Fractal uncertainty principle

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joint work with Joshua Zahl (MIT)

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Discrete uncertainty principle

We use the discrete case for simplicity of presentation

$$\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z} = \{0, \dots, N-1\}$$

$$\ell^2_N = \{u : \mathbb{Z}_N \to \mathbb{C}\}, \quad \|u\|^2_{\ell^2_N} = \sum\nolimits_j |u(j)|^2$$

$$\mathcal{F}_N u(j) = \frac{1}{\sqrt{N}} \sum\nolimits_k e^{-2\pi ijk/N} u(k)$$

The Fourier transform $\mathcal{F}_N : \ell^2_N \to \ell^2_N$ is a unitary operator

Take $X = X(N)$, $Y = Y(N) \subset \mathbb{Z}_N$. Want a bound for some $\beta > 0$

$$\|\mathbf{1}_X \mathcal{F}_N \mathbf{1}_Y\|_{\ell^2_N \to \ell^2_N} \leq CN^{-\beta}, \quad N \to \infty \tag{1}$$

Here $\mathbf{1}_X, \mathbf{1}_Y : \ell^2_N \to \ell^2_N$ are multiplication operators

If (1) holds, say that X, Y satisfy uncertainty principle with exponent β

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Basic properties

$$\|\mathbf{1}_X \mathcal{F}_N \mathbf{1}_Y\|_{\ell^2_N \to \ell^2_N} \leq C N^{-\beta}, \quad N \to \infty; \quad \beta > 0 \tag{2}$$

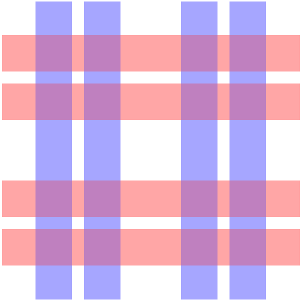
Why uncertainty principle?

Basic properties

$$\|\mathbf{1}_X \mathcal{F}_N \mathbf{1}_Y \mathcal{F}_N^{-1}\|_{\ell^2_N \to \ell^2_N} \leq C N^{-\beta}, \quad N \to \infty; \quad \beta > 0 \qquad (2)$$

$\mathbf{1}_X$ localizes to X in position, $\mathcal{F}_N \mathbf{1}_Y \mathcal{F}_N^{-1}$ localizes to Y in frequency

(2) $\implies$ these localizations are incompatible



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Volume bound using Hölder's inequality:

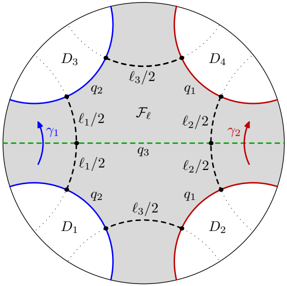
$$\begin{aligned}\|\mathbf{1}_X \mathcal{F}_N \mathbf{1}_Y\|_{\ell^2_N \to \ell^2_N} &\le \|\mathbf{1}_X\|_{\ell^\infty_N \to \ell^2_N} \|\mathcal{F}_N\|_{\ell^1_N \to \ell^\infty_N} \|\mathbf{1}_Y\|_{\ell^2_N \to \ell^1_N} \\ &\le \sqrt{\frac{|X| \cdot |Y|}{N}}\end{aligned}$$

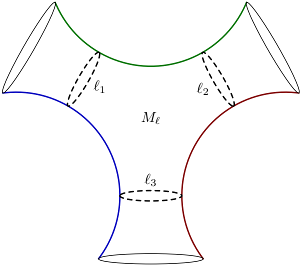
This norm is < 1 when $|X| \cdot |Y| < N$. Cannot be improved in general:

$$N = MK, \ X = M\mathbb{Z}/N\mathbb{Z}, \ Y = K\mathbb{Z}/N\mathbb{Z} \quad \implies \quad \|\mathbf{1}_X \mathcal{F}_N \mathbf{1}_Y\|_{\ell^2_N \to \ell^2_N} = 1$$

