, N-1\}$, $N \gg 1$. Fourier transform:

$$\mathcal{F}_N : \ell^2_N \to \ell^2_N, \quad \mathcal{F}_N u(j) = \frac{1}{\sqrt{N}} \sum_\ell e^{-2\pi i j \ell / N} u(\ell)$$

Quantization of observables on the torus $\mathbb{T}^2 = \mathbb{S}^1_x \times \mathbb{S}^1_\xi$, $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$:

$$a \in C^\infty(\mathbb{T}^2) \quad \mapsto \quad \mathrm{Op}_N(a) : \ell^2_N \to \ell^2_N$$

$\mathrm{Op}_N(a)$ can localize in both position x and frequency ξ

Properties

- $a = a(x) \quad \Longrightarrow \quad \mathrm{Op}_N(a) = a_N, \quad a_N(j) = a(j/N)$
- $a = a(\xi) \quad \Longrightarrow \quad \mathrm{Op}_N(a) = \mathcal{F}_N^* a_N \mathcal{F}_N$
- $[\mathrm{Op}_N(a), \mathrm{Op}_N(b)] = -\frac{i}{2\pi N} \mathrm{Op}_N(\{a, b\}) + \mathcal{O}(N^{-2})_{\ell^2_N \to \ell^2_N}$

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Open quantum baker's maps

Example: $M = 3$, $\mathcal{A} = \{0, 2\}$. We put $N := M^k$ and

$$B_N = \mathcal{F}_N^* \begin{pmatrix} \chi_{N/3}\,\mathcal{F}_{N/3}\,\chi_{N/3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \chi_{N/3}\,\mathcal{F}_{N/3}\,\chi_{N/3} \end{pmatrix} : \ell_N^2 \to \ell_N^2$$

where we fix $\chi \in C_0^\infty((0,1);[0,1])$, $\chi_N(j) = \chi(j/N)$

- Why is B_N a quantization of $\varkappa_{M,\mathcal{A}}$? It satisfies Egorov's theorem:

$$B_N \operatorname{Op}_N(a) = \operatorname{Op}_N(b) B_N + \mathcal{O}(N^{-1})_{\ell_N^2 \to \ell_N^2}$$

if $\quad a(x,\xi) = b(y,\eta) \quad$ when $\varkappa_{M,\mathcal{A}}(x,\xi) = (y,\eta)$, $\xi, y \in \operatorname{supp}\chi$

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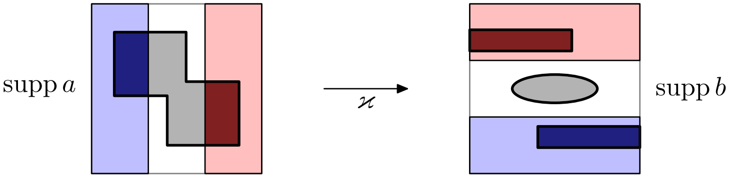

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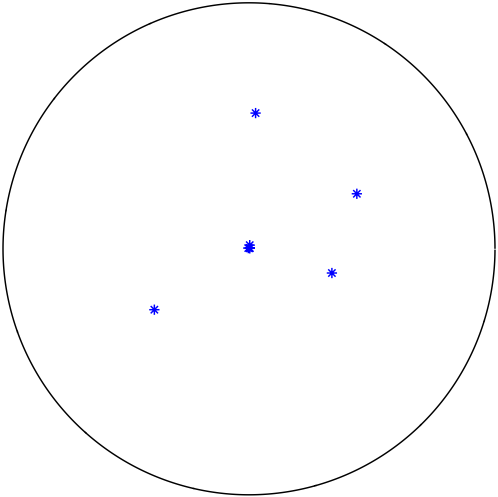
- Resonances = eigenvalues of B_N
  $\operatorname{Spec}(B_N) \subset D(0,1)$
- Similar procedure works for any $M, \mathcal{A}$

