Pollicott–Ruelle resonances for hyperbolic manifolds

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joint work with Frédéric Faure (Grenoble) and Colin Guillarmou (ENS)

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- Anosov flows: hyperbolic/chaotic smooth dynamical systems
- Pollicott–Ruelle resonances: complex characteristic frequencies of decay of correlations
- Our case: geodesic flow on a hyperbolic Riemannian manifold

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Anosov flows

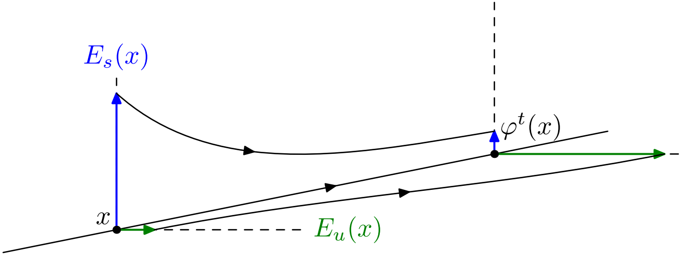
A flow $\varphi^t = e^{tX} : \mathcal{M} \to \mathcal{M}$ is Anosov if ($\theta > 0$)

$$T_x\mathcal{M} = \mathbb{R}X(x) \oplus E_s(x) \oplus E_u(x)$$

$$|d\varphi^t(x) \cdot v| \leq Ce^{-\theta|t|}, \quad t \geq 0, \ v \in E_s(x)$$

$$|d\varphi^t(x) \cdot v| \leq Ce^{-\theta|t|}, \quad t \leq 0, \ v \in E_u(x)$$

Fundamental example: $\mathcal{M} = SM$, $\varphi^t : \mathcal{M} \to \mathcal{M}$ geodesic flow, (M, g) has sectional curvature < 0



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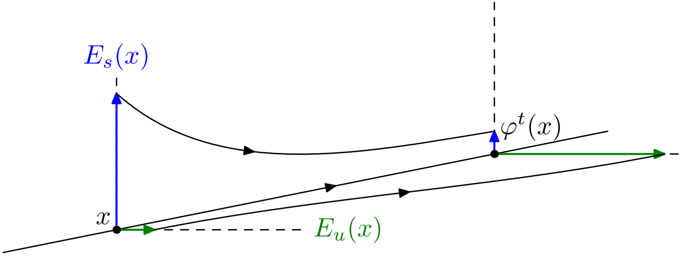

Correlations and resonances

Assume that φ^t is contact $\Longrightarrow$ exists a smooth invariant measure μ

$$f, g \in L^2(\mathcal{M}) \quad \Longrightarrow \quad \rho_{f,g}(t) = \int_{\mathcal{M}} [f \circ \varphi^{-t}] \bar{g} \, d\mu$$

Dolgopyat '98, Liverani '04, Tsujii '10, Nonnenmacher–Zworski '13:

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Resonances of φ^t: poles of the meromorphic extension to $\mathbb{C}$ of

$$\hat{\rho}_{f,g}(\lambda) = \int_0^\infty e^{-\lambda t} \rho_{f,g}(t) \, dt, \quad \operatorname{Re} \lambda > 0, \quad f, g \in C^\infty$$

Pollicott '86, Ruelle '86, Baladi '05, Liverani '05, Gouëzel–Liverani '06, Baladi–Tsujii '07, Butterley–Liverani '07, Faure–Sjöstrand '11

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$$f \circ \varphi^{-t} = e^{-tX} f, \quad X : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M}), \quad \varphi^t = e^{tX}$$

$$\hat{\rho}_{f,g}(\lambda) = \int_0^\infty \langle e^{-t(X+\lambda)} f, g \rangle_{L^2} = \langle (X+\lambda)^{-1} f, g \rangle_{L^2}$$

Resonances = eigenvalues of $-X$ on anisotropic Sobolev space

$$\mathcal{H}^r, \quad r \gg -\operatorname{Re}\lambda; \quad H^r(\mathcal{M}) \subset \mathcal{H}^r \subset H^{-r}(\mathcal{M})$$

Useful criterion: $\lambda \in \mathbb{C}$ a resonance $\iff$ the space

$$\operatorname{Res}_X(\lambda) = \{u \in \mathcal{D}'(\mathcal{M}) \mid (X+\lambda)u = 0,\ \operatorname{WF}(u) \subset E_u^*\}$$

of resonant states is nontrivial. Here $E_u^*(x) = (\mathbb{R}X(x) \oplus E_u(x))^\perp \subset T_x^*\mathcal{M}$.

