PROPAGATION OF SINGULARITIES AND NONTRAPPING ESTIMATES

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Abstract. In this expository note, we prove the semiclassical propagation of singularities estimate in the presence of complex absorbtion, using the positive commutator method of Hörmander. As an application, we show a nontrapping estimate for one-dimensional semiclassical potential scattering using the method of complex scaling. We also explain how nontrapping estimates in certain other situations (such as a nontrapping obstacle in $\mathbb{R}^3$) lead to exponential decay for solutions of the wave equation.

1. Motivation

Let $\mathcal{O} \subset \mathbb{R}^3$ be an obstacle (a domain with smooth boundary and connected complement) and $\mathcal{E} = \mathbb{R}^3 \setminus \mathcal{O}$ be its exterior domain. Consider the wave equation in $\mathcal{E}$ with Dirichlet boundary conditions[1]

$$\begin{gathered}(\partial_t^2 - \Delta_x)u(t,x) = f(t,x),\ t > 0,\ x \in \mathcal{E};\\ u|_{t=0} = \partial_t u|_{t=0} = 0;\\ u|_{x\in\mathcal{E}} = 0.\end{gathered}$$

Here f has bounded support. For simplicity, we will assume that $f(t,x) \in C^\infty((0,\infty)\times\mathcal{E})$ and thus u lies in the same class, but the estimates work naturally in certain Sobolev classes. In this note, we concentrate on the nontrapping property and not on the general properties of wave equations, which can be found for example in [Ta, Chapter 8].

The global energy

$$E(t) = \frac{1}{2}\int_{\mathcal{E}} |u_t|^2 + |\nabla_x u|^2\,dx$$

is conserved in time. However, we are interested in the behavior of $u(t,x)$ when $t \to +\infty$ and x stays in a fixed compact set $K \subset \mathcal{E}$. We define local energy as

$$E_K(t) = \frac{1}{2}\int_K |u_t|^2 + |\nabla_x u|^2\,dx.$$

[1]A classical treatment of obstacle scattering can be found in [LaPhi]. A shorter introduction following similar methods is found in [Ta]. Our approach is perhaps closest to [Sj].

We wish to prove *exponential decay of local energy*; i.e., that there exists a constant $\nu > 0$ and an f-dependent constant C such that

$$E_K(t) \leq Ce^{-\nu t}. \tag{1.1}$$

It turns out that this exponential decay property is related to the phenomenon of *trapping*. Namely, consider a billiard ball trajectory $\gamma(t)$ in $\mathcal{E}$; that is, $\gamma(t)$ moves along a straight line until it hits the obstacle, at which point it is reflected and continued further by the same principle.[2] It is convenient to picture γ as living in the phase space $\mathcal{E} \times \mathbb{R}^n = T^*\mathcal{E}$, with coordinates $(x \in \mathcal{E}, \xi \in \mathbb{R}^n)$; x corresponds to position and ξ to momentum (or speed, in our situation); then away from $\mathcal{O}$, $(x, \xi) = \gamma(t)$ solves Hamilton's equations for the Hamiltonian $p = \frac{1}{2}|\xi|^2$:

$$\dot{x} = \partial_\xi p = \xi,\ \dot{\xi} = -\partial_x p = 0. \tag{1.2}$$

Such a γ is called *forward trapped* if there exists a compact set K such that $\gamma(t) \in K$ for all $t > 0$. Otherwise, we say that γ *escapes* as $t \to \infty$. (In fact, once $\gamma(t)$ leaves the convex hull of $\mathcal{O}$, it moves in a straight line to infinity.) Similarly we define backwards trapped trajectories; the compact *trapped set* consists of trajectories that are trapped in both directions. The obstacle $\mathcal{O}$ is called *nontrapping* if the trapped set is nonempty; in fact, in this case each trajectory escapes in both directions (exercise). For a nontrapping obstacle, the exponential decay estimate (1.1) holds.[3] We will not prove this fact here, but rather indicate how it follows from a certain *nontrapping estimate* from scattering theory and prove this estimate in a simpler model case. (The simplifications have to do with the structure of infinity, not of the trapped set; in fact, we present the key propagation of singularities estimate in great generality.)

