Chaos in dynamical systems

Semyon Dyatlov

IAP Mathematics Lecture Series
January 26, 2015

Can you see the difference?

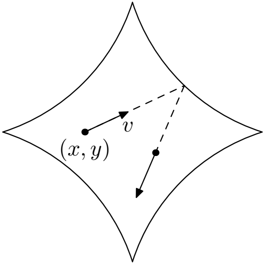
Can you see it now?

Can you see it now?

predictable chaotic

Billiards as dynamical systems

- $\Omega \subset \mathbb{R}^2$ is the billiard domain
- $\mathcal{X} = \{(x, y, v_x, v_y) \mid (x, y) \in \Omega, v_x^2 + v_y^2 = 1\}$ is the phase space
- $\Phi_t : \mathcal{X} \to \mathcal{X}$, $t \in \mathbb{R}$ is the billiard ball flow



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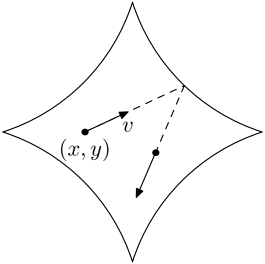

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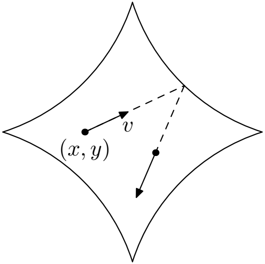


There is no unique proper way to define what chaos means. . .

But one way is using statistics of many trajectories for long times

Correlations

Consider a discrete time dynamical system with phase space $\mathcal{X}$

$$x_{j+1} = F(x_j), \quad F : \mathcal{X} \to \mathcal{X}; \quad x_j = F^{(j)}(x_0).$$

Measures

- A measure μ assigns a number $\mu(A) \geq 0$ to any measurable set $A \subset \mathcal{X}$
- If $\mathcal{X} = \mathbb{R}^n$, then a natural choice of $\mu(A)$ is the volume of A
- Assume that μ is invariant: $\mu(F^{-1}(A)) = \mu(A)$
- Assume also that μ is a probability measure: $\mu(\mathcal{X}) = 1$

For measurable $A, B \subset \mathcal{X}$, define the correlation

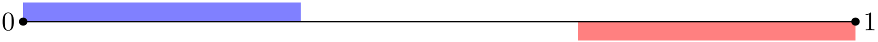
$$\rho_{A,B}(j) = \mu(F^{-(j)}(A) \cap B) = \mu\big(\{x \mid F^{(j)}(x) \in A,\ x \in B\}\big), \quad j \in \mathbb{N}_0.$$

Mixing

$$\rho_{A,B}(j) = \mu(F^{-(j)}(A) \cap B), \quad j \in \mathbb{N}_0.$$

Example: $\mathcal{X} = [0,1]$, $F(x) = (2x) \mod 1$, $x_{j+1} = (2x_j) \mod 1$

Take $A = [0, \frac{1}{3}], B = [\frac{2}{3}, 1]$; showing $F^{-(j)}(A)$ and B



$j = 0$, correlation $= 0.0000\ldots$

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$j = 1$, correlation $= 0.0000\ldots$

Mixing

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Example: $\mathcal{X} = [0,1]$, $F(x) = (2x) \mod 1$, $x_{j+1} = (2x_j) \mod 1$

Take $A = [0, \frac{1}{3}], B = [\frac{2}{3}, 1]$; showing $F^{-(j)}(A)$ and B

$j = 2$, correlation $= 0.0833\ldots$

Mixing

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Take $A = [0, \frac{1}{3}], B = [\frac{2}{3}, 1]$; showing $F^{-(j)}(A)$ and B

$j = 3$, correlation $= 0.0833\ldots$

Mixing

$$\rho_{A,B}(j) = \mu(F^{-(j)}(A) \cap B), \quad j \in \mathbb{N}_0.$$

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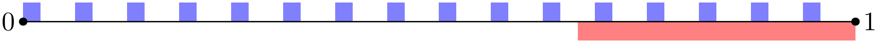

$j = 4$, correlation $= 0.1041\dots$

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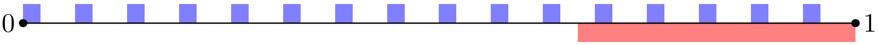


$j = 4$, correlation $= 0.1041\dots$

Definition

The dynamical system is mixing, if for all A, B

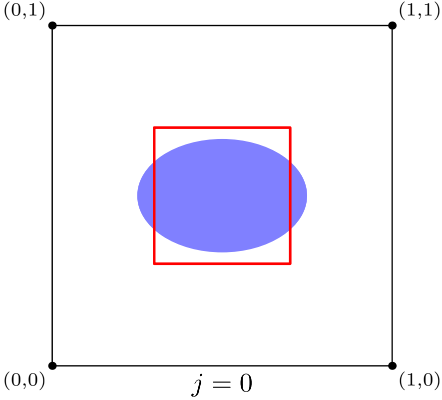
$$\rho_{A,B}(j) \to \mu(A)\mu(B) \quad \text{as } j \to \infty. \tag{1}$$