The "outgoing" condition $\operatorname{WF}(u) \subset E_u^*$ is invariant under differential operators

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Hyperbolic manifolds

Assumptions

- $\mathcal{M} = SM = \{(x, \xi) \mid x \in M,\ \xi \in T_x M,\ |\xi|_g = 1\}$; φ^t geodesic flow
- (M, g) has sectional curvature -1 and dimension $n + 1$

- $M = \Gamma \backslash \mathbb{H}^{n+1}$, $\Gamma \subset \mathrm{PSO}(1, n+1)$
- In the Minkowski hyperboloid model $\mathbb{H}^{n+1} \subset \mathbb{R}^{1,n+1}$,

$$
\begin{aligned}
X(x, \xi) &= \xi \partial_x + x \partial_\xi \\
\varphi^t(x, \xi) &= (x \cosh t + \xi \sinh t, x \sinh t + \xi \cosh t) \\
E_u(x, \xi) &= \{\eta \partial_x + \eta \partial_\xi \mid \eta \in T_x \mathbb{H}^{n+1},\ \eta \perp \xi\} \\
E_s(x, \xi) &= \{\eta \partial_x - \eta \partial_\xi \mid \eta \in T_x \mathbb{H}^{n+1},\ \eta \perp \xi\}
\end{aligned}
$$

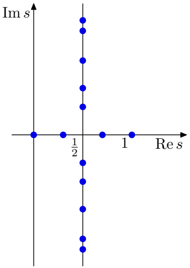
Gelfand–Fomin '55, Guillemin '77, Moore '87, Ratner '87, Zelditch '87, Flaminio–Forni '03, Leboeuf '04, Faure–Tsujii '13...

Results

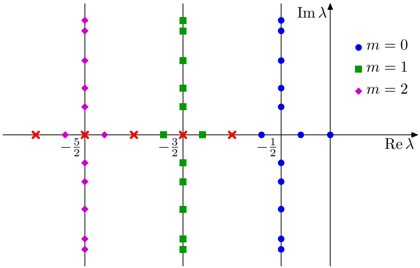
Theorem, $\dim M = 2$

Except at $-1 - \frac{1}{2}\mathbb{N}_0$, λ is a P–R resonance iff

$$-(\lambda + m)(\lambda + m + 1) \in \mathrm{Spec}(-\Delta) \quad \text{for some } m \in \mathbb{N}_0.$$



$s(1-s) \in \mathrm{Spec}(-\Delta)$



$\lambda \in \mathrm{Spec}_{\mathrm{PR}}(X)$

Theorem [DFG '14], $\dim M = n+1 \geq 2$

Except at $-\frac{n}{2} - \frac{1}{2}\mathbb{N}_0$ (+ special case: $\lambda \in -2\mathbb{N}$), λ is a resonance iff

$$-\Big(\lambda + m + \frac{n}{2}\Big)^2 + \frac{n^2}{4} + m - 2\ell \in \operatorname{Spec}^{m-2\ell}(-\Delta), \quad m \in \mathbb{N}_0,\ \ell \in [0, m/2],$$

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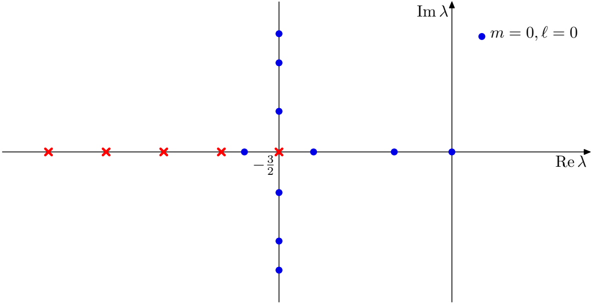