Numerical example: $M = 5$, $\mathcal{A} = \{1, 3\}$



$\mathrm{Spec}(B_N)$ for $k = 2$, $N = M^k$

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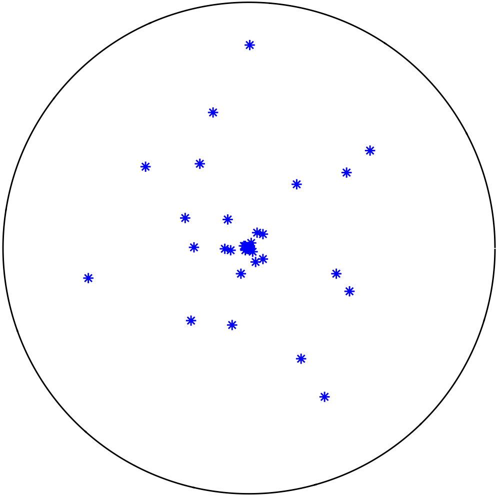

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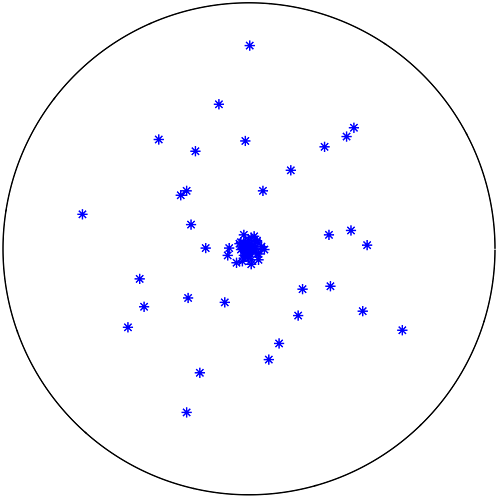

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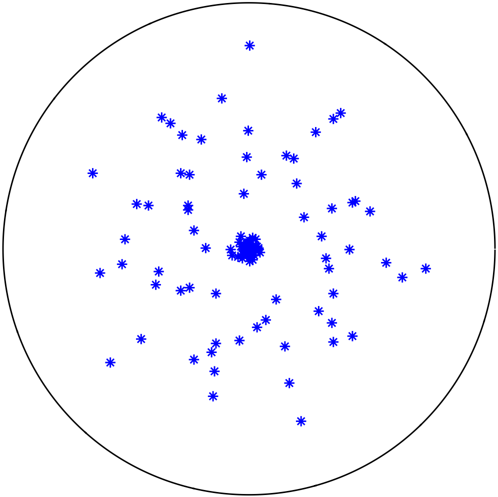

$\mathrm{Spec}(B_N)$ for $k = 5$, $N = M^k$

Results: spectral gaps

Define the spectral radius of B_N:

$$R_N := \max \{ |\lambda| : \lambda \in \operatorname{Spec}(B_N) \}, \quad N := M^k$$

Theorem 1 [D–Jin '16]

There exists (explicitly computable!)

$$\beta = \beta(M, \mathcal{A}) > \max\left(0, \frac{1}{2} - \delta\right)$$

such that B_N has an asymptotic spectral gap of size β:

$$\limsup_{N\to\infty} R_N \le M^{-\beta} < 1 \tag{1}$$

The convention $M^{-\beta} = e^{-\beta \log M}$ is due to $\varkappa$ having expansion rate M

The bound (1) with $\beta = -P(1/2) = \frac{1}{2} - \delta$ is the pressure gap, valid under the pressure condition $\delta < \frac{1}{2}$

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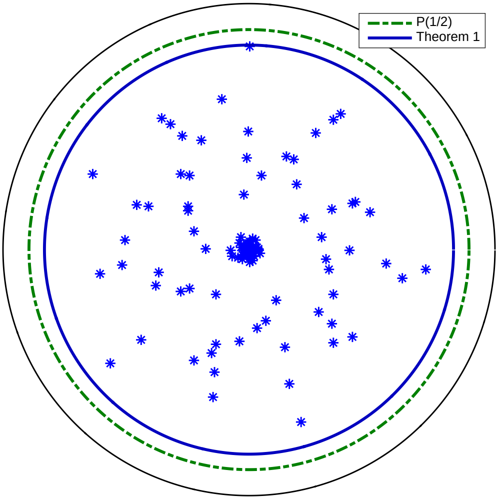

For some cases the gap of Theorem 1 approximates the spectral radius well

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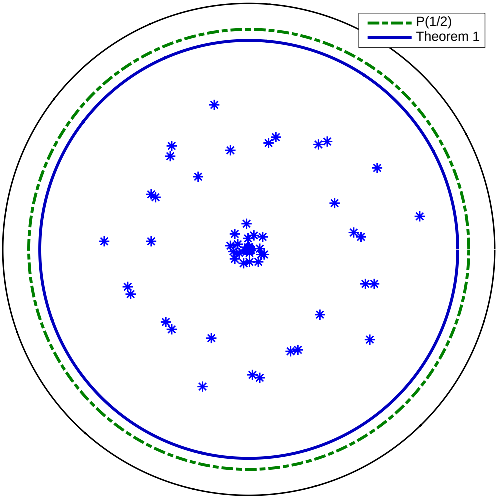