Application: spectral gaps for hyperbolic surfaces

$(M,g)=\Gamma\backslash\mathbb{H}^2$ convex co-compact hyperbolic surface





Resonances: poles of the Selberg zeta function (with a few exceptions)

$$Z_M(\lambda)=\prod_{\ell\in\mathcal{L}_M}\prod_{k=0}^{\infty}\left(1-e^{-(s+k)\ell}\right),\quad s=\frac{1}{2}-i\lambda$$

where $\mathcal{L}_M$ is the set of lengths of primitive closed geodesics on M

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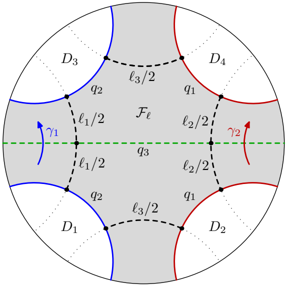
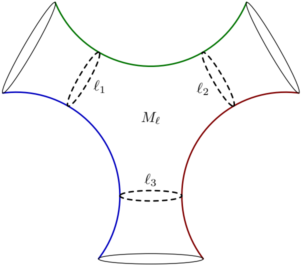




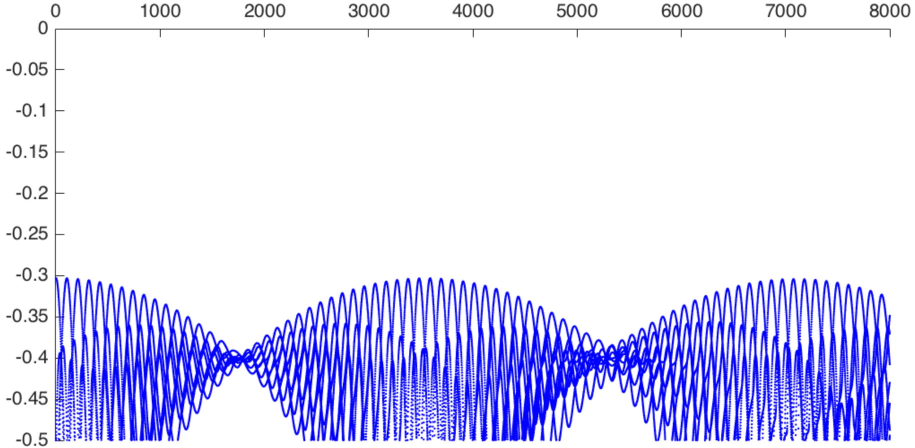
Resonances: poles of the scattering resolvent

$$R(\lambda) = \Big(-\Delta_g - \frac{1}{4} - \lambda^2 \Big)^{-1} : \begin{cases} L^2(M) \to L^2(M), & \operatorname{Im}\lambda > 0 \\ L^2_{\mathrm{comp}}(M) \to L^2_{\mathrm{loc}}(M), & \operatorname{Im}\lambda \le 0 \end{cases}$$

Existence of meromorphic continuation: Patterson '75,'76, Perry '87,'89, Mazzeo–Melrose '87, Guillopé–Zworski '95, Guillarmou '05, Vasy '13

