Resonances in classical and quantum dynamics

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March 30, 2015

What are resonances?

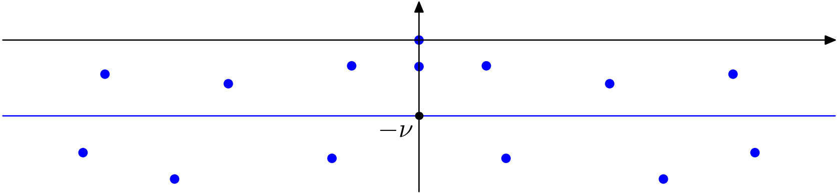
Resonances: complex characteristic frequencies associated to open or dissipative systems

real part = rate of oscillation, imaginary part = rate of decay

For an observable $u(t)$, the resonance expansion is

$$u(t) = \sum_{\substack{\omega_j \text{ resonance} \\ \operatorname{Im} \omega_j \geq -\nu}} e^{-it\omega_j} u_j + \mathcal{O}(e^{-\nu t}), \quad t \to +\infty$$

which is analogous to eigenvalue expansions for closed systems



Motivation: statistics for billiards

One billiard ball in a Sinai billiard with finite horizon

10000 billiard balls in a Sinai billiard with finite horizon

#(balls in the box)	$\rightarrow$	volume of the box
velocity angles distribution	$\rightarrow$	uniform measure

10000 billiard balls in a three-disk system

#(balls in the box)	$\to$	0 exponentially
velocity angles distribution	$\to$	some fractal measure

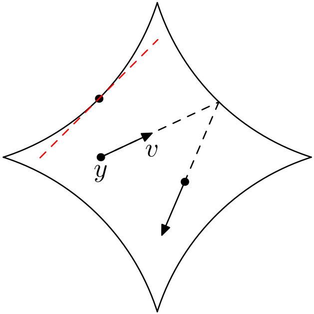
Dynamical systems

$\mathcal{U}$ phase space of the dynamical system

$\varphi^t : \mathcal{U} \to \mathcal{U}$ flow of the system

Correlations: $f, g \in C^\infty(\mathcal{U})$

$$\rho_{f,g}(t) = \int_{\mathcal{U}} (f \circ \varphi^{-t}) g \, dx dv$$



Examples

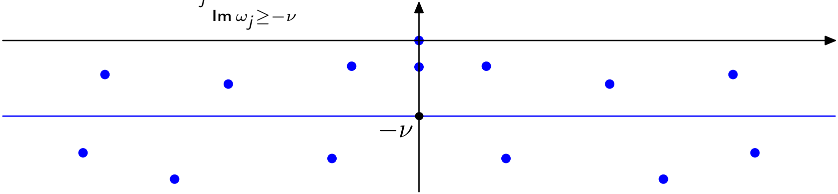
- Billiard ball flow on $\mathcal{U} = \{(y, v) \mid y \in M, \ |v| = 1\}$, $M \subset \mathbb{R}^2$
- Geodesic flow on $\mathcal{U} = \{(y, v) \mid y \in M, \ |v|_g = 1\}$, (M, g) a negatively curved Riemannian manifold

Pollicott–Ruelle resonances

$$\rho_{f,g}(t) = \int_{\mathcal{U}} (f \circ \varphi^{-t}) g \, dxdv$$

Pollicott–Ruelle resonances would appear in resonance expansions of $\rho_{f,g}$ for smooth hyperbolic systems and are independent of f, g:

$$\rho_{f,g}(t) = \sum_{\substack{\omega_j \text{ PR resonance} \\ \operatorname{Im}\omega_j \geq -\nu}} e^{-it\omega_j} c_j(f,g) + \mathcal{O}(e^{-\nu t}), \quad t \to +\infty$$