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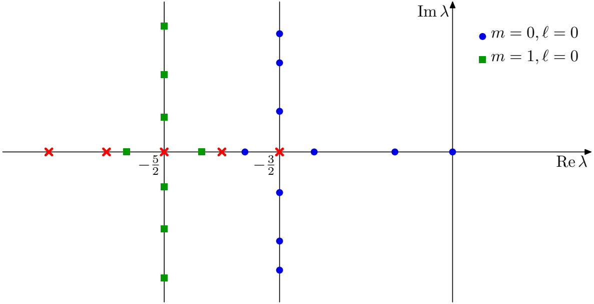

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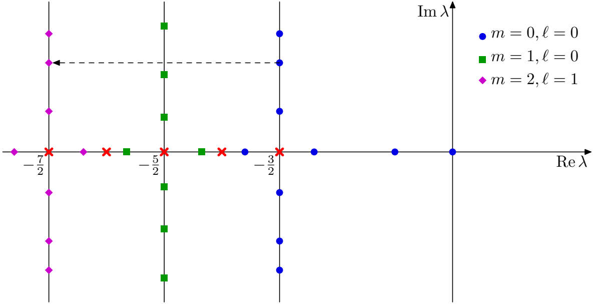

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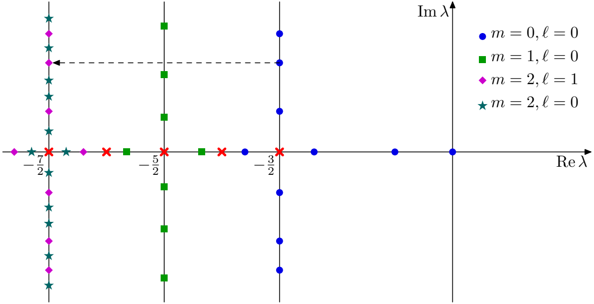

Resonant states and multiplicities

Resonant states of X $\sim$ eigenstates of Δ

$$\mathrm{Res}_X(\lambda) = \{u \in \mathcal{D}'(SM) \mid (X+\lambda)u = 0,\ \mathrm{WF}(u) \subset E_u^*\}$$
$$\mathrm{Res}_X^*(\lambda) = \{u^* \in \mathcal{D}'(SM) \mid (X^*+\bar\lambda)u^* = 0,\ \mathrm{WF}(u^*) \subset E_s^*\}$$
$$\mathrm{Res}_X^\#(\lambda) = \{u \in \mathcal{D}'(SM) \mid \exists k : (X+\lambda)^k u = 0,\ \mathrm{WF}(u) \subset E_u^*\}$$

Theorem [DFG '14]: resonances are semisimple

For $\lambda \notin -\frac{n}{2} - \frac{1}{2}\mathbb{N}_0$, $\mathrm{Res}_X(\lambda) = \mathrm{Res}_X^\#(\lambda)$

Proof relies on pairing formula:

$$\langle u, u^*\rangle_{L^2(SM)} = \mathcal{F}(\lambda)\langle f, f^*\rangle_{L^2(M)}, \quad u \in \mathrm{Res}_X(\lambda),\ u^* \in \mathrm{Res}_X^*(\lambda)$$

where f, f^* are the eigenstates of $-\Delta$ on tensors which correspond to u, u^*
Anantharaman–Zelditch '07, Hansen–Hilgert–Schröder '11

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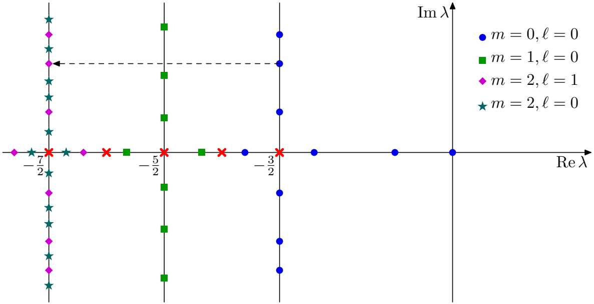