Plots of resonances

Three-funnel surface with $\ell_1 = \ell_2 = \ell_3 = 7$

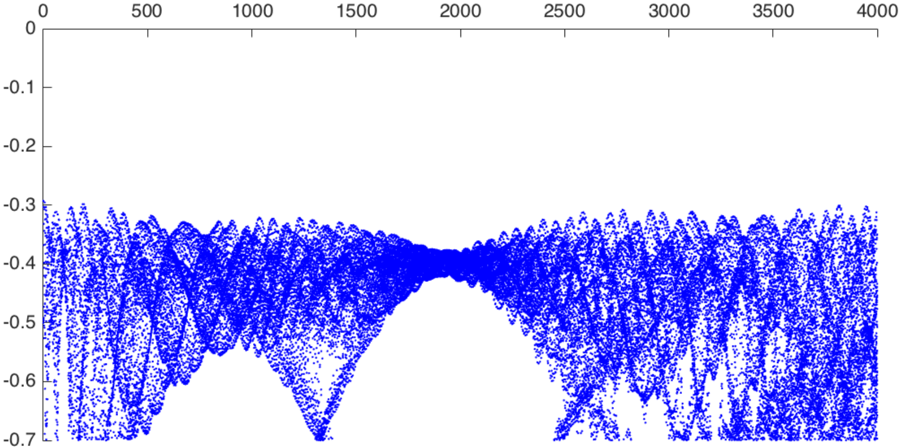


Data courtesy of David Borthwick and Tobias Weich

See `arXiv:1305.4850` and `arXiv:1407.6134` for more

Plots of resonances

Three-funnel surface with $\ell_1 = 6$, $\ell_2 = \ell_3 = 7$



Data courtesy of David Borthwick and Tobias Weich

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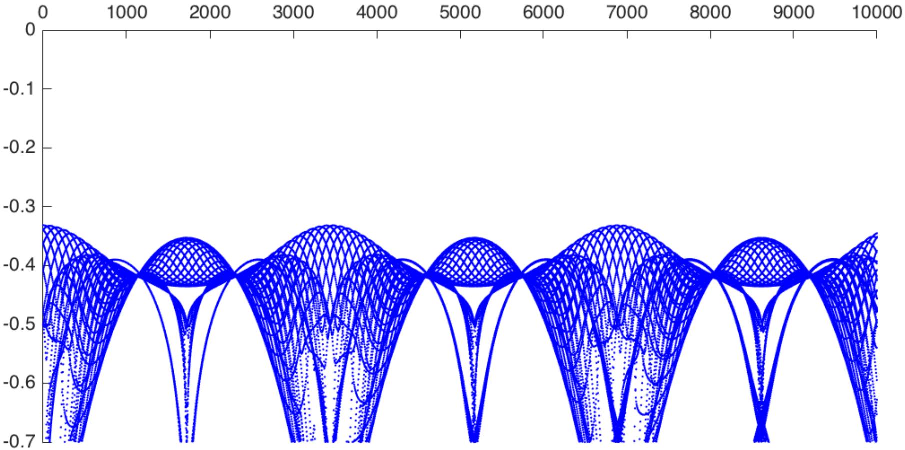
Plots of resonances

Torus-funnel surface with $\ell_1 = \ell_2 = 7$, $\varphi = \pi/2$, trivial representation



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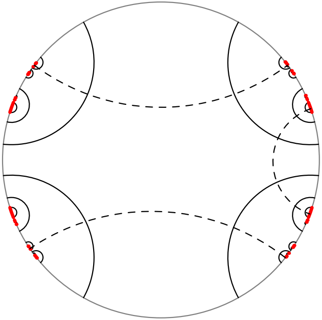
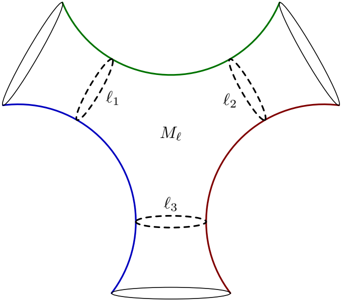
The limit set and δ

$M = \Gamma\backslash\mathbb{H}^2$ hyperbolic surface

$\Lambda_\Gamma \subset \mathbb{S}^1$ the limit set

$\delta := \dim_H(\Lambda_\Gamma) \in (0,1)$





Trapped geodesics: those with endpoints in Λ_Γ

Spectral gaps

Essential spectral gap of size $\beta > 0$:

only finitely many resonances with $\operatorname{Im}\lambda > -\beta$

Application: exponential decay of waves (modulo finite dimensional space)

Patterson–Sullivan theory: the topmost resonance is at $\lambda = i(\delta - \frac{1}{2})$, where $\delta = \dim_H \Lambda_\Gamma \in (0,1)$ $\Rightarrow$ gap of size $\beta = \max\left(0, \frac{1}{2} - \delta\right)$

$\delta > \frac{1}{2}$ $\delta < \frac{1}{2}$

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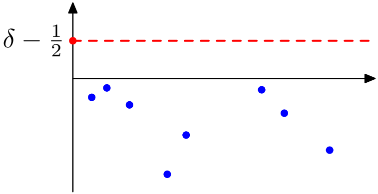

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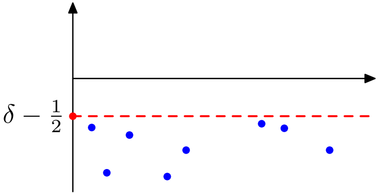

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Improved gap $\beta = \frac{1}{2} - \delta + \varepsilon$ for $\delta \leq 1/2$:
Dolgopyat '98, Naud '04, Stoyanov '11,'13, Petkov–Stoyanov '10

Bourgain–Gamburd–Sarnak '11, Oh–Winter '14: gaps for the case of congruence quotients

However, the size of ε is hard to determine from these arguments

Spectral gaps via uncertainty principle

$$M = \Gamma\backslash\mathbb{H}^2, \quad \Lambda_\Gamma \subset \mathbb{S}^1 \text{ limit set}, \quad \dim_H \Lambda_\Gamma = \delta \in (0,1)$$

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Theorem [D–Zahl '15]

Assume that Λ_Γ satisfies hyperbolic uncertainty principle with exponent β. Then M has an essential spectral gap of size $\beta-$.