...and for some cases, this upper bound is far from sharp

Previous work

Nonnenmacher–Zworski '07, Walsh quantization of open quantum baker's maps which uses the Fourier transform on $\otimes^k \mathbb{Z}_M$ instead of $\mathbb{Z}_N$:
gap for $M = 3$, $\mathcal{A} = \{0, 2\}$, but no gap for $M = 4$, $\mathcal{A} = \{0, 2\}$

General hyperbolic systems:

- Patterson '76, Sullivan '79, Ikawa '88, Gaspard–Rice '89, Nonnenmacher–Zworski '09: pressure gap $\beta = -P(\frac{1}{2})$ for $P(\frac{1}{2}) < 0$
- Naud '05, Petkov–Stoyanov '10, Stoyanov '11, '12, Bourgain–Gamburd–Sarnak '11, Oh–Winter '16: improved gap $\beta = -P(\frac{1}{2}) + \varepsilon$ for some systems with $P(\frac{1}{2}) \leq 0$, where $\varepsilon > 0$ depends on the system in an unspecified way. Build on Dolgopyat '98
- D–Zahl '16: improved gap $\beta > 0$ for hyperbolic surfaces with $P(\frac{1}{2}) = 0$ and nearby surfaces, some with $P(\frac{1}{2}) > 0$.
  Bounds on β in terms of constants in Ahlfors–David regularity of the limit set. Uses fractal uncertainty principle and additive combinatorics

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Reduction to fractal uncertainty principle

Let $(B_N - \lambda)u = 0$, $\|u\|_{\ell^2_N} = 1$, $|\lambda| \geq c > 0$

Iterate Egorov's theorem ρk times, where $N = M^k$, $0 < 1 - \rho \ll 1$

$$B_N^k \operatorname{Op}_N(a) u = \operatorname{Op}_N(b) B_N^k u + \mathcal{O}(N^{-\infty})$$

$$\text{if} \quad a(x, \xi) = b(y, \eta) + \text{L.O.T.} \quad \text{when } \varkappa^k(x, \xi) = (y, \eta)$$

This is still possible since the resulting symbols vary on the scale N^{-1}

Recall $\Gamma_k^- = \text{Domain}(\varkappa^k)$, $\Gamma_k^+ = \text{Range}(\varkappa^k)$

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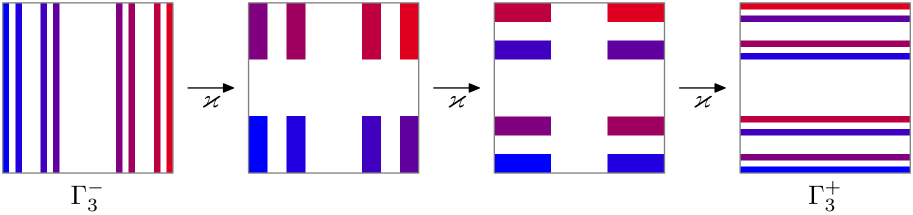

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- $a \equiv 1,\ b = \mathbf{1}_{\Gamma_k^+} \quad \Longrightarrow \quad u = \operatorname{Op}_N(\mathbf{1}_{\Gamma_k^+})u + \mathcal{O}(N^{-\infty})$
- $b \equiv 1,\ a = \mathbf{1}_{\Gamma_k^-} \quad \Longrightarrow \quad \|\operatorname{Op}_N(\mathbf{1}_{\Gamma_k^-})u\| \geq |\lambda|^k$
- Contradiction if $|\lambda| \geq M^{-\beta+\varepsilon}$ and the fractal uncertainty principle holds with exponent β:

$$\|\operatorname{Op}_N(\mathbf{1}_{\Gamma_k^-})\operatorname{Op}_N(\mathbf{1}_{\Gamma_k^+})\|_{\ell^2_N \to \ell^2_N} \leq CN^{-\beta}$$

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Recall $\Gamma_k^- = \text{Domain}(\varkappa^k)$, $\Gamma_k^+ = \text{Range}(\varkappa^k)$