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- Zeta function approach + techniques for exceptional points: Juhl '01, Bunke–Olbrich '95, '96, '99, '01
- Rate of decay of correlations depends on the small eigenvalues of Δ
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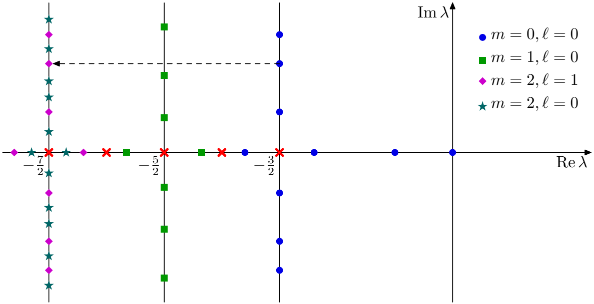

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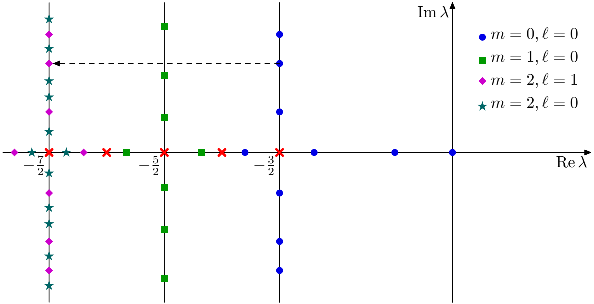

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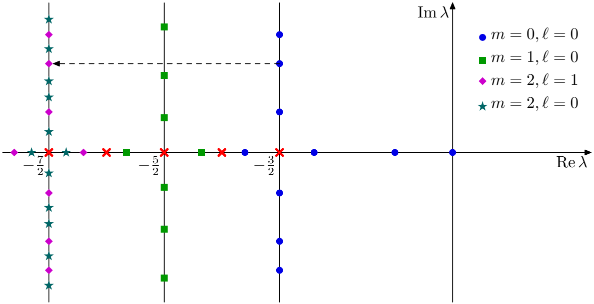

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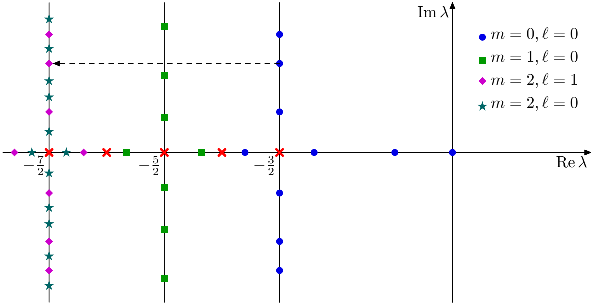

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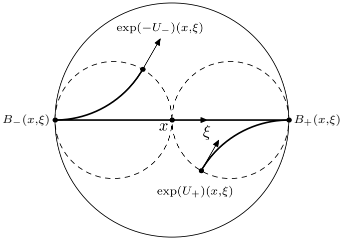
Horocyclic operations (dimension 2)

$M = \Gamma\backslash\mathbb{H}^2, \quad SM = \Gamma\backslash S\mathbb{H}^2, \quad \varphi^t = e^{tX}$

U_+, U_- horocyclic vector fields on SM:

$$[X, U_\pm] = \pm U_\pm, \quad [U_+, U_-] = 2X$$

$$E_u = \mathbb{R}U_-, \quad E_s = \mathbb{R}U_+$$



λ is a resonance $\iff \exists u \in \mathcal{D}'(SM)\setminus 0: \quad (X+\lambda)u = 0, \quad \mathrm{WF}(u) \subset E_u^*$

$$m \geq 0 \Longrightarrow (X + \lambda + m)U_-^m u = 0, \quad \mathrm{WF}(U_-^m u) \subset E_u^*$$