Proof

- Enough to show $e^{-\beta t}$ decay of waves at frequency $\sim h^{-1}$, $0 < h \ll 1$
- Microlocal analysis + hyperbolicity of geodesic flow $\Rightarrow$ description of waves at times $\log(1/h)$ using stable/unstable Lagrangian states
- Hyperbolic UP $\Rightarrow$ a superposition of trapped unstable states has norm $\mathcal{O}(h^\beta)$ on trapped stable states

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Theorem [D–Zahl '15]

Assume that Λ_Γ satisfies hyperbolic uncertainty principle with exponent β. Then M has an essential spectral gap of size $\beta-$.

The Patterson–Sullivan gap $\beta = \frac{1}{2} - \delta$ corresponds to the volume bound:

$$|X| \sim |Y| \sim N^\delta \quad \Longrightarrow \quad \sqrt{\frac{|X| \cdot |Y|}{N}} \sim N^{\delta - 1/2}$$

Discrete UP with β for discretizations of Λ_Γ

$\Downarrow$

Hyperbolic UP with $\beta/2$

Regularity of limit sets

The sets X, Y coming from convex co-compact hyperbolic surfaces are δ-regular with some constant $C > 0$:

$$C^{-1}n^\delta \leq \big|X \cap [j-n, j+n]\big| \leq Cn^\delta, \quad j \in X,\ 1 \leq n \leq N$$

Conjecture 1

If X, Y are δ-regular with constant C and $\delta < 1$, then

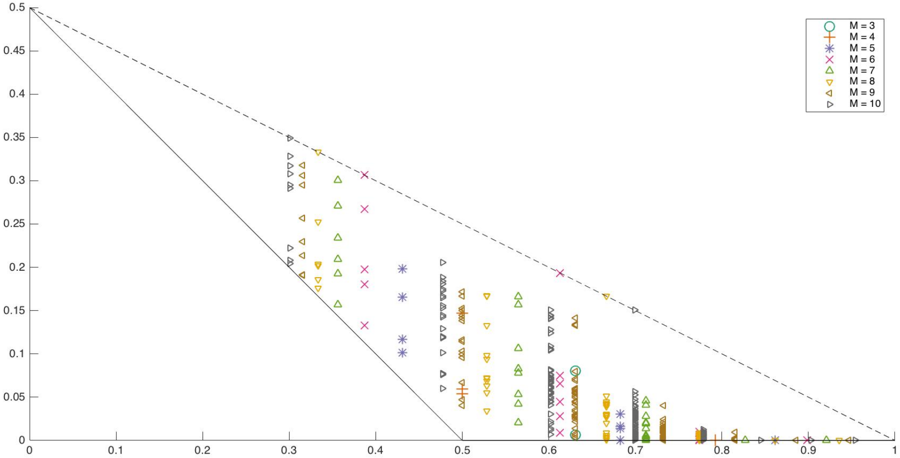
$$\|\mathbf{1}_X \mathcal{F}_N \mathbf{1}_Y\|_{\ell^2_N \to \ell^2_N} \leq CN^{-\beta}, \quad \beta = \beta(\delta, C) > \max\Big(0, \frac{1}{2} - \delta\Big)$$

Implies that each convex co-compact M has essential spectral gap > 0

Conjecture holds for discrete Cantor sets with $N = M^k$, $k \to \infty$

$$X = Y = \Big\{ \sum_{0 \leq \ell < k} a_\ell M^\ell \,\Big|\, a_0, \dots, a_{k-1} \in \mathcal{A} \Big\}, \quad \mathcal{A} \subset \{0, \dots, M-1\}$$

Uncertainty principle for Cantor sets (numerics)



