Resonances in chaotic scattering

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Overview

Resonances: complex characteristic frequencies describing exponential decay of waves in open systems

$\operatorname{Re}\lambda_j$ = rate of oscillation, $-\operatorname{Im}\lambda_j$ = rate of decay

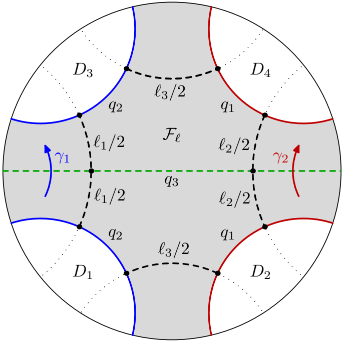
Our setting: convex co-compact hyperbolic surfaces

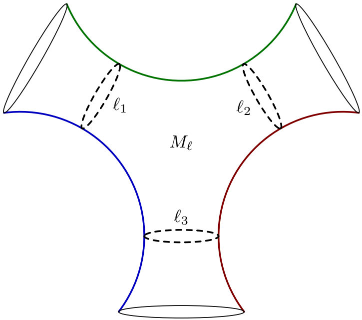
The high frequency régime $|\operatorname{Im}\lambda| \leq C$, $|\operatorname{Re}\lambda| \gg 1$ is governed by the set of trapped trajectories, which in our case is determined by the limit set Λ_Γ

We give a new spectral gap and fractal Weyl bound for resonances using a “fractal uncertainty principle”

Hyperbolic surfaces

$(M, g) = \Gamma\backslash\mathbb{H}^2$ convex co-compact hyperbolic surface





An example: three-funnel surface with neck lengths ℓ_1, ℓ_2, ℓ_3

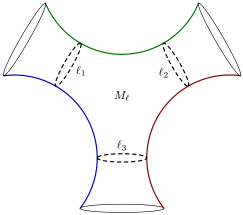
Resonances of hyperbolic surfaces

(M, g) convex co-compact hyperbolic surface

Δ_g Laplace–Beltrami operator on $L^2(M)$

The L^2 spectrum of $-\Delta_g$ consists of

- eigenvalues in $(0, \frac{1}{4})$
- continuous spectrum $[\frac{1}{4}, \infty)$



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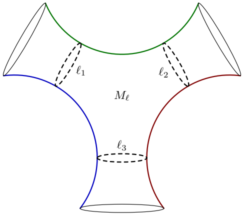
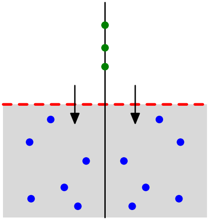


Resonances are poles of the meromorphic continuation

$$R(\lambda) = \left(-\Delta_g - \lambda^2 - \frac{1}{4}\right)^{-1} : \begin{cases} L^2 \to H^2, & \operatorname{Im}\lambda > 0 \\ L^2_{\mathrm{comp}} \to H^2_{\mathrm{loc}}, & \operatorname{Im}\lambda \leq 0 \end{cases}$$



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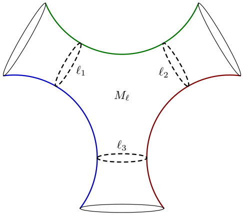

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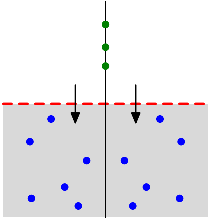
Existence of meromorphic continuation:
Patterson '75,'76, Perry '87,'89, Mazzeo–Melrose '87, Guillopé–Zworski '95, Guillarmou '05, Vasy '13



Resonances can be defined in many other situations, such as Euclidean scattering or black hole scattering

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- As poles of the resolvent, resonances participate in resonance expansions of waves (under additional assumptions)

$$\chi e^{-it\sqrt{-\Delta_g - 1/4}}\chi f = \sum_{\substack{\lambda_j \, resonance \\ \operatorname{Im}\lambda_j \ge -\nu}} e^{-it\lambda_j} u_j(x) + \mathcal{O}(e^{-\nu t})$$