They are defined as poles of meromorphic continuations of

$$\hat{\rho}_{f,g}(\omega) = \int_0^\infty e^{it\omega} \rho_{f,g}(t) \, dt$$

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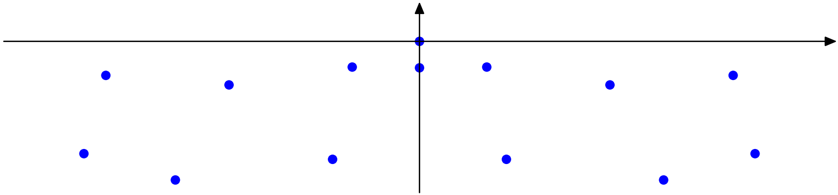
Closed system: $\rho_{f,g}(t) = c\Big(\int_{\mathcal{U}} f \, dxdv \Big) \Big(\int_{\mathcal{U}} g \, dxdv \Big) + o(1)$



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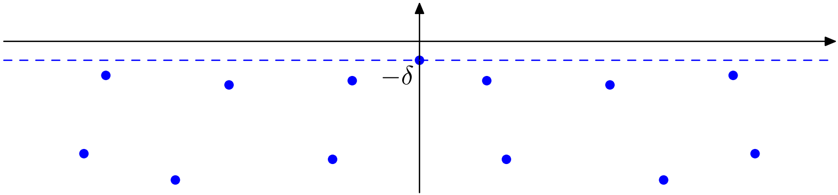
Open system: $\rho_{f,g}(t) = e^{-\delta t} \Big(\int_{\mathcal{U}} f \, d\mu_- \Big) \Big(\int_{\mathcal{U}} g \, d\mu_+ \Big) + o(e^{-\delta t})$



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Ruelle '76,'86,'87, Pollicott '85,'86, Parry–Pollicott '90, Rugh '92, Fried '95, Kitaev '99, Blank–Keller–Liverani '02, Liverani '04,'05, Gouëzel–Liverani '06, Baladi–Tsujii '07, Butterley–Liverani '07, Faure–Roy–Sjöstrand '08, Faure–Sjöstrand '11, D–Guillarmou '14

Climate models: Chekroun–Neelin–Kondrashov–McWilliams–Ghil '14

Inverse problems: Guillarmou '14

Ruelle zeta function

$$\zeta_R(\omega) = \prod_\gamma (1 - e^{i\omega T_\gamma}), \quad \operatorname{Im}\omega \gg 1$$

where T_γ are periods of primitive closed trajectories γ

Theorem [Giulietti–Liverani–Pollicott '12, D–Zworski '13, D–Guillarmou '14]

For a hyperbolic dynamical system (open or closed)*, the Ruelle zeta function continues meromorphically to $\omega \in \mathbb{C}$.

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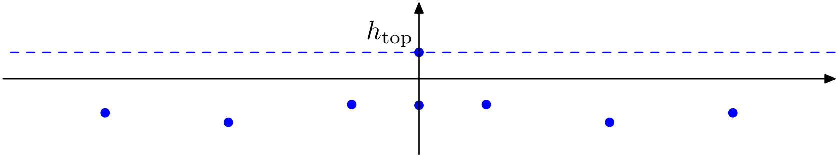
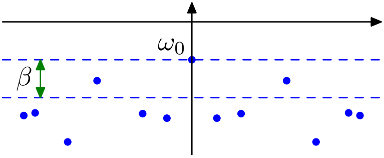
Prime orbit theorem (POT): $\#\{\gamma \mid T_\gamma \leq T\} = \frac{e^{h_{top}T}}{h_{top}T}(1 + o(1))$

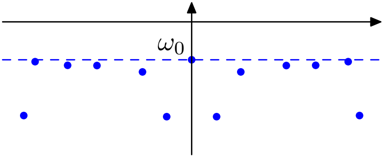


Margulis, Parry–Pollicott '90

Spectral gaps

Essential spectral gap of size $\beta > 0$:
there are finitely many resonances in $\{\operatorname{Im}\omega \geq \operatorname{Im}\omega_0 - \beta\}$,
where ω_0 is the top resonance





Spectral gap* $\Longrightarrow$ resonance expansion:

$$\rho_{f,g}(t) = \sum_{\substack{\omega_j \text{ PR resonance} \\ \operatorname{Im}\omega_j \geq -\nu}} e^{-it\omega_j} c_j(f,g) + \mathcal{O}(e^{-\nu t}), \quad \nu := \operatorname{Im}\omega_0 - \beta$$