- $a \equiv 1$, $b = \mathbf{1}_{\Gamma_k^+} \quad \Longrightarrow \quad u = \operatorname{Op}_N(\mathbf{1}_{\Gamma_k^+})u + \mathcal{O}(N^{-\infty})$
- $b \equiv 1$, $a = \mathbf{1}_{\Gamma_k^-} \quad \Longrightarrow \quad \|\operatorname{Op}_N(\mathbf{1}_{\Gamma_k^-})u\| \geq |\lambda|^k$
- Contradiction if $|\lambda| \geq M^{-\beta+\varepsilon}$ and the fractal uncertainty principle holds with exponent β:

$$\|\operatorname{Op}_N(\mathbf{1}_{\Gamma_k^-})\operatorname{Op}_N(\mathbf{1}_{\Gamma_k^+})\|_{\ell^2_N \to \ell^2_N} \leq CN^{-\beta}$$

Reduction to fractal uncertainty principle

Let $(B_N - \lambda)u = 0$, $\|u\|_{\ell^2_N} = 1$, $|\lambda| \geq c > 0$

Iterate Egorov's theorem ρk times, where $N = M^k$, $0 < 1 - \rho \ll 1$

$$B_N^k \operatorname{Op}_N(a)u = \operatorname{Op}_N(b)\lambda^k u + \mathcal{O}(N^{-\infty})$$

$$\text{if} \quad a(x,\xi) = b(y,\eta) + \text{L.O.T.} \quad \text{when } \varkappa^k(x,\xi) = (y,\eta)$$

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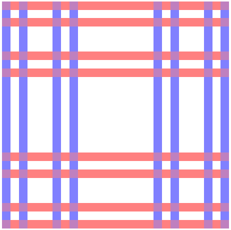
Want to prove the fractal uncertainty principle

$$\|\mathrm{Op}_N(\mathbf{1}_{\Gamma_k^-})\mathrm{Op}_N(\mathbf{1}_{\Gamma_k^+})\|_{\ell_N^2\to\ell_N^2} \le CN^{-\beta}$$

Using the relation of $\Gamma_k^\pm$ with the Cantor set $\mathcal{C}_k \subset \mathbb{Z}_N$, rewrite this as

$$\|\mathbf{1}_{\mathcal{C}_k}\mathcal{F}_N\mathbf{1}_{\mathcal{C}_k}\|_{\ell_N^2\to\ell_N^2} \le CN^{-\beta} \quad (2)$$

(2) $\Rightarrow$ no function can be localized on $\mathcal{C}_k$ in both position and frequency



Volume bound: $N = M^k$, $|\mathcal{C}_k| = |\mathcal{A}|^k = N^\delta$, $\|\mathcal{F}_N\|_{\ell_N^1\to\ell_N^\infty} \le N^{-1/2}$

$\Rightarrow$ (2) with $\beta = \frac{1}{2} - \delta$, recovering the pressure gap

To prove Theorem 1, we need to improve over $\beta = 0$ and the volume bound

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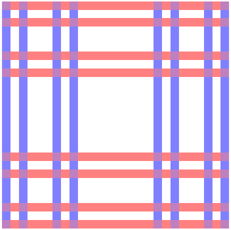

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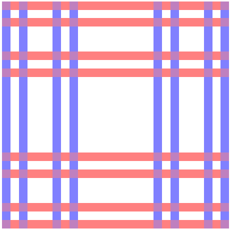

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Proof of fractal uncertainty principle

Theorem 2 [D–Jin '16]

We have $\|\mathbf{1}_{\mathcal{C}_k}\mathcal{F}_N\mathbf{1}_{\mathcal{C}_k}\|_{\ell^2_N\to\ell^2_N}\leq N^{-\beta}$ for some

$$\beta=\beta(M,\mathcal{A})>\max\left(0,\frac{1}{2}-\delta\right)$$

- Submultiplicativity: if $r_k:=\|\mathbf{1}_{\mathcal{C}_k}\mathcal{F}_N\mathbf{1}_{\mathcal{C}_k}\|_{\ell^2_N\to\ell^2_N}$ then $r_{k+\ell}\leq r_k\cdot r_\ell$
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- Thus enough to show that $r_k < \min(1, N^{\delta-1/2})$ for some k
- $r_k < 1$: if not, then find nonzero $u = \mathbf{1}_{\mathcal{C}_k}u$, $\mathcal{F}_N u = 0$ on $\mathbb{Z}_N \setminus \mathcal{C}_k$
  By cyclic shift, may assume that $M-1 \notin \mathcal{A}$. The polynomial

  $$p(z) = \sum_j u(j)z^j$$

  has degree at most $\max \mathcal{C}_k \leq (M-1)M^{k-1}$ and at least $|\mathbb{Z}_N \setminus \mathcal{C}_k| \geq M^k - (M-1)^k$ roots. Contradiction for large k