- As poles of the Selberg zeta function, resonances play a role in counting closed geodesics on M
- As poles of the scattering operator, resonances can be computed from experimental data, see for instance Potzuweit–Weich–Barkhofen–Kuhl–Stöckmann–Zworski '12, Barkhofen–Weich–Potzuweit–Stöckmann–Kuhl–Zworski '13

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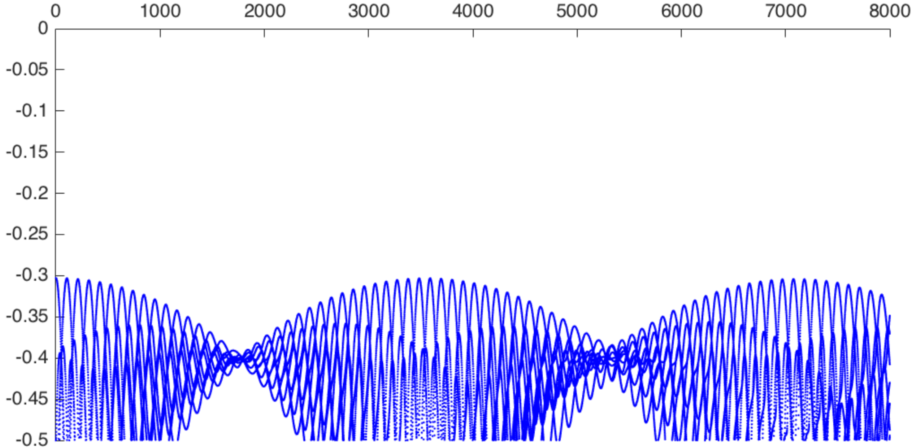
Plots of resonances

Three-funnel surface with $\ell_1 = \ell_2 = \ell_3 = 7$

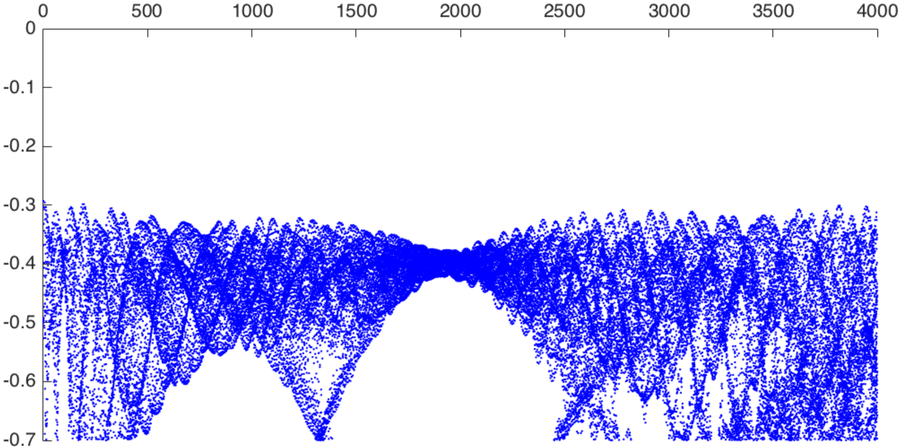


Data courtesy of David Borthwick and Tobias Weich

See arXiv:1305.4850 and arXiv:1407.6134 for more

Plots of resonances

Three-funnel surface with $\ell_1 = 6$, $\ell_2 = \ell_3 = 7$



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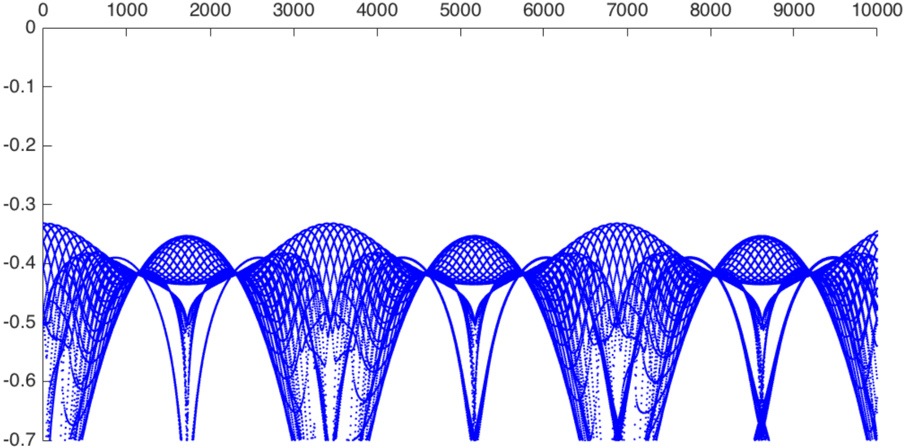
Plots of resonances