Estimates of type (1.1) have been discovered by Lax and Phillips, see [LaPhi]. They used an object now called Lax–Phillips semigroup; however, most modern treatments use instead the contour deformation argument, which we briefly present here. Let us take the Fourier–Laplace transform of u in time:

$$\hat{u}(\omega)(x) = \int_0^\infty e^{it\omega} u(t, x)\, dt. \tag{1.3}$$

If $\operatorname{Im}\omega > 0$, then by conservation of energy and thanks to the exponential decay of $e^{it\omega}$, the integral (1.3) converges and gives a function in $L^2(\mathcal{E})$. Moreover, we have the differential

[2]We cannot afford to consider the microlocal properties of boundary value problems in this note; in particular, we ignore the problems of glancing or obstacles with corners. Both propagation of singularities and the nontrapping estimate for the model case in Section 4 are proved in a situation without boundary.

[3]If the obstacle has a trapped set of full measure (for example, imagine a three-dimensional version of the letter C, with the ends very close to each other), then a lot of energy will stay near the trapped set and we cannot expect exponential decay. However, the estimate (1.1) is still true in certain cases when the trapping is mild; i.e. almost all trajectories near the trapped set move away from it exponentially fast. See [Ik] for the case of several convex obstacles and [NoZw] for a general statement under a certain pressure condition.

equation

$$(-\Delta_x - \omega^2)\hat{u}(\omega) = \hat{f}(\omega). \tag{1.4}$$

The equation (1.4) makes sense for $\operatorname{Im}\omega \leq 0$ as well. In fact, one can show that there exists a family of operators $R(\omega) : L^2_{\mathrm{comp}}(\mathcal{E}) \to L^2_{\mathrm{loc}}(\mathcal{E})$ (i.e., from compactly supported to locally square integrable functions) such that $\hat{u}(\omega) = R(\omega)\hat{f}(\omega)$ for $\operatorname{Im}\omega > 0$, and $R(\omega)$ is *meromorphic* in ω, with poles of finite rank. The construction of $R(\omega)$ depends on the structure of spatial infinity; for the obstacle problem, one can either construct a parametrix from the free resolvent and the cutoff resolvent of the Laplacian on $\mathcal{E}$ using the framework of black box scattering [Sj, Section 2.3], or use the method of layer potentials [Ta, Section 9.7]. We will not give a construction of $R(\omega)$ here; we only note that in the absence of the obstacle, it would be given by the *free resolvent*

$$R_0(\omega)f(x) = \frac{1}{4\pi}\int \frac{e^{i\omega|x-y|}}{|x-y|} f(y)\,dy. \tag{1.5}$$

The operator $R(\omega)$ has no poles on the real line; however, it does have infinitely many poles in $\{\operatorname{Im}\omega < 0\}$, called *resonances*.

Coming back to analysing the behavior of $u(t)$ for large t, we write by Fourier inversion formula

$$u(t) = \frac{1}{2\pi}\int_{\operatorname{Im}\omega=1} e^{-it\omega}R(\omega)\hat{f}(\omega)\,d\omega.$$

We now want to deform the contour of integration to the lower half-plane, where $e^{-it\omega}$ is decaying exponentially in $t \to +\infty$.[4] For this, we need to have an estimate on $R(\omega)$ in some strip below the real line; we claim that under the nontrapping assumption, we have the following *nontrapping estimate*: for each $\chi \in C_0^\infty(\mathcal{E})$ and each $\nu > 0$,

$$\|\chi R(\omega)\chi\|_{L^2\to L^2} \leq C|\omega|^{-1},\ \operatorname{Im}\omega > -\nu,\ |\operatorname{Re}\omega| \gg 1 \text{ depending on } \nu. \tag{1.6}$$

Given (1.6), we see that for each ν, the set of resonances in $\{\operatorname{Im}\omega \geq -\nu\}$ is finite; therefore, we can write

$$\begin{aligned} u(t) &= \sum_{\operatorname{Im}\hat{\omega}\geq-\nu} \operatorname{Res}_{\hat{\omega}}(e^{-it\omega}R(\omega)\hat{f}(\omega)) + \frac{1}{2\pi}\int_{\operatorname{Im}\omega=-\nu} e^{-it\omega}R(\omega)\hat{f}(\omega)\,d\omega \\ &= \sum_{\operatorname{Im}\hat{\omega}\geq-\nu} \operatorname{Res}_{\hat{\omega}}(e^{-it\omega}R(\omega)\hat{f}(\omega)) + O(e^{-\nu t}), \end{aligned} \tag{1.7}$$

where we estimated the integral by (1.6) and used the fact that $\hat{f}(\omega)$ is Schwartz in $\operatorname{Re}\omega$ for bounded $\operatorname{Im}\omega$. The sum in (1.7) is over resonances $\hat{\omega}$ and each of its terms has a time profile $t^k e^{-it\hat{\omega}}$, with $k \geq 0$ an integer and $e^{-it\hat{\omega}}$ exponentially decaying. The formula (1.7)