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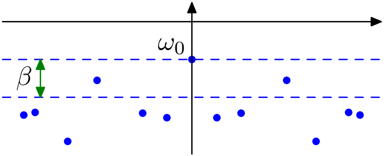
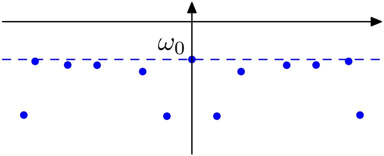


gap



no gap

Spectral gap for ζ_R $\implies$ exponential remainder in POT:

$$\#\{\gamma \mid T_\gamma \leq T\} = \frac{e^{h_{\mathrm{top}} T}}{h_{\mathrm{top}} T}(1 + \mathcal{O}(e^{-\tilde{\beta} T}))$$

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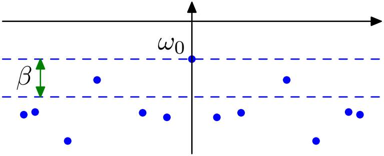

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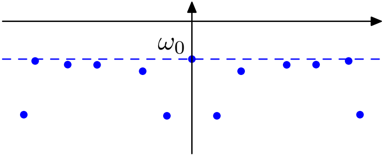

no gap

Gaps known for geodesic flows on compact negatively curved manifolds: Dolgopyat '98, Liverani '04, Tsujii '12, Giulietti–Liverani–Pollicott '12, Nonnenmacher–Zworski '13, Faure–Tsujii '13

and some special noncompact cases: Naud '05, Petkov–Stoyanov '10, Stoyanov '11,'13

We now switch to a different case of quantum resonances, featured in expansions of solutions to wave equations rather than classical correlations

Examples

- Potential scattering (Schrödinger operators)
- Obstacle scattering
- Black hole ringdown

Questions

- Can resonances be defined?
- Is there a spectral gap?
- How fast does the number of resonances grow?

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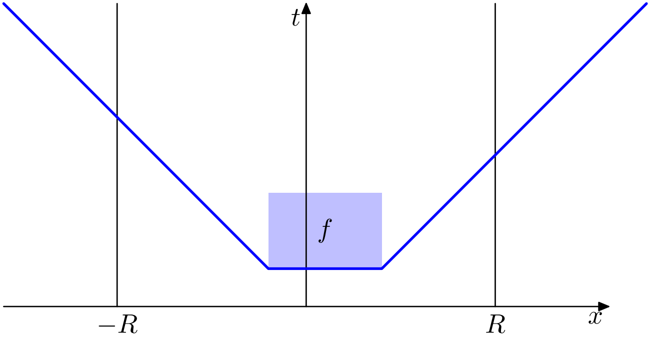
Example: scattering on the line

Wave equation: $\begin{cases} (\partial_t^2 - \partial_x^2)u &= f \in C_0^\infty((0,\infty)_t \times \mathbb{R}_x) \\ u|_{t<0} &= 0 \end{cases}$

Question: how does $u(t,x)$ behave for $t \to \infty$ and $|x| \le R$?



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Fourier–Laplace transform in time:

$$\hat{u}(\omega)(x) := \int_0^\infty e^{it\omega} u(t,x)\,dt \;\in\; L^2(\mathbb{R}), \quad \operatorname{Im}\omega > 0$$

$$(-\partial_x^2 - \omega^2)\hat{u}(\omega) = \hat{f}(\omega), \quad \operatorname{Im}\omega > 0$$

Resolvent: $\hat{u}(\omega) = R(\omega)\hat{f}(\omega)$, where

$$R(\omega) := (-\partial_x^2 - \omega^2)^{-1} : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad \operatorname{Im}\omega > 0$$

Fourier inversion formula:

$$u(t) = \frac{1}{2\pi} \int_{\operatorname{Im}\omega = 1} e^{-it\omega} R(\omega)\hat{f}(\omega)\,d\omega$$

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$$u(t) = \frac{1}{2\pi} \int_{\operatorname{Im}\omega = 1} e^{-it\omega} R(\omega) \hat{f}(\omega)\, d\omega$$