Horizontal axis: the dimension δ; vertical axis: the FUP exponent β

Uncertainty principle via additive energy

For $X \subset \mathbb{Z}_N$, its additive energy is (note $|X|^2 \leq E_A(X) \leq |X|^3$)

$$E_A(X) = \big|\{(a,b,c,d) \in X^4 \mid a+b=c+d \mod N\}\big|$$

$$\|\mathbf{1}_X \mathcal{F}_N \mathbf{1}_Y\|_{\ell^2_N \to \ell^2_N} \leq \frac{E_A(X)^{1/8}|Y|^{3/8}}{N^{3/8}} \qquad (3)$$

In particular, if $|X| \sim |Y| \sim N^\delta$ and $E_A(X) \leq C|X|^3 N^{-\beta_E}$, then X, Y satisfy uncertainty principle with

$$\beta = \frac{3}{4}\Big(\frac{1}{2} - \delta\Big) + \frac{\beta_E}{8}$$

Proof of (3): use Schur's Lemma and a T^*T argument to get

$$\|\mathbf{1}_X \mathcal{F}_N \mathbf{1}_Y\|^2_{\ell^2_N \to \ell^2_N} \leq \frac{1}{\sqrt{N}} \max_{j \in Y} \sum_{k \in Y} \big|\mathcal{F}_N(\mathbf{1}_X)(j-k)\big|$$

The sum in the RHS is bounded using L^4 norm of $\mathcal{F}_N(\mathbf{1}_X)$

Estimating additive energy

Theorem [D–Zahl '15]

If $X \subset \mathbb{Z}_N$ is δ-regular with constant C_R and $\delta \in (0,1)$, then

$$E_A(X) \leq C|X|^3 N^{-\beta_E}, \quad \beta_E = \delta \exp\big[-K(1-\delta)^{-28}\log^{14}(1+C_R)\big]$$

Here K is a global constant

Proof

- X is δ-regular $\implies$ X cannot contain long arithmetic progressions
- A version of Freĭman's Theorem $\implies$ X cannot have maximal additive energy on a large enough intermediate scale
- Induction on scale $\implies$ a power improvement in $E_A(X)$

Additive portraits

For $X \subset \mathbb{Z}_N$, take $f_X : \mathbb{Z}_n \to \mathbb{N}_0$, $j \mapsto \left|\{(a,b) \in X^2 \colon a - b = j \mod N\}\right|$

Sort $f_X(0), \dots, f_X(N-1)$ in decreasing order $\implies$ additive portrait of X

$|X|^2 = f_X(0) + \cdots + f_X(N-1), \quad E_A(X) = f_X(0)^2 + \cdots + f_X(N-1)^2$

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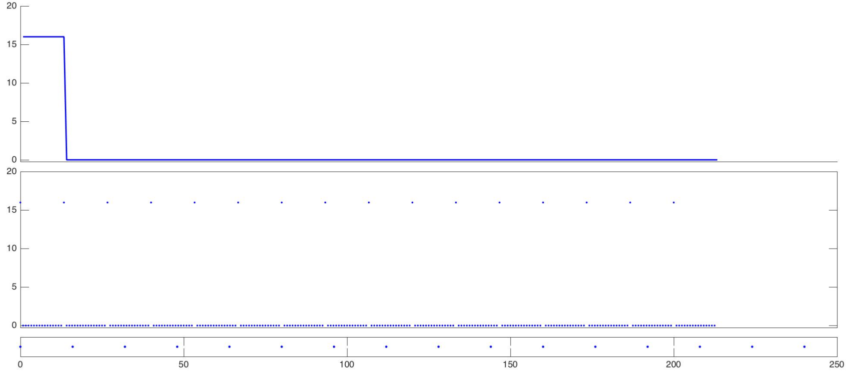

A subgroup $16\mathbb{Z}/256\mathbb{Z}$

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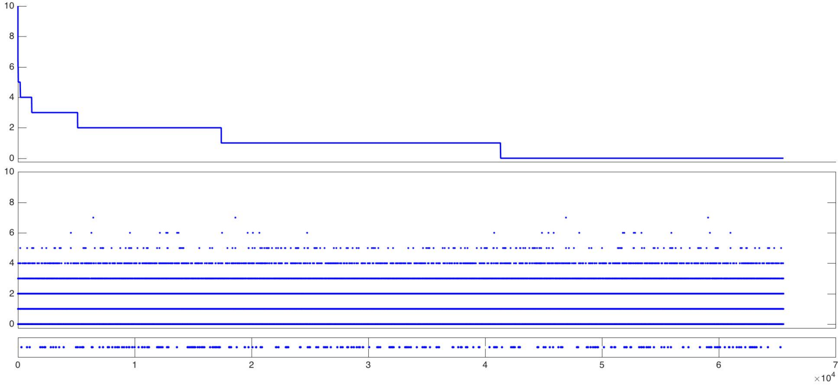