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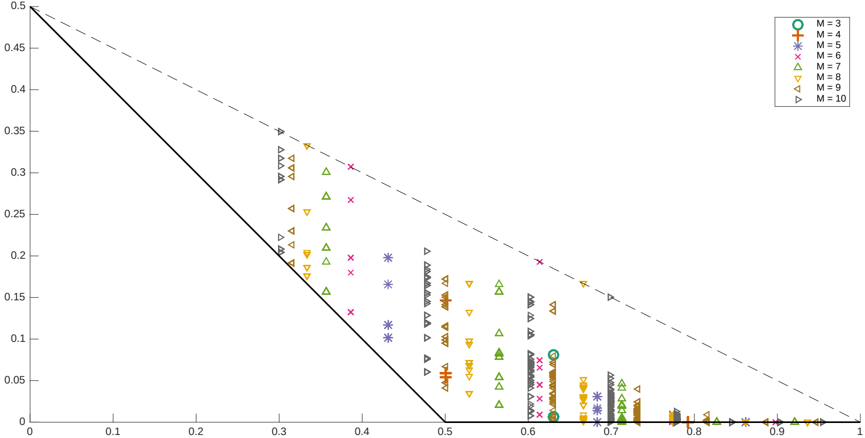
- Thus enough to show that $r_k<\min(1,N^{\delta-1/2})$ for some k
- $r_k<N^{\delta-1/2}=|\mathcal{C}_k|/\sqrt{N}$: if not, then

$$\|\mathbf{1}_{\mathcal{C}_k}\mathcal{F}_N\mathbf{1}_{\mathcal{C}_k}\|_{\ell^2_N\to\ell^2_N}=\frac{|\mathcal{C}_k|}{\sqrt{N}}=\|\mathbf{1}_{\mathcal{C}_k}\mathcal{F}_N\mathbf{1}_{\mathcal{C}_k}\|_{\mathrm{HS}}$$

Then $\mathbf{1}_{\mathcal{C}_k}\mathcal{F}_N\mathbf{1}_{\mathcal{C}_k}$ has rank 1, so all 2×2 minors are zero.
Contradiction when $|\mathcal{A}|>1$, $k=2$

More on fractal uncertainty exponents



X axis: δ; Y axis: FUP exponent β (numerics); all alphabets with $M \leq 10$

Solid line: $\beta = \max(0, \frac{1}{2} - \delta)$, dashed line: $\beta = -\frac{P(1)}{2} = \frac{1-\delta}{2}$

More on fractal uncertainty exponents

Bounds on β as $M \to \infty$:

$\delta \leq 1/2$:

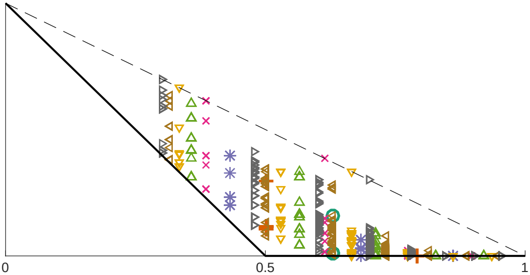
$$\beta - \left(\tfrac{1}{2} - \delta\right) \gtrsim \frac{1}{M^8 \log M}$$

$\delta \approx 1/2$: using additive energy,

$$\beta \gtrsim \frac{1}{\log M}$$

$\delta \geq 1/2$:

$$\beta \gtrsim \exp\left(-M^{\frac{\delta}{1-\delta}+o(1)}\right)$$



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- Examples of alphabets (arithmetic progressions) with $\delta \leq 1/2$ and

$$\beta - \left(\tfrac{1}{2} - \delta\right) \lesssim \frac{M^{2\delta-1}}{\log M}$$