Torus-funnel surface with $\ell_1 = \ell_2 = 7$, $\varphi = \pi/2$, trivial representation



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High frequency asymptotics and geometric optics

- We will study resonances in the high frequency limit

$$\operatorname{Re}\lambda_j \to \infty, \quad |\operatorname{Im}\lambda_j| \leq C$$

They correspond to waves with bounded rate of exponential decay

- At high frequency, waves approximately travel along geodesics of M. We use microlocal analysis, the mathematical theory behind geometric optics, as well as classical/quantum correspondence
- Long living waves have to localize on geodesics which do not escape
- In our case, the flow is hyperbolic and the trapped set is fractal. Need to understand the interplay between
  - dispersion of waves living on individual geodesics
  - interferences between different geodesics

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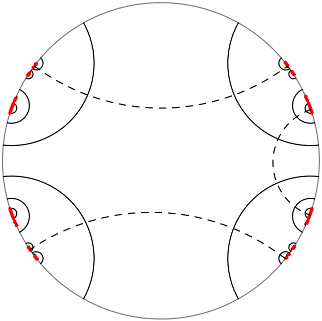
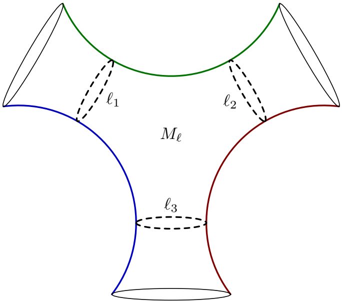
The limit set and δ

$M = \Gamma\backslash\mathbb{H}^2$ hyperbolic surface

$\Lambda_\Gamma \subset \mathbb{S}^1 = \partial\overline{\mathbb{H}^2}$ the limit set

$\delta := \dim_H(\Lambda_\Gamma) \in [0, 1]$





Trapped geodesics: both endpoints in Λ_Γ

Forward/backward trapped: one endpoint in Λ_Γ

Essential spectral gap

Essential spectral gap of size $\beta > 0$:

only finitely many resonances with $\operatorname{Im}\lambda > -\beta$

One application: resonance expansions of waves with $\mathcal{O}(e^{-\beta t})$ remainder

Patterson–Sullivan: the topmost resonance is $\lambda = i(\delta - \frac{1}{2})$, therefore there is a gap of size $\beta = \max\left(0, \frac{1}{2} - \delta\right)$

See also Ikawa '88, Gaspard–Rice '89, Nonnenmacher–Zworski '09

$\delta > \frac{1}{2}$ $\delta < \frac{1}{2}$

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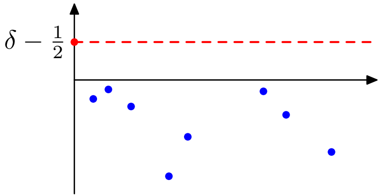

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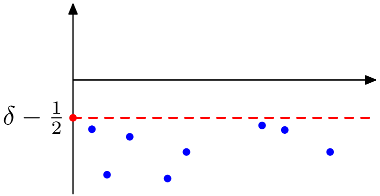

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Essential spectral gap: finitely many resonances with $\operatorname{Im}\lambda > -\beta$
Standard gap: $\beta_{\mathrm{std}} = \max(0, \frac{1}{2} - \delta)$

Naud '04, Stoyanov '11 (inspired by Dolgopyat '98):
gap of of size $\frac{1}{2} - \delta + \varepsilon$ for $0 < \delta \leq \frac{1}{2}$ and $\varepsilon > 0$ depending on M

Theorem 1 [D–Zahl '15]

There is a gap of size

$$\beta = \frac{3}{8}\Big(\frac{1}{2} - \delta\Big) + \frac{\beta_E}{16}$$

where $\beta_E \in [0, \delta]$ is the improvement in the asymptotic of additive energy of the limit set Λ_Γ. Furthermore

$$\beta_E > \delta \exp\big[-K(1-\delta)^{-28}\log^{14}(1+C)\big] > 0$$

where C is the δ-regularity constant of Λ_Γ and K a global constant.