Our case: $\mu(A)\mu(B) = \frac{1}{9} = 0.1111\dots$

Another example of mixing: Arnold's cat map

$\mathcal{X} = \{(x,y) \mid x \in [0,1],\ y \in [0,1]\}$; showing $F^{-(j)}(A)$ and B

$$x_{j+1} = (2x_j + y_j) \mod 1,$$
$$y_{j+1} = (x_j + y_j) \mod 1.$$



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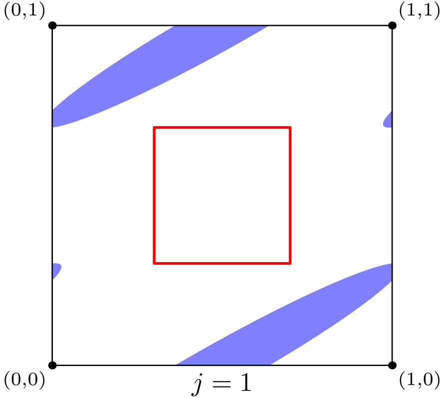

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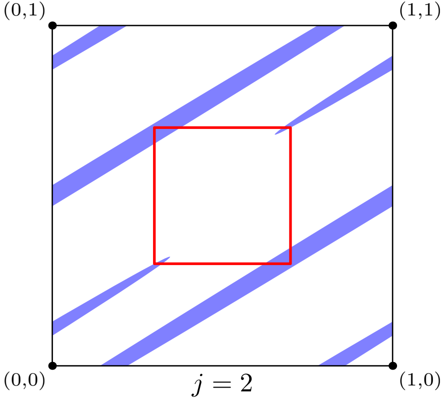

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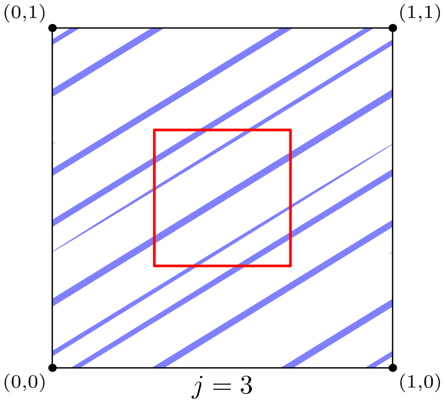

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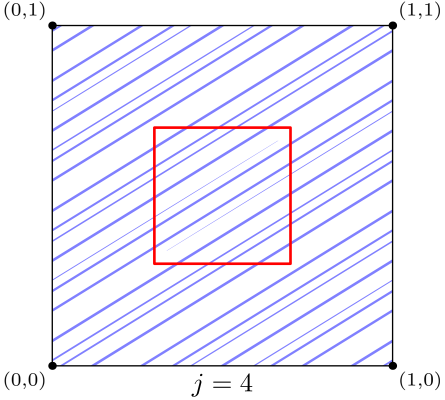

10000 billiard balls in a Sinai billiard

$$\#(\text{balls in the box}) \quad \to \quad \text{volume of the box}$$
$$\text{velocity angles distribution} \quad \to \quad \text{uniform measure}$$

Ergodicity

Let $x_{j+1} = F(x_j)$, $F : \mathcal{X} \to \mathcal{X}$ be a discrete time dynamical system and μ be an invariant measure on the phase space $\mathcal{X}$

Definition

- Let $f : \mathcal{X} \to \mathbb{R}$ be a integrable function. Define the ergodic average

$$\langle f \rangle_m(x) = \frac{1}{m} \sum_{j=0}^{m-1} f(F^{(j)}(x)), \quad x \in \mathcal{X},\ m \in \mathbb{N}$$

- A dynamical system $(\mathcal{X}, F, \mu)$ is ergodic if for all f and almost every x,

$$\langle f \rangle_m(x) \to \int_{\mathcal{X}} f \, d\mu \quad \text{as } m \to \infty \tag{2}$$

Here 'almost every x' means 'the set of all x where (2) does not hold has μ-measure zero' and is important because of special trajectories

Ergodicity and mixing

Birkhoff's Ergodic Theorem

Assume that for each $A \subset \mathcal{X}$ which is invariant (i.e. $F^{-1}(A) = A$) we have either $\mu(A) = 0$ or $\mu(A) = 1$. Then the dynamical system $(\mathcal{X}, F, \mu)$ is ergodic, i.e. $\langle f \rangle_m \to \int f \, d\mu$ for all f.