2^8 points chosen at random with $N = 2^{16}$

Additive portraits

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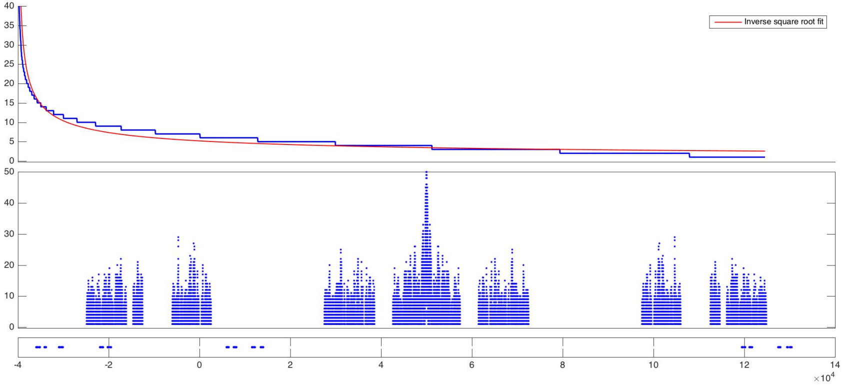

Discretized limit set with $\delta = 1/2$, $N = 2^{16}$ (data by Arjun Khandelwal)

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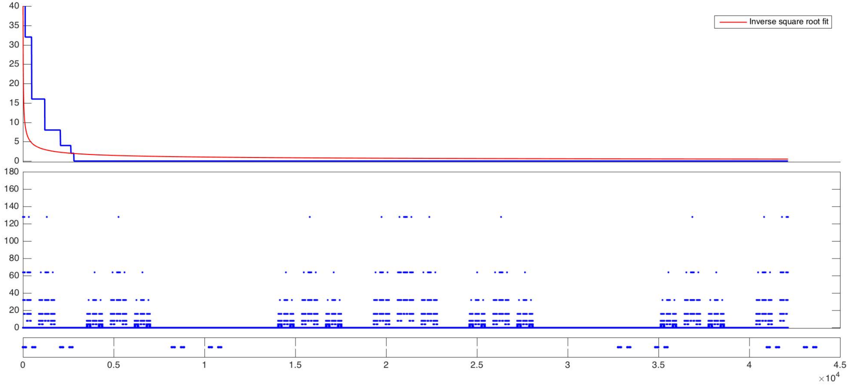

Cantor set with $M = 4$, $\mathcal{A} = \{0, 2\}$, $k = 8$, $N = 2^{16}$

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Numerics for $\delta = 1/2$ indicate: j-th largest value of f_X is $\sim \sqrt{\frac{N}{j}}$.
This would give additive energy $\sim N \log N$

Conjecture 2

Let X be a discretization on scale $1/N$ of a limit set Λ_Γ of a convex co-compact surface with $\dim \Lambda_\Gamma = \delta \in (0,1)$. (Note $|X| \sim N^\delta$.) Then

$$E_A(X) = \mathcal{O}(N^{3\delta - \beta_E +}), \quad \beta_E := \min(\delta, 1 - \delta).$$

What does this give for hyperbolic surfaces?

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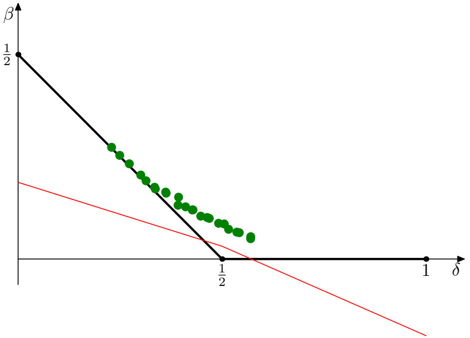

Numerics by Borthwick–Weich '14 + gap under Conjecture 2

Thank you for your attention!