Meromorphically continue $R(\omega) : L^2_{\text{comp}}(\mathbb{R}) \to L^2_{\text{loc}}(\mathbb{R})$

$$R(\omega) g(x) = \frac{i}{2\omega} \int_{\mathbb{R}} e^{i\omega|x-y|} g(y)\, dy, \quad \omega \in \mathbb{C}$$

and deform the contour, with the integral being $\mathcal{O}(e^{-\nu t})$ in $L^2(-R, R)$:

$$u(t) = c_f + \frac{1}{2\pi} \int_{\operatorname{Im}\omega = -\nu} e^{-it\omega} R(\omega) \hat{f}(\omega)\, d\omega$$

Potential scattering on the line

Introduce a potential $V \in L^\infty(\mathbb{R})$

$$R(\omega) = (-\partial_x^2 + V - \omega^2)^{-1} : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad \operatorname{Im}\omega > 0$$

continues meromorphically to a family of operators

$$R(\omega) : L^2_{\mathrm{comp}}(\mathbb{R}) \to L^2_{\mathrm{loc}}(\mathbb{R}), \quad \omega \in \mathbb{C}$$

The poles of $R(\omega)$, called resonances, are featured in resonance expansions for the wave equation $(\partial_t^2 - \partial_x^2 + V)u = f$, and sound like this:

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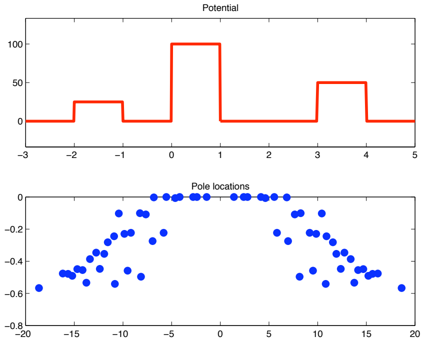


Computed using codes by David Bindel

Obstacle scattering

$\Delta_{\mathcal{E}}$: the Laplacian on $\mathcal{E} = \mathbb{R}^3 \setminus \mathscr{O}$ with Dirichlet boundary conditions, where $\mathscr{O} \subset \mathbb{R}^3$ is an obstacle

$$R(\omega) = (-\Delta_{\mathcal{E}} - \omega^2)^{-1} : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3), \quad \operatorname{Im} \omega > 0$$

continues meromorphically to a family of operators

$$R(\omega) : L^2_{\mathrm{comp}}(\mathbb{R}^3) \to L^2_{\mathrm{loc}}(\mathbb{R}^3), \quad \omega \in \mathbb{C}$$

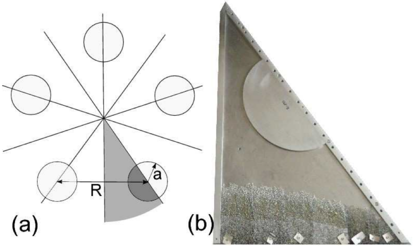
and the poles of $R(\omega)$ are called resonances

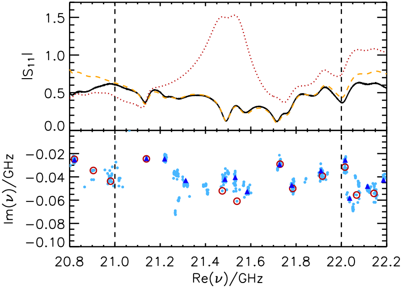
A rich mathematical theory dating back to Lax–Phillips '69, Vainberg '73, Melrose, Sjöstrand

D–Zworski, *Mathematical theory of scattering resonances*, available online

A real experimental example

Microwave experiments:





Potzuweit–Weich–Barkhofen–Kuhl–Stöckmann–Zworski ’12

Essential spectral gap for obstacles

Essential spectral gap: $R(\omega)$ has finitely many poles in $\{\operatorname{Im}\omega > -\beta\}$

Implies* exponential decay of local energy of waves modulo a finite dimensional space

Is there a gap? Depends on the structure of trapped billiard ball trajectories

Essential spectral gap for obstacles

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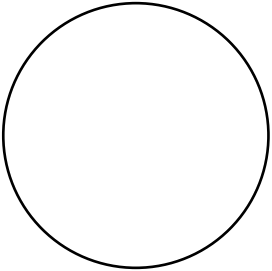
One convex obstacle:



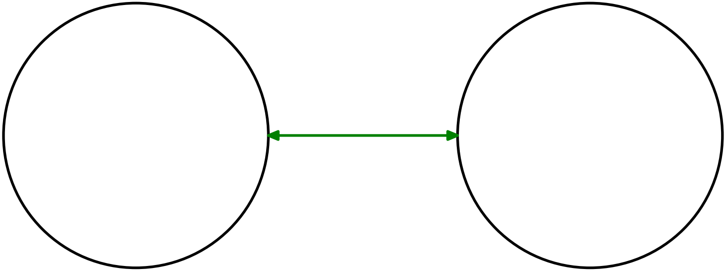
No trapping $\implies$ gap of any size

Lax–Phillips '69, Morawetz–Ralston–Strauss '77, Vainberg '89, Melrose–Sjöstrand '82, Sjöstrand–Zworski '91...

Essential spectral gap for obstacles

Is there a gap? Depends on the structure of trapped billiard ball trajectories

Two convex obstacles:



One trapped trajectory $\implies$ a lattice of resonances and gap of fixed size

Ikawa '82, Gérard–Sjöstrand '87, Christianson '06

Related case of black holes: Wunsch–Zworski '10, Nonnenmacher–Zworski '13, Dyatlov '13,'14

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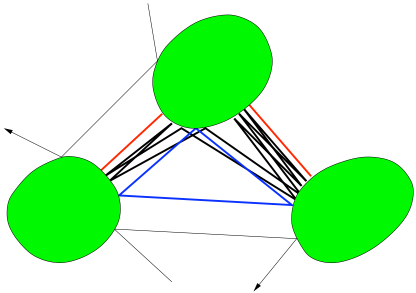
Three convex obstacles:

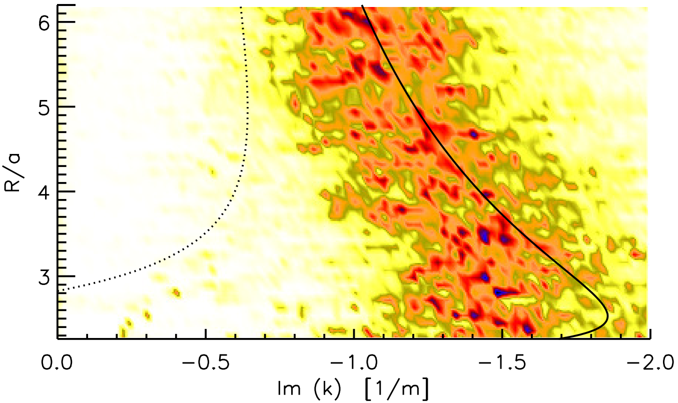


Fractal set of trapped trajectories $\implies$ gap under a pressure condition

Ikawa '88, Gaspard–Rice '89, Naud '04, Nonnenmacher–Zworski '09, Petkov–Stoyanov '10. . .

Experimental observation of the gap

Three-disk system:



Barkhofen–Weich–Potzuweit–Stöckmann–Kuhl–Zworski '13

Fractal Weyl laws

Weyl law for $-\Delta u_j = \lambda_j^2 u_j$ on a compact manifold M of dimension n:

$$\#\{\lambda_j \leq R\} = c_n \operatorname{Vol}(M) R^n (1 + o(1)), \quad R \to \infty$$

On a noncompact manifold with a hyperbolic trapped set, for each $\nu > 0$

$$\#\{\omega_j \in \mathrm{Res} : |\operatorname{Re} \omega_j| \leq R,\ \operatorname{Im} \omega_j \geq -\nu\} \leq C R^{1+\delta},$$

where $2\delta + 2$ is the upper Minkowski dimension of the trapped set

Melrose '83, Sjöstrand '90, Zworski '99, Wunsch–Zworski '00, Guillopé–Lin–Zworski '04, Sjöstrand–Zworski '07, Nonnenmacher–Sjöstrand–Zworski '11, Datchev–Dyatlov '12, Datchev–D–Zworski '12

Weyl laws and band structure for some cases with smooth trapped sets:

- Black holes (Kerr–de Sitter): Dyatlov '13
- Closed hyperbolic systems (contact Anosov): Faure–Tsujii '11,'13

Thank you for your attention!