- Examples of special alphabets with $\beta = \frac{1-\delta}{2}$

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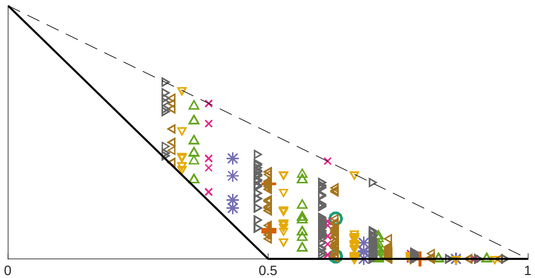

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Special alphabets with $\beta = \frac{1-\delta}{2}$

We call $\mathcal{A}$ a special alphabet, if

$$\text{for all } \ j, \ell \in \mathcal{A}, \ j \neq \ell, \quad \text{we have } \ \mathcal{F}_M(\mathbf{1}_{\mathcal{A}})(j-\ell) = 0 \qquad (3)$$

Such $\mathcal{A}$ have $\beta = \frac{1-\delta}{2} = -\frac{P(1)}{2}$, which is the largest possible value of β and all nonzero singular values of $\mathbf{1}_{\mathcal{C}^k}\mathcal{F}_N\mathbf{1}_{\mathcal{C}^k}$ are equal to $N^{-\beta}$

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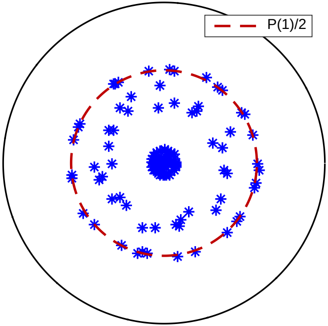

Example: $M = 6$, $\mathcal{A} = \{1, 4\}$, $N = M^5$

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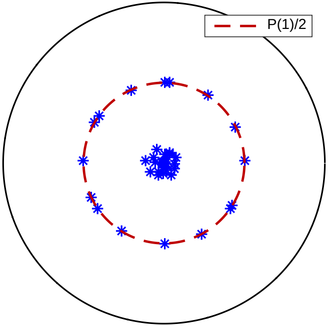

Example: $M = 8$, $\mathcal{A} = \{2, 4\}$, $N = M^4$

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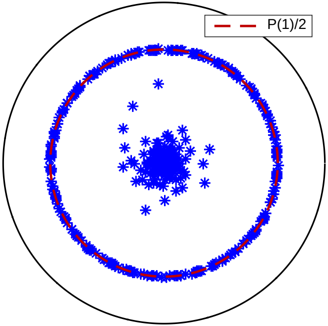

Example: $M = 8$, $\mathcal{A} = \{1, 2, 5, 6\}$, $N = M^4$

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Conjecture 1 (band structure)

Assume $(M, \mathcal{A})$ satisfies (3). Then there exists $\mu > \frac{1-\delta}{2}$ such that:

- For any $\varepsilon > 0$ and N large, there is a second gap

$$\operatorname{Spec}(B_N) \cap \{M^{-\mu} \leq |\lambda| \leq M^{-\frac{1-\delta}{2}-\varepsilon}\} = \emptyset$$

- Eigenvalues in the first band satisfy exact fractal Weyl law:

$$\big|\operatorname{Spec}(B_N) \cap \{|\lambda| \geq M^{-\mu}\}\big| = |\mathcal{A}|^k = N^{\delta}$$

Conjecture 1 is confirmed by numerics

Results: resonance counting

We count eigenvalues of B_N in annuli:

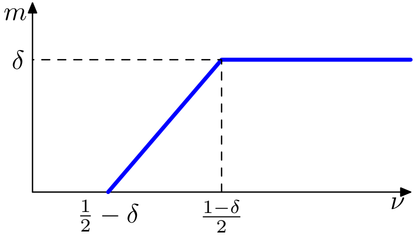
$$\#(N,\nu) = \big|\operatorname{Spec}(B_N) \cap \{|\lambda| \geq M^{-\nu}\}\big|$$

Theorem 3 [D–Jin '16]

For each $\varepsilon > 0$ and $\nu > 0$ we have the fractal Weyl upper bound

$$\#(N,\nu) \leq C_{\nu,\varepsilon} N^{m(\delta,\nu)+\varepsilon}, \quad m(\delta,\nu) = \min(\delta, 2\nu + 2\delta - 1)$$

Note: $m = \delta$ for $\nu \geq \frac{1-\delta}{2} = -\frac{P(1)}{2}$, $m < 0$ for $\nu < \frac{1}{2} - \delta = -P(\frac{1}{2})$