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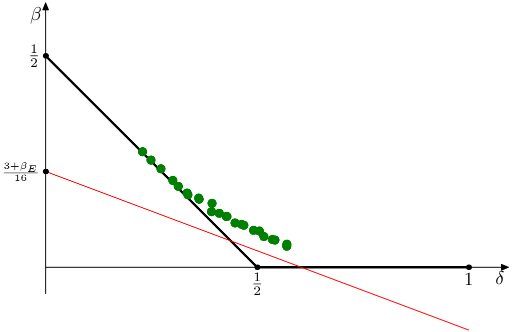
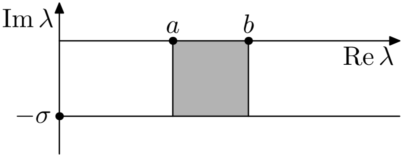


Numerics for 3- and 4-funneled surfaces by Borthwick–Weich '14
+ our gap for $\beta_E := \delta$ (representing some wishful thinking)

Counting resonances

Denote by $N_{[a,b]}(\sigma)$ the number of resonances with

$$\operatorname{Re}\lambda \in [a,b], \quad \operatorname{Im}\lambda \geq -\sigma$$



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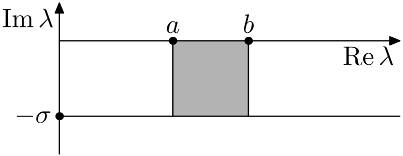

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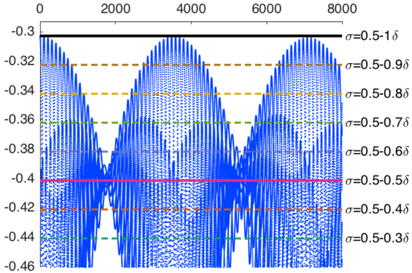

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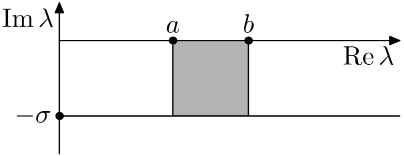

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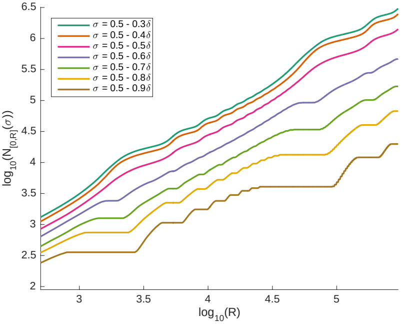

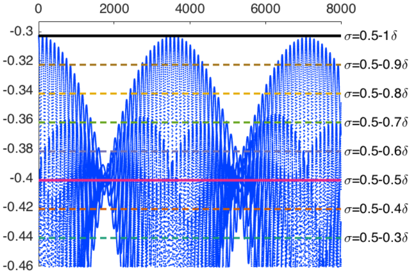

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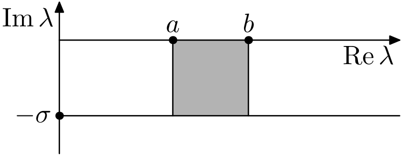

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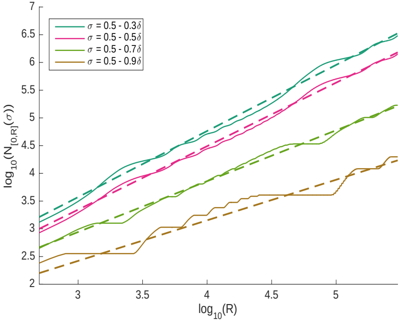