No resonances for $\operatorname{Re}\lambda > 0 \Longrightarrow U_-^m u = 0$ for $m > -\operatorname{Re}\lambda$

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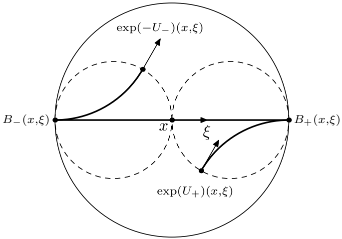

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Reduction to the first band (dimension 2)

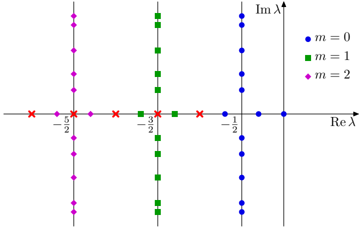
$$(X+\lambda)u = 0, \quad \mathrm{WF}(u) \subset E_u^*, \quad U_-^{m+1}u = 0 \quad \text{for } m \gg 1$$

$$u,\ U_-u,\ U_-^2u,\ \dots,\ v := U_-^m u \neq 0,\ U_-^{m+1}u = 0$$

$$(X+\lambda+m)v = 0, \quad U_-v = 0; \quad \mathrm{WF}(v) \subset E_u^* \text{ automatically}$$

Other direction: given v, put $(\lambda \notin -1 - \frac{1}{2}\mathbb{N})$

$$u := \frac{(2\lambda+m)!}{m!(2\lambda+2m)!} U_+^m v; \quad (X+\lambda)u = 0, \quad U_-^m u = v$$



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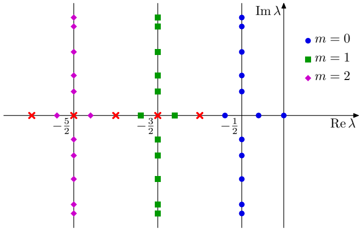
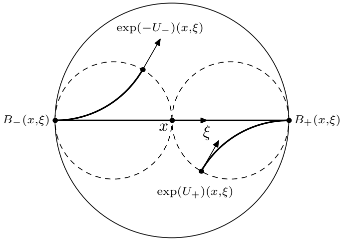


First band (dimension 2)

$$(X+\lambda)v = 0, \quad U_- v = 0$$

$$f(x) := \int_{S_xM} v(x,\xi)\,d\xi$$

$$(-\Delta + \lambda(\lambda+1))f = 0$$



$$B_- : S\mathbb{H}^2 \to \mathbb{S}^1,\ \Phi_- : S\mathbb{H}^2 \to (0,\infty)$$

$$U_-B_- = XB_- = 0,\ U_-\Phi_- = 0,\ X\Phi_- = -\Phi_-$$

$$v(x,\xi) = \Phi_-(x,\xi)^\lambda w(B_-(x,\xi)), \quad w \in \mathcal{D}'(\mathbb{S}^1)$$

$$w(\gamma.\nu) = |d\gamma(\nu)|^\lambda w(\nu), \quad \nu \in \mathbb{S}^1, \quad \gamma \in \Gamma$$

$$f(x) = \int_{\mathbb{S}^1} P(x,\nu)^{1+\lambda} w(\nu)\,d\nu, \quad P(x,\nu) = \frac{1-|x|^2}{|x-\nu|^2},\ x \in \mathbb{B}^2$$

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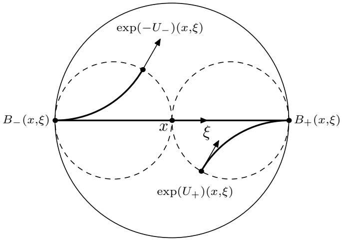