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- Lu–Sridhar–Zworski '03: concentration of decay rates near $\nu = -P(1)/2$. Jakobson–Naud '12 conjectured gap of this size
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No matching lower bounds are known, except

Nonnenmacher–Zworski '07: Exact fractal Weyl law for Walsh quantization

Conjecture 2 (fractal Weyl law)

For each $\nu > \frac{1-\delta}{2}$, we have $\#(N,\nu) \geq c_\nu N^\delta > 0$

Conjecture 2 is also supported by numerics

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Ideas of the proof

- Recall that for $(B_N - \lambda)u = 0$, $\|u\| = 1$, $|\lambda| \geq M^{-\nu}$,

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- The first statement $\Rightarrow$ $\#(N,\nu) \lesssim \mathrm{Rank}(\operatorname{Op}_N(\mathbf{1}_{\Gamma_+^k})) = N^\delta$
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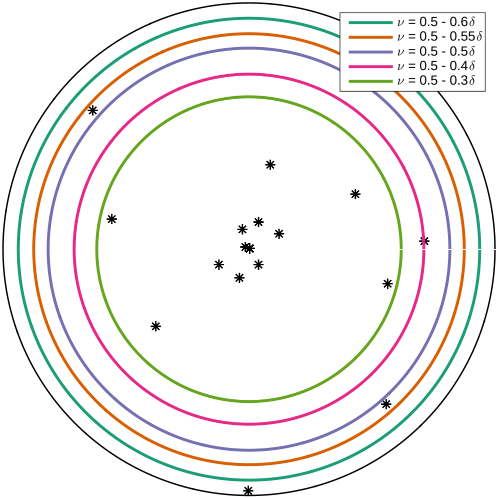
Numerical example: $M = 6$, $\mathcal{A} = \{1, 2, 3, 4\}$



$k = 2$

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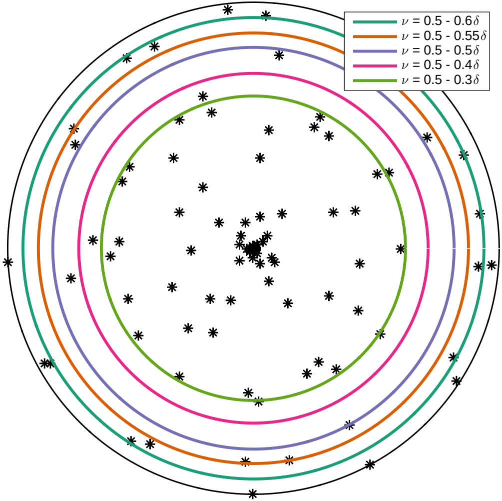

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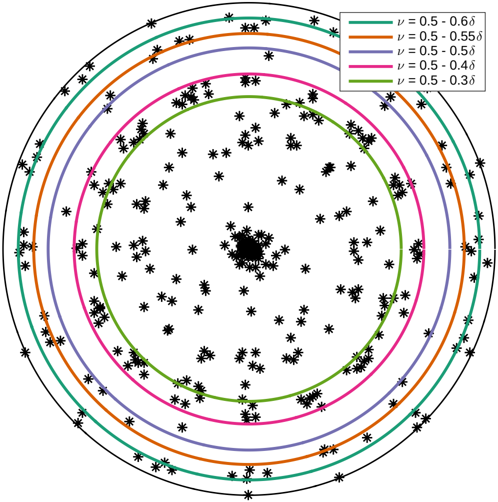