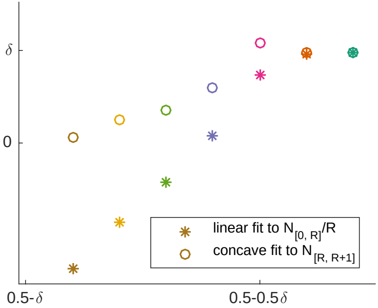

Fractal Weyl bounds

$N_{[a,b]}(\sigma) = \#\{\text{resonances with } \operatorname{Re}\lambda \in [a,b],\ \operatorname{Im}\lambda > -\sigma\}$

Theorem 2 [D '15]

For σ fixed and $R \to \infty$, $N_{[R,R+1]}(\sigma) = \mathcal{O}(R^{m(\sigma,\delta)+})$, where

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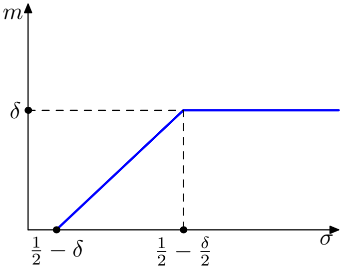
Note that $m = 0$ at $\sigma = \frac{1}{2} - \delta$ and $m = \delta$ starting from $\sigma = \frac{1}{2} - \frac{\delta}{2}$



Zworski '99, Guillopé–Lin–Zworski '04, Datchev–D '13: $N_{[R,R+1]}(\sigma) = \mathcal{O}(R^{\delta})$

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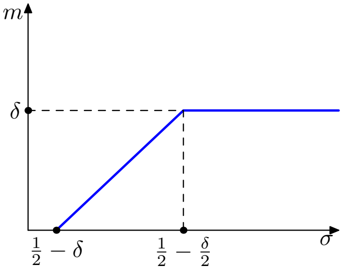

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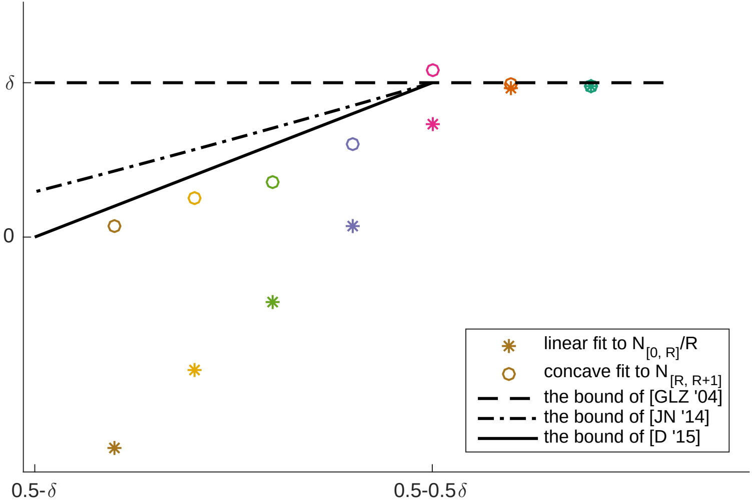
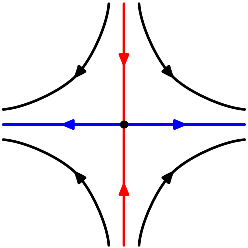
Fractal Weyl bounds in pictures



A comparison of numeric fits with the bounds of Guillopé–Lin–Zworski '04, Jakobson–Naud '14, and D '15

Dynamics of the geodesic flow



$M = \Gamma\backslash\mathbb{H}^2$ convex co-compact hyperbolic surface

The homogeneous geodesic flow

$$\varphi^t : T^*M \setminus 0 \to T^*M \setminus 0$$

is hyperbolic with weak (un)stable foliations L_u/L_s

Incoming/outgoing tails:

$$\Gamma_+ = \{(x,\xi) \mid \varphi^t(x,\xi) \not\to \infty \text{ as } t \to -\infty\}$$
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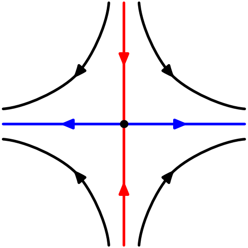