Mixing implies ergodicity: if $F^{-1}(A) = A$, then

$$\rho_{A,A}(j) = \mu(F^{-(j)}(A) \cap A) = \mu(A)$$

converges as $j \to \infty$ to $\mu(A)^2$. Therefore, $\mu(A)$ is either 0 or 1.

Ergodicity does **not** imply mixing: consider the irrational shift

$$\mathcal{X} = [0, 1], \quad x_{j+1} = (x_j + \alpha) \mod 1, \quad \alpha \in \mathbb{R} \setminus \mathbb{Q}.$$

It is not mixing, but it is ergodic with respect to the length measure μ

Ergodicity of the irrational shift

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$$\langle f \rangle_m(x) = \frac{1}{m} \sum_{j=0}^{m-1} f\big((x + \alpha j) \mod 1\big)$$

Case 1: $f \equiv 1$. Then $\langle f \rangle_m \equiv 1$ as well.

Case 2: $f(x) = \cos(2\pi kx)$, $k \in \mathbb{Z}$, $k > 0$. Then

$$\langle f \rangle_m(x) = \frac{1}{m} \sum_{j=0}^{m-1} \cos(k(x + 2\pi\alpha j))$$

$$= \frac{\sin(kx + (2m-1)\pi\alpha k) - \sin(kx - \pi\alpha k)}{2m \sin(\pi\alpha k)} \to 0 = \int_0^1 f(x)\,dx$$

where $\sin(\pi\alpha k) \neq 0$ because α is irrational

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We are proving that for almost every x,

$$\langle f\rangle_m(x) := \frac{1}{m}\sum_{j=0}^{m-1} f\big((x+\alpha j) \mod 1\big) \to \int f\,d\mu \tag{3}$$

Write the Fourier series:

$$f(x) = \sum_{k=0}^{\infty} a_k \cos(2\pi kx) + \sum_{k=1}^{\infty} b_k \sin(2\pi kx), \quad a_0 = \int f\,d\mu.$$

Assume first that the series converges absolutely, i.e. $\sum_k |a_k| + |b_k| < \infty$. Let f_N be the sum of the first N terms of (both) series. Then

$$\langle f\rangle_m(x) - \langle f_N\rangle_m(x) \ \leq\ \varepsilon_N := \sum_{k>N} |a_k| + |b_k| \ \to 0 \quad \text{as } N \to \infty$$

Cases 1–3 above show that for each x and N,

$$\langle f_N\rangle_m(x) \to a_0 \quad \text{as } m \to \infty$$

Together, these prove (3). The case of general f needs an additional argument using L^2 theory omitted here

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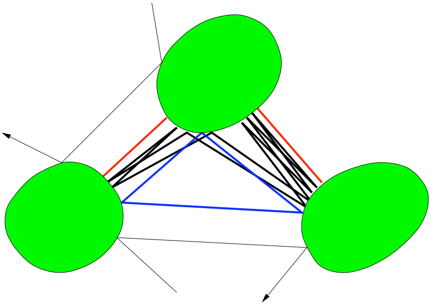
10000 billiard balls in a three-disk system

#(balls in the box)	$\rightarrow$	0 exponentially
velocity angles distribution	$\rightarrow$	some fractal measure

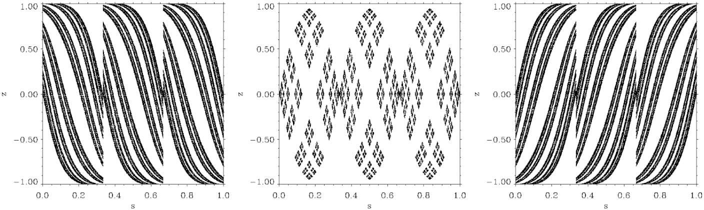
Open chaotic systems

The three-disk system is open meaning that we allow escape to infinity. The key objects are the

- incoming set $\Gamma_- \subset \mathcal{X}$, consisting of trajectories trapped as $t \to +\infty$;
- outgoing set $\Gamma_+ \subset \mathcal{X}$, consisting of trajectories trapped as $t \to -\infty$;
- trapped set $K := \Gamma_- \cap \Gamma_+$