$k = 4$

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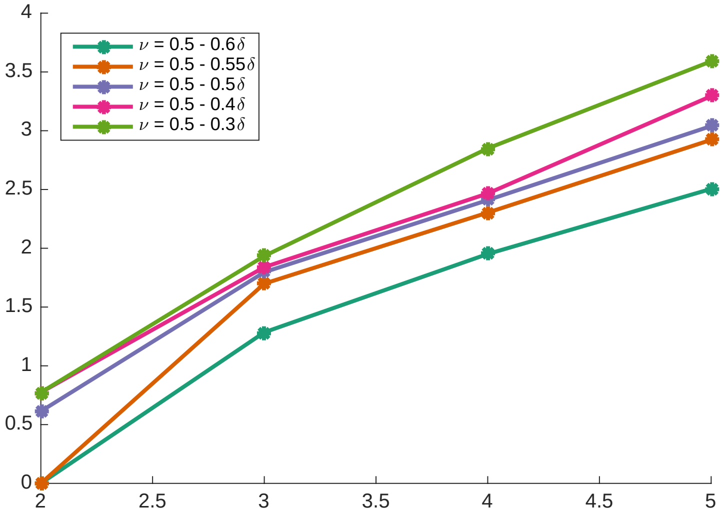
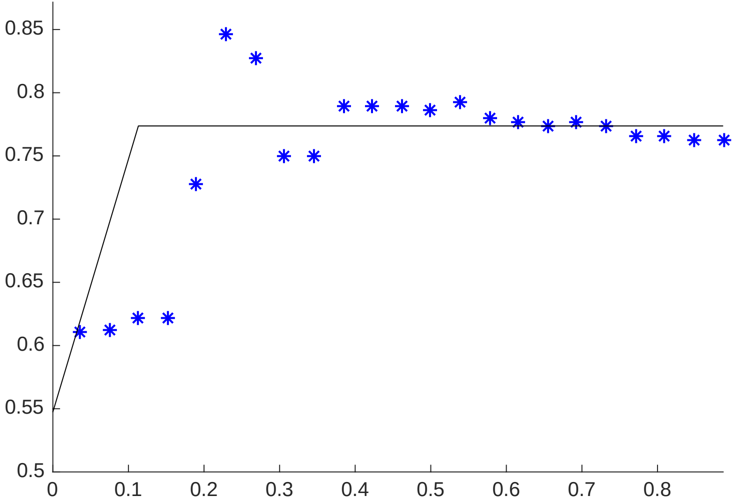


Plot of $\log \#(M^k, \nu)/\log M$ as a function of k

Numerical example: $M = 6$, $\mathcal{A} = \{1, 2, 3, 4\}$



Linear fits for the growth exponent of $\#(N, \nu)$ and the bound of Theorem 3

Summary

- We obtain results on spectral gap which lie well beyond what is known for more general systems
- We use fractal uncertainty principle, the fine structure of the associated Cantor sets, and simple tools from harmonic analysis, algebra, combinatorics, and number theory
- We also show a fractal Weyl upper bound
- We discover that the studied systems form a rich class with a variety of different types of behavior

Thank you for your attention!

Results: dependence on cutoff

Recall that the defininition of $B_N = B_{N,\chi}$ involved a cutoff function

$$\chi \in C_0^\infty((0,1);[0,1])$$

e.g. for $M = 3$, $\mathcal{A} = \{0, 2\}$

$$B_N = \mathcal{F}_N^* \begin{pmatrix} \chi_{N/3}\,\mathcal{F}_{N/3}\,\chi_{N/3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \chi_{N/3}\,\mathcal{F}_{N/3}\,\chi_{N/3} \end{pmatrix}$$

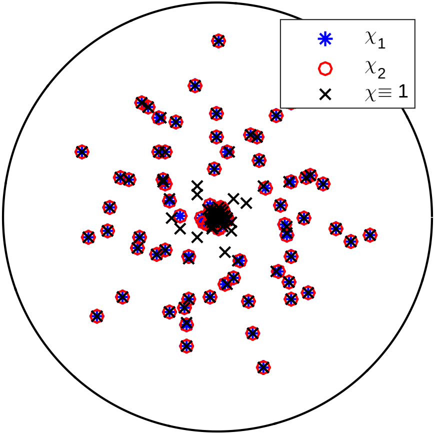
Theorem 4 [D–Jin '16]

Assume that $\chi_1, \chi_2 \in C_0^\infty((0,1);[0,1])$ and $\chi_1 = \chi_2$ near the Cantor set $\mathcal{C}_\infty \subset [0,1]$. Then for each ν, eigenvalues of B_{N,χ_1} in $\{|\lambda| \geq M^{-\nu}\}$ are $\mathcal{O}(N^{-\infty})$ quasimodes of B_{N,χ_2}.

Dependence on cutoff

If $0, M-1 \notin \mathcal{A}$ it is natural to take $\chi = 1$ near $\mathcal{C}_\infty$.

However we cannot take $\chi \equiv 1$:



$M = 5$, $\mathcal{A} = \{1, 3\}$, $N = M^5$, $\chi_1 = \chi_2 = 1$ near $\mathcal{C}_\infty$