Microlocal limits of plane waves

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joint work with Colin Guillarmou (ENS)

(M, g) a compact Riemannian manifold of dimension $n + 1$

$$\Delta_g u_j = z_j^2 u_j,\ z_j \geq 0,\ \|u_j\|_{L^2} = 1.$$

Weyl Law

- Hörmander: $\#\{j \mid z_j \leq z\} = c\, z^{n+1} + \mathcal{O}(z^n)$.
- Duistermaat–Guillemin: $o(z^n)$ if periodic trajectories form a set of measure zero.
- Bérard: $\mathcal{O}(z^n / \log z)$ if sectional curvature < 0.

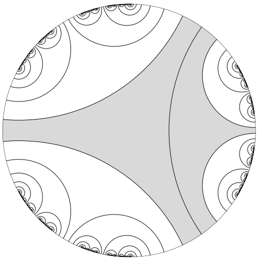
Local Weyl Law

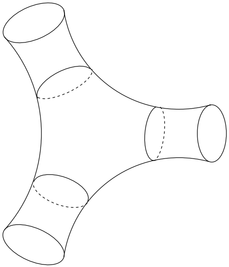
$$\sum_{z_j \leq z} \int_M a|u_j|^2\, d\,\mathrm{Vol} = c\, z^{n+1} \int_M a\, d\,\mathrm{Vol} + \dots$$

for each $a \in C^\infty(M)$, where $\dots$ is the remainder in the Weyl Law.

Our case: (M, g) a noncompact Riemannian manifold, either Euclidean or asymptotically hyperbolic with curvature -1 outside of a compact set.

Example: convex co-compact hyperbolic quotients $\Gamma \backslash \mathbb{H}^{n+1}$.





Application of our work: improved remainders in the Weyl Law

The remainders depend on the structure of the trapped set K, the union of all (unit speed) geodesics which stay in some fixed compact set for all times. For hyperbolic quotients, the trapped set is fractal of Hausdorff dimension $\dim_H(K) = 2\delta + 1$, $0 < \delta < n$.

We assume that the trapped set K has measure zero. The remainders below can be expressed via classical escape rate and maximal expansion rate. To simplify the statements, we assume that M has sectional curvature -1 near K and $\dim_H(K) = 2\delta + 1$.

"Local Weyl Law" in the noncompact case [D–Guillarmou '12]

$$a \in C_0^\infty(M) \Longrightarrow \operatorname{Tr}(a \cdot \mathbf{1}_{[0,z^2]}(\Delta_g)) = \sum_j c_j z^{n+1-j} + \mathcal{O}(z^{\delta+}).$$

"Weyl Law" in the noncompact case [D–Guillarmou '12]

If (M, g) is Euclidean near infinity and $s(z)$ is the spectral shift function:

$$\varphi \in C_0^\infty(\mathbb{R}) \Longrightarrow \int_0^\infty \varphi(z) s'(z)\, dz = \operatorname{Tr}_{\mathrm{bb}}(\varphi(\sqrt{\Delta_g}) - \varphi(\sqrt{\Delta_{\mathbb{R}^{n+1}}})),$$

$$\text{then} \qquad s(z) = \sum_j c_j z^{n+1-j} + \mathcal{O}(z^{\delta+}).$$

Birman–Krein '62: spectral shift function = scattering phase

Asymptotics of $s(z)$: Buslaev '75, Majda–Ralston '78, Jensen–Kato '78, Petkov–Popov '82, Melrose '88, Robert '92, Guillopé–Zworski '97, Christiansen '98, Bruneau–Petkov '03, Dimassi '05 ...
Limits of plane waves: Guillarmou–Naud '11, D '11.

Our result is the first one on manifolds without boundary with a fractal remainder depending on dynamical information.

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Further results

- The remainder is bounded by the number of resonances, i.e. poles of $s(z)$, in a logarithmic region $\{|z_j - z| \leq c \log z\}$. If the fractal Weyl Law of Sjöstrand–Zworski '07, Datchev–D '12 were known to hold there, we would get $\mathcal{O}(z^{\delta+})$, with $\dim_M(K) = 2\delta + 1$.
- For hyperbolic surfaces, the remainder is related to Selberg zeta function. Guillopé–Lin–Zworski '04: $\mathcal{O}(z^{\delta})$. Jakobson–Naud '10: for $\delta > 3/4$, the remainder is **not** $\mathcal{O}(z^{2\delta-3/2-})$.
- Guillarmou–Naud '11: $\mathcal{O}(z^{-\infty})$ for $\Gamma\backslash\mathbb{H}^{n+1}$ when $\delta < n/2$.

Microlocal limits of eigenfunctions

Semiclassical quantization: each $a(m,\nu)\in C_0^\infty(T^*M)$ is mapped to a compactly supported operator

$$\mathrm{Op}_h(a)=a(m,hD_m):C^\infty(M)\to C_0^\infty(M).$$

Here $h\to 0$ is proportional to z^{-1}, so that $\lambda:=hz\sim 1$.

Quantum Ergodicity [Shnirelman, Colin de Verdière, Zelditch]

For compact (M,g) with ergodic geodesic flow, a density one subsequence of eigenfunctions converges to the Liouville measure:

$$a\in C_0^\infty(T^*M)\Longrightarrow \langle \mathrm{Op}_{z_{j_k}^{-1}}(a)u_{j_k},u_{j_k}\rangle\to c\int_{S^*M}a\,d\mu_L.$$

Integrated semiclassical form [Helffer–Martinez–Robert '87]:

$$h^n\sum_{hz_j\in[1,1+h]}\Big|\langle \mathrm{Op}_h(a)u_j,u_j\rangle-c\int_{S^*M}a\,d\mu_L\Big|\to 0\qquad\text{as } h\to 0.$$

We study noncompact manifolds (M, g) which are either Euclidean or asymptotically hyperbolic with curvature -1 outside of a compact set. We refer to the Euclidean case as a model case here.