Open chaotic systems

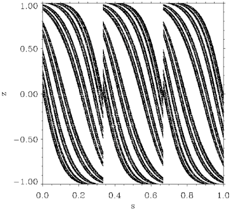


incoming set | trapped set | outgoing set

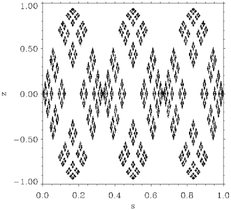
Poon–Campos–Ott–Grebogi '96

The trapped set has a fractal structure. . .
and supports a fractal measure

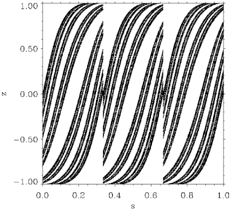
Open chaotic systems



incoming set



trapped set



outgoing set

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The trapped set has a fractal structure...
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A mixed system: the Nosé–Hoover oscillator

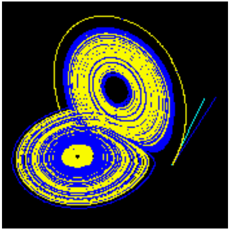
This system on $\mathcal{X} = \mathbb{R}^3$ has both chaotic and predictable behavior:

$$\dot{x} = y, \quad \dot{y} = yz - x, \quad \dot{z} = 1 - y^2.$$

Here dots represent the time derivatives of $x = x(t)$, $y = y(t)$, $z = z(t)$

An invariant measure is $\mu = e^{-\frac{1}{2}(x^2+y^2+z^2)}\, dxdydz$

This oscillator follows the line of research started with the Lorenz system/butterfly effect (1969, MIT; pictures courtesy of Wikipedia)



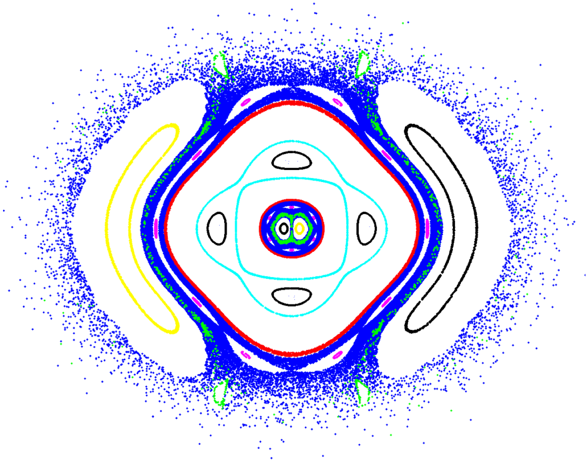
A mixed system: the Nosé–Hoover oscillator

A regular trajectory

A mixed system: the Nosé–Hoover oscillator

A chaotic trajectory

A mixed system: the Nosé–Hoover oscillator



The Poincaré section $\{z = 0\}$. Each color represents a different trajectory

Entropy: a measure of chaos

For a dynamical system $x_{j+1} = F(x_j)$, $x \in \mathcal{X}$, the topological entropy $h_{\rm top} \geq 0$ is a measure of the complexity of the system

In the case of hyperbolic systems, the number of primitive closed orbits of period at most T (i.e. the sets $\gamma = \{F^{(j)}(x) \mid j \in \mathbb{N}_0\}$ where $F^{(m)}(x) = x$ for some $m \in [1, T]$; we denote minimal such m by T_γ) grows like

$$N(T) = \frac{e^{h_{\rm top} T}}{h_{\rm top} T}(1 + o(1)), \quad T \to \infty$$

Can be proved using the dynamical zeta function (whose first pole is $h_{\rm top}$)

$$\zeta_R(s) = \prod_{\gamma \text{ primitive closed orbit}} (1 - e^{-sT_\gamma})^{-1}, \quad \operatorname{Re} s \gg 1,$$

which should be compared to the Riemann zeta function

$$\zeta(s) = \prod_{p \text{ prime}} (1 - e^{-s \log p})^{-1}, \quad \operatorname{Re} s > 1$$

An area of active research (including by yours truly)...

Further reading

- A.Katok and B.Hasselblatt, Introduction to the modern theory of dynamical systems, Cambridge University Press, 1995
- M.Hirsch, S.Smale, and R.Devaney, Differential equations, dynamical systems, and an introduction to chaos, Academic Press, 2013
- D.Ruelle, Chance and Chaos, Princeton University Press, 1991
- M.Gutzwiller, Chaos in classical and quantum mechanics, Springer, 1990

Exercises

1. Show that the map $x \mapsto (2x) \mod 1$ on $\mathcal{X} = [0,1]$ is mixing when the sets A, B in (1) on slide 6 are finite unions of intervals
2. Show that the map $x \mapsto (x + \alpha) \mod 1$ on $\mathcal{X} = [0,1]$ is not ergodic for rational α and not mixing for any α
3. Work out Case 3 on slide 11. (Bonus: simplify the treatment of Cases 2 and 3 using complex numbers.)
4. Find all the primitive closed orbits of the map $x \mapsto (2x) \mod 1$ on $[0,1]$

Thank you for your attention!