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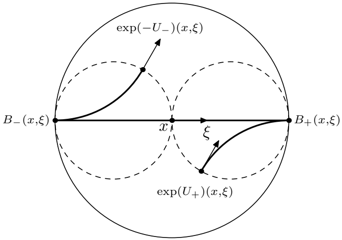

$$v(x,\xi) = \Phi_-(x,\xi)^\lambda w(B_-(x,\xi)), \quad w \in \mathcal{D}'(\mathbb{S}^1)$$

$$w(\gamma.\nu) = |d\gamma(\nu)|^\lambda w(\nu), \quad \nu \in \mathbb{S}^1, \quad \gamma \in \Gamma$$

$$f(x) = \int_{\mathbb{S}^1} P(x,\nu)^{1+\lambda} w(\nu)\, d\nu, \quad P(x,\nu) = \frac{1-|x|^2}{|x-\nu|^2},\ x \in \mathbb{B}^2$$

First band (dimension 2)

$v \in \mathcal{D}'(SM)$
$(X+\lambda)v = 0$
$U_- v = 0$

$f(x) = \int_{S_x M} v(x,\xi)\, d\xi$ →

$f \in C^\infty(M)$
$(-\Delta - z)f = 0$
$z = -\lambda(\lambda+1)$

$v = \Phi_-^\lambda (w \circ B_-)$

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Poisson operator $w \mapsto f$ bijective ($\lambda \notin -1 - \mathbb{N}$):
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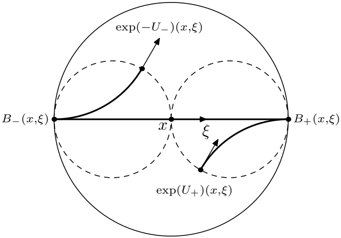

This finishes the proof in dimension 2

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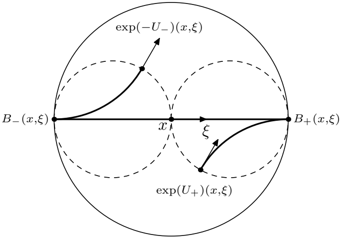

This finishes the proof in dimension 2

Higher dimensions

Problem: $U_\pm$ are no longer well-defined vector fields on SM

Solution: in dimension 2, U_- spanned E_u

$$E_u(x,\xi) = \theta_-(\mathcal{E}(x,\xi)), \quad \theta_- : \eta \mapsto (\eta, \eta)$$

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$$\mathcal{U}_- : \mathcal{D}'(SM) \to \mathcal{D}'(SM; \mathcal{E}^*), \quad \mathcal{U}_- u(x,\xi)\cdot\eta = du(x,\xi)\cdot\theta_-(\eta)$$

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Analogy with polynomials on $\mathbb{R}^n$: $\mathcal{U}_- \sim d$, $\mathcal{U}_+ \sim x$

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is an isomorphism for $\lambda \notin -\frac{n}{2} - \frac{1}{2}\mathbb{N}_0$, $n > 1$

Injectivity

Weak expansion ($x = r\nu \in \mathbb{B}^{n+1}$, $\rho = 1 - r \to 0$, $\nu \in \mathbb{S}^n$)

$$\mathscr{P}_\lambda w(x) \sim c_\lambda w(\nu)\rho^{-\lambda} + \sum_{k\in\mathbb{N}} f_k(\nu)\rho^{-\lambda+k} + \sum_{k\in\mathbb{N}_0} f'_k(\nu)\rho^{n+\lambda+k}$$

Surjectivity

- $\exists$ weak expansion for each trace-free divergence-free eigenstate v
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Pairing formula and semisimplicity ($m = 0$)

$$(X+\lambda)v = 0,\quad \mathcal{U}_- v = 0,\quad f(y) = \int_{S_xM} v(y,\eta_-)\,d\eta_-$$

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Change of variables $\Psi : (x,\xi,\eta) \mapsto (y,\eta_-,\eta_+)$

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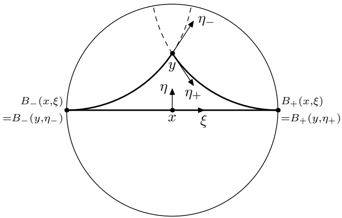

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Thank you for your attention!