Resonances in chaotic dynamics: beyond the pressure gap

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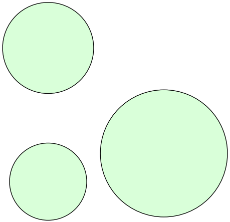
July 29, 2015

Resonances in obstacle scattering

$\mathcal{E} = \mathbb{R}^3 \setminus \mathcal{O}$ exterior of several convex obstacles



$$u_{tt} - \Delta_x u = 0$$
$$u|_{t=0} = \chi f_0, \quad u_t|_{t=0} = \chi f_1$$
$$u|_{x\in\partial\mathcal{E}} = 0, \quad \chi \in C_0^\infty(\mathbb{R}^3)$$

Does $\|\chi u(t)\|_{L^2}$ decay exponentially as $t \to \infty$?

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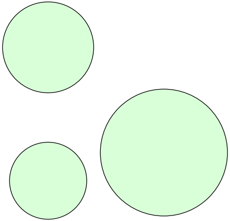

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Resonances: poles of the meromorphic continuation

$$R(\omega) = (-\Delta_{\mathcal{E}} - \omega^2)^{-1} : \begin{cases} L^2(\mathcal{E}) \to L^2(\mathcal{E}), & \operatorname{Im}\omega > 0 \\ L^2_{\mathrm{comp}}(\mathcal{E}) \to L^2_{\mathrm{loc}}(\mathcal{E}), & \operatorname{Im}\omega \le 0 \end{cases}$$

Essential spectral gap of size $\beta > 0$:

only finitely many resonances with $\operatorname{Im}\omega > -\beta$

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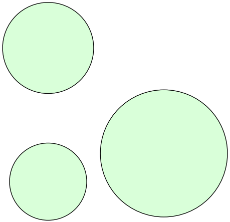

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$\Downarrow$

resonance expansion:

$$\chi u(t) = \sum_{\substack{\omega_j \text{ resonance} \\ \operatorname{Im}\omega_j \geq -\beta}} e^{-it\omega_j} u_j + \mathcal{O}(e^{-\beta t}), \quad t \to +\infty$$

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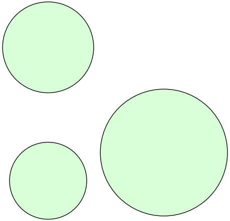

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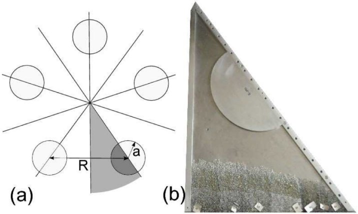
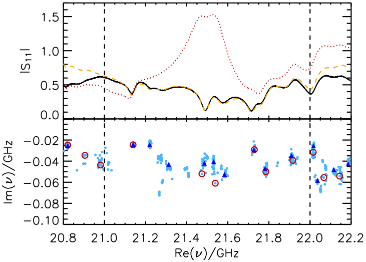
Resonances in microwave experiments:





Potzuweit–Weich–Barkhofen–Kuhl–Stöckmann–Zworski '12

The pressure gap

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Is there a gap? The answer depends on trapped trajectories

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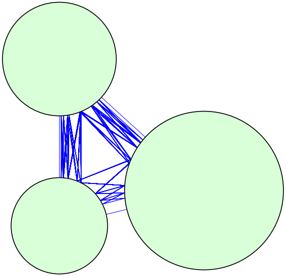
Topological pressure $P(\frac{1}{2}) \in \mathbb{R}$
measures the 'thickness' of the trapped set

Fewer trapped trajectories $\Longrightarrow$ smaller $P(\frac{1}{2})$



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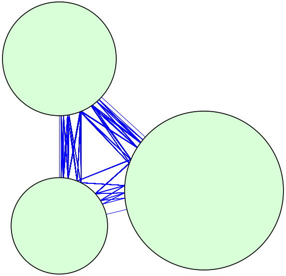
Pressure gap: $\beta = -P(\frac{1}{2})$ under the pressure condition

$$P(1/2) < 0$$



Ikawa '88, Gaspard–Rice '89, Nonnenmacher–Zworski '09...

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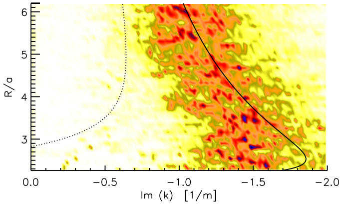

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Experiment: Barkhofen–Weich–Potzuweit–Stöckmann–Kuhl–Zworski '13

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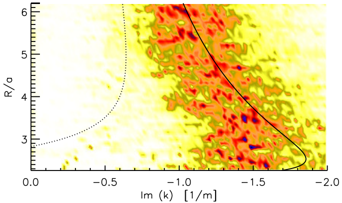