[4]The general contour deformation argument can be found in [Sj, Section 3]. For a simpler one-dimensional situation, the reader is directed to [Zw, Section 2.2]; in fact, the nontrapping estimate there follows from our discussion below, as after semiclassical rescaling the operator $D_x^2 + V(x)$ becomes $h^2D_x^2 + h^2V(x)$ instead of $h^2D_x^2 + V(x)$ that we study; thus, the semiclassical prinicipal symbol is just $|\xi|^2$ and its flow is nontrapping regardless of V.

is known as *resonance expansion* of linear waves. In particular, if we take $\nu > 0$ small enough so that there are no resonances in $\{\operatorname{Im}\omega > -\nu\}$, we arrive to the exponential decay estimate (1.1).

We now need to understand why the geometric nontrapping condition on the classical flow implies the nontrapping estimate (1.6). The estimate (1.6) was proved by Morawetz for a star-shaped obstacle, see [LaPhi, Appendix 3 and Theorem 5.3.2]. For that, she used the vector field

$$F = x \cdot \partial_x.$$

The symbol of (i times) the multiplier F is the function $f(x, \xi) = x \cdot \xi$. For $p = |\xi|^2$ the symbol of the Laplacian, one computes the derivative of f along solutions to (1.2): $H_p f = 2|\xi|^2 > 0$ for $\xi \neq 0$. Moreover, if $\mathcal{O}$ is star-shaped, then f strictly increases with each reflection; therefore, f grows along the billiard ball trajectories. Such f is called an *escape function*; its existence is equivalent to the obstacle being nontrapping. For a general nontrapping obstacle, we might not be able to find an escape function f which is polynomial in ξ and would thus correspond to a differential operator F; however, we can quantize an arbitrary smooth function f to a pseudodifferential operator. The nontrapping estimate in this general case is then proved as follows: we first handle the infinity, for example by complex scaling (presented in Section 4 for a model one-dimensional case) and then use the propagation of singularities estimate, based on Hörmander's positive commutator argument; the latter is presented in Section 3.

2. Semiclassical quantization

In this section, we briefly review semiclassical notation as used in [EvZw]. We consider the Weyl quantization (see [EvZw, Chapter 4] for details)

$$a \in S^m \mapsto \operatorname{Op}_h(a) = a^w(x, hD_x) : H_h^s(\mathbb{R}^n) \to H_h^{s-m}(\mathbb{R}^n)$$

We will often denote semiclassical quantization of a symbol by the corresponding uppercase letter, e.g. $A := \operatorname{Op}_h(a)$. Here a smooth function $a(x, \xi)$ lies in the symbol class S^m if and only if

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha\beta} \langle \xi \rangle^{m-|\beta|}.$$

(We restrict to Kohn–Nirenberg symbols as in [EvZw, Section 10.3] to make it possible to quantize functions on manifolds.) For m a nonnegative integer, the class S^m includes polynomials $a(x, \xi)$ in ξ (with coefficients functions of x bounded with all their derivatives), and the corresponding operators $\operatorname{Op}_h(a)$ are semiclassical differential operators. (We use the Weyl quantization to fix notation; any other quantization would work equally as well.) The H_h^s are semiclassical Sobolev spaces; however, we will often concentrate on the behavior of symbols in a compact subset of (x, ξ); for functions localized there, all semiclassical Sobolev norms are equivalent. We also assume that all studied symbols $a(x, \xi; h) \in S^m$ are *classical* in their dependence on h in the following sense: there exists a sequence $a_j(x,\xi) \in S^{m-j}$ such that for each N,

$$a(x,\xi;h) = \sum_{0\leq j<N} h^j a_j(x,\xi) + O(h^N)_{S^{m-N}}.$$

We write $a \sim \sum_{j\geq 0} h^j a_j$. It turns out that the space of operators with classical symbols is invariant under the basic operations of semiclassical analysis (products, adjoints, and changes of variables). The component a_0 is called the *principal symbol*.