Eigenfunctions are replaced by plane waves $E_h(\lambda, \xi)$, $\lambda > 0$, $\xi \in \mathbb{S}^n$:

$$(h^2\Delta_g - \lambda^2)E_h(\lambda, \xi) = 0, \qquad E_h(\lambda, \xi; m) = e^{\frac{i\lambda}{h} m\cdot\xi} + (\text{incoming})$$

ξ is the direction of the outgoing part of the wave at infinity.

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Theorem [D–Guillarmou '12]

If the trapped set has measure zero, then for each $a \in C_0^\infty(M)$,

$$h^{-1} \int_{[1,1+h]\times\mathbb{S}^n} \left| \langle \mathrm{Op}_h(a)E_h(\lambda, \xi), E_h(\lambda, \xi)\rangle - \int_{S^*M} a\, d\mu_\xi \right| \to 0 \text{ as } h \to 0.$$

Here μ_ξ is a measure on S^*M depending on ξ, and

$$\int \mu_\xi \, d\xi = \mu_L.$$

Remainder estimates

The remainder can be estimated by the measure of the set of trajectories that stay in a fixed compact set for a multiple of the Ehrenfest time $t_e \sim \log(1/h)$. If M has curvature -1 near K and $\dim_H(K) = 2\delta + 1$, then for $f \in C^\infty(\mathbb{S}^n)$, ignoring subprincipal terms,

$$h^{-1} \int_{[1,1+h]\times\mathbb{S}^n} \Big| \langle \mathrm{Op}_h(a) E_h, E_h \rangle - \int_{S^*M} a\, d\mu_\xi \Big| = \mathcal{O}(h^{\frac{n-\delta}{2}-}),$$

$$h^{-1} \int_{[1,1+h]} \Big| \int_{\mathbb{S}^n} f(\xi) \langle \mathrm{Op}_h(a) E_h, E_h \rangle \, d\xi - \int_{S^*M} a\, d\mu_L \Big| = \mathcal{O}(h^{n-\delta-}).$$

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Remainder bounds in local Weyl Law by a Tauberian argument and

$$\mathbf{1}_{[a^2,b^2]}(h^2\Delta_g) = \frac{1}{(2\pi h)^{n+1}} \int_a^b \lambda^n \int_{\mathbb{S}^n} E_h(\lambda,\xi) \otimes E_h(\lambda,\xi)\, d\xi d\lambda.$$

Use Robert '92 to express the spectral shift function as a trace.

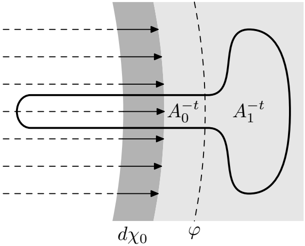
Outline of the proof

Using the Schrödinger group $U(t) = e^{\frac{ith\Delta}{2}}$, we have modulo cutoffs

$$\langle \mathrm{Op}_h(a)E_h, E_h\rangle = \langle A^{-t}E_h, E_h\rangle, \qquad A^{-t} = U(-t)\,\mathrm{Op}_h(a)U(t).$$

By Egorov's Theorem, $A^{-t} = \mathrm{Op}_h(a \circ g^{-t}) + \mathcal{O}(h)$. Here g^t is the geodesic flow. We take $\lim_{t\to+\infty} \lim_{h\to 0}\langle A^{-t}E_h, E_h\rangle$.

- Write $A^{-t} = A_0^{-t} + A_1^{-t}$, with A_0^{-t} in a compact set and A_1^{-t} near infinity.
- In $\langle A_1^{-t}E_h, E_h\rangle$, we can replace E_h by the incoming wave $e^{\frac{i\lambda}{h} m\cdot\xi}$, getting μ_ξ.
- $\|\langle A_0^{-t}E_h, E_h\rangle\|_{L^1_{\lambda,\xi}}$ can be estimated by the measure of the set of trajectories trapped for time t. Goes to zero since $\mu_L(K) = 0$.



To get a remainder estimate, take t before the Ehrenfest time:

$$t = \Lambda_0^{-1} \log(1/h)/2, \quad \Lambda_0 > \Lambda_{\max} := \limsup_{t\to\infty} \frac{1}{t} \log \sup_{S^*M} \|dg^t\|.$$

Then A^{-t} is still pseudodifferential in a mildly exotic class.

Ingredients of the proof

- Egorov's Theorem up to Ehrenfest time for noncompact manifolds, using iteration and cutoffs
- A weak form of propagation of singularities for the scattering resolvent, to handle $\langle A_1^{-t} E_h, E_h \rangle$. Vasy '10 for hyperbolic infinities
- Hilbert–Schmidt norm estimates for spectral projectors (via Fourier integral operators) to estimate $\langle A_0^{-t} E_h, E_h \rangle$
- Going to twice the Ehrenfest time by propagating in both directions
- Propagating the kernel $\int e^{\frac{i\lambda}{h}(m-m')\cdot\xi} f(\xi)\, d\xi$ up to Ehrenfest time by parametrizing it as $\int U(t) B_t\, dt$, for pseudodifferential B_t

Thank you for your attention!