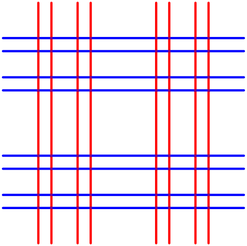

Microlocalization of resonant states

Assume $\lambda = h^{-1} - i\nu$ is a resonance, $0 < h \ll 1$. There is a resonant state

$$\Big(-\Delta_g - \frac{1}{4} - \lambda^2\Big)u = 0, \quad u \text{ outgoing at infinity}, \quad \|u\| = 1$$

Microlocally, u lives near Γ_+, has positive mass on Γ_-, and

$$u = e^{i\lambda t}U(t)u; \quad U(t) = e^{-it\sqrt{-\Delta_g - 1/4}} \text{ quantizes } \varphi^t$$

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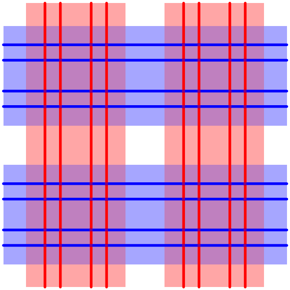
Outgoing condition implies:

$$u = \mathrm{Op}_h(\chi_+) u + \mathcal{O}(h^\infty),$$
$$\|\mathrm{Op}_h(\chi_-) u\| \geq C^{-1}$$
$$\operatorname{supp} \chi_\pm \subset \ \varepsilon\text{-neighborhood of } \Gamma_\pm$$



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Propagation for time t:

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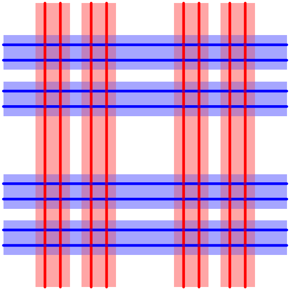

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Propagation for time $t = \log(1/h)$:

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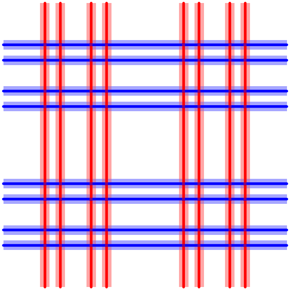
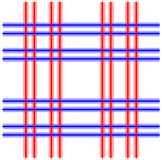
Use second microlocal calculi associated to L_u / L_s

In practice, we take $t = \rho \log(1/h)$, $\rho = 1 - \varepsilon$





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Proof of Theorem 1 (gaps)

- To get a gap of size β, enough to show a fractal uncertainty principle:
$$\|\mathrm{Op}_h^{L_s}(\chi_-)\mathrm{Op}_h^{L_u}(\chi_+)\|_{L^2\to L^2} \ll h^\beta$$
- A basic bound gives the standard gap $\beta = \frac{n-1}{2} - \delta$:
$$\|\mathrm{Op}_h^{L_s}(\chi_-)\mathrm{Op}_h^{L_u}(\chi_+)\|_{\mathrm{HS}} \leq Ch^{\frac{n-1}{2}-\delta} \qquad (1)$$
- The bound via additive energy is obtained by harmonic analysis in L^4

Proof of Theorem 2 (counting)

- First write for each resonant state, $u = \mathcal{A}(\lambda)u$,
$\mathcal{A}(\lambda) = Y(\lambda)\mathrm{Op}_h^{L_s}(\chi_-)\mathrm{Op}_h^{L_u}(\chi_+) + \mathcal{O}(h^\infty)$, $\|Y(\lambda)\| \leq Ch^{-\nu}$
- Next estimate $\det(I - \mathcal{A}(\lambda)^2) \leq \exp(\|\mathcal{A}(\lambda)\|_{\mathrm{HS}}^2)$ using (1)

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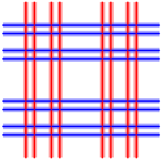

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Thank you for your attention!