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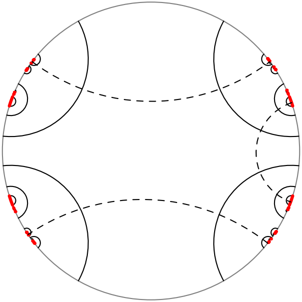
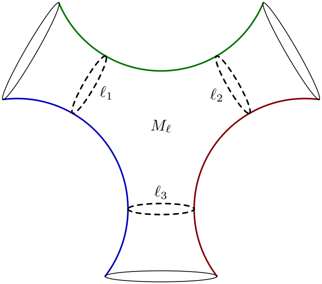
Experiment: Barkhofen–Weich–Potzuweit–Stöckmann–Kuhl–Zworski '13

There seems to be a bigger gap when $P(\frac{1}{2}) \approx 0\ldots$

Resonances for hyperbolic surfaces

We now switch to the mathematically cleaner case of convex co-compact hyperbolic surfaces $M = \Gamma\backslash\mathbb{H}^2$





On the cover $\mathbb{H}^2$, trapped trajectories have endpoints in the limit set

$$\Lambda_\Gamma \subset \mathbb{S}^1, \quad \dim_H(\Lambda_\Gamma) = \delta \in (0, 1)$$

Spectral gap for hyperbolic surfaces

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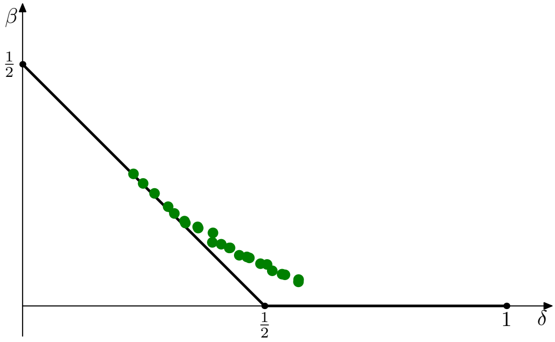
The pressure gap is $-P(\frac{1}{2}) = \frac{1}{2} - \delta$



Borthwick–Weich '14: numerics for symmetric 3- and 4-funneled surfaces

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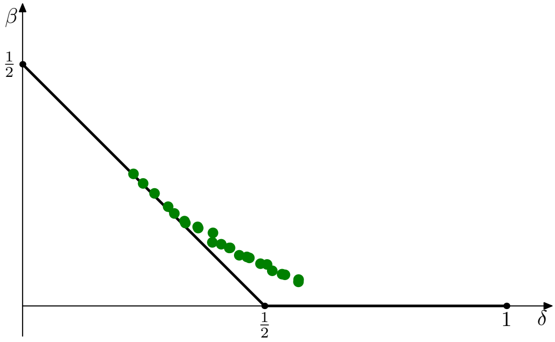

Borthwick–Weich '14: numerics for symmetric 3- and 4-funneled surfaces
Once again, the gap seems to be bigger than $\frac{1}{2} - \delta$ when $\delta \approx \frac{1}{2}$...

Improved gap for hyperbolic surfaces

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Dolgopyat '98, Naud '04, Stoyanov '11,'13, Petkov–Stoyanov '10: for $\delta \leq \frac{1}{2}$, gap of size $\frac{1}{2} - \delta + \varepsilon$, where $\varepsilon > 0$ depends on the surface

Theorem [D–Zahl '15]

There is an essential spectral gap of size

$$\beta = \frac{3}{8}\Big(\frac{1}{2} - \delta\Big) + \frac{\beta_E}{16}$$

where $\beta_E \in (0, \delta)$ is the improvement in the asymptotic of additive energy of the limit set. We have

$$\beta_E > \delta \exp\big[-K(1-\delta)^{-14}\log^{14}(1+C)\big]$$

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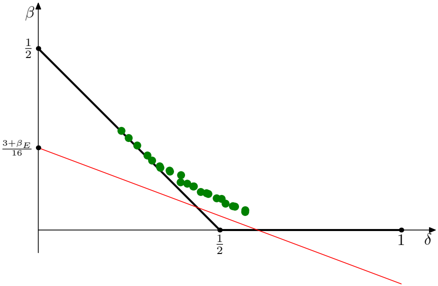

C depends continuously on the surface $\Longrightarrow$ examples of cases when the pressure condition fails (i.e. $\delta > \frac{1}{2}$), but there is a gap

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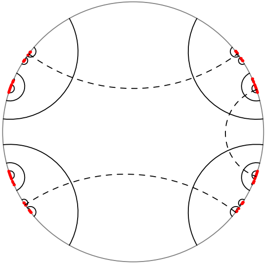

Additive energy: $X \subset \{-N, \dots, N\}$, $|X| \sim N^\delta$ obtained by discretizing the stereographically projected limit set Λ_Γ on the scale $1/N \to 0$

$$E_A(X) = \#\{(a,b,c,d) \in X^4 \mid a+b = c+d\} \leq CN^{3\delta - \beta_E}$$

Main idea: additive energy captures the fractal structure of Λ_Γ and gives a quantitative bound on cancellations in a fractal uncertainty principle associated to Λ_Γ

Thank you for your attention!