The main property of semiclassical quantization that we use is the multiplication formula:

$$a \in S^{m_1},\ b \in S^{m_2} \Longrightarrow \mathrm{Op}_h(a)\,\mathrm{Op}_h(b) = \mathrm{Op}_h(a\#b),$$

where $a\#b \in S^{m_1+m_2}$ and

$$(a\#b)(x,\xi) \sim \sum_{j\geq 0} \frac{h^j}{(2i)^j j!}(\partial_\xi \cdot \partial_x - \partial_\eta \cdot \partial_y)^j (a(x,\xi)b(y,\eta))|_{\substack{y=x\\ \eta=\xi}}.$$

In particular, we derive (with remainders bounded between appropriate semiclassical Sobolev spaces)

$$\mathrm{Op}_h(a)\,\mathrm{Op}_h(b) = \mathrm{Op}_h(ab) + O(h), \tag{2.1}$$

$$[\mathrm{Op}_h(a), \mathrm{Op}_h(b)] = \frac{h}{i}\,\mathrm{Op}_h(\{a,b\}) + O(h^2). \tag{2.2}$$

Here $\{a,b\}$ is the Poisson bracket:

$$\{a,b\} = \partial_\xi a \cdot \partial_x b - \partial_x a \cdot \partial_\xi b.$$

For a real-valued, we can write $\{a,b\} = H_a b$, where H_a is the Hamilton vector field of a:

$$H_a = \partial_\xi a \cdot \partial_x - \partial_x a \cdot \partial_\xi.$$

Finally, we need the following sharp Gårding inequality [EvZw, Theorem 4.32]: if $a \in C_0^\infty$ and $a \geq 0$ everywhere, then there exists a constant C such that for each $u \in L^2$,

$$\langle \mathrm{Op}_h(a)u, u\rangle \geq -Ch\|u\|_{L^2}^2. \tag{2.3}$$

Here $\langle\cdot,\cdot\rangle$ denotes the L^2 inner product.

3. Propagation of singularities

The results of this section apply in fact to any manifold, not just $\mathbb{R}^n$, if we use semiclassical quantization on manifolds as in [EvZw, Chapter 14].

We let $p(x,\xi;h) \in S^m$, with p_0 its principal part, and let $P = \mathrm{Op}_h(p)$ be the associated operator. Our goal is to get estimates on solutions to the equation $Pu = v$. We start with

Theorem 1. *(Elliptic estimate) Assume that $a(x,\xi)$ is compactly supported and such that*

$$\operatorname{supp} a \subset \{p_0 \neq 0\}. \tag{3.1}$$

Then there exists a constant C such that for each u,

$$\|Au\|_{L^2} \leq C\|Pu\|_{L^2} + O(h^\infty)\|u\|_{L^2}.$$

Proof. If one is content with $O(h)$ in place of $O(h^\infty)$, the estimate is proved by taking any $q_0(x,\xi) \in C_0^\infty$ such that $a = q_0 \cdot p_0$; its existence follows immediately from (3.1). Then by (2.1),

$$A = \operatorname{Op}_h(q_0)P + O(h);$$

thus

$$\begin{aligned} \|Au\|_{L^2} &\leq \|\operatorname{Op}_h(q_0)Pu\|_{L^2} + O(h)\|u\|_{L^2} \\ &\leq C\|Pu\|_{L^2} + O(h)\|u\|_{L^2}. \end{aligned} \tag{3.2}$$

In order to get the $O(h^\infty)$ remainder, we consider an h-dependent q:

$$q \sim \sum_{j\geq 0} h^j q_j,$$

such that, with $Q = \operatorname{Op}_h(q)$,

$$A = Q\#P + O(h^\infty). \tag{3.3}$$

This is done by inductively solving for each q_j. In fact, q_0 has been found above, and we will be able to arrange it so that each q_j is supported in $\{p_0 \neq 0\}$. Assume that for some J, we found $q_1, \dots, q_{J-1}$ such that

$$a - \Big(\sum_{j<J} h^j q_j\Big)\#p = O(h^J). \tag{3.4}$$

We can write the right-hand side of (3.4) as $h^J c_J(x,\xi) + O(h^{J+1})$, with c_J supported in $\{p_0 \neq 0\}$. Then, if we take q_J such that $p_0 q_J = c_J$, we have (3.4) for $J+1$ in place of J. By induction in J, we arrive to (3.3). It remains to argue analogously to (3.2), with Q taking the place of $\operatorname{Op}_h(q_0)$. □

The main fact to be proved in this section is

Theorem 2. *(Propagation of singularities) Assume that the principal symbol p_0 is real valued. We can then consider the flow lines of the Hamiltonian vector field H_{p_0}, which we call bicharacteristics. Also, assume that $q \in S^m$ is such that q_0 is real-valued. Let a, b, g be compactly supported functions on $\mathbb{R}^{2n}$ such that $\operatorname{supp} b \subset \{g \neq 0\}$ and one of the following dynamical assumptions holds:*

1. *for each bicharacteristic $\gamma(t)$ with $\gamma(0) \in \operatorname{supp} a \cap \{p_0 = 0\}$, there exists $T > 0$ such that $\gamma(T) \in \{b \neq 0\}$ and $\gamma([0,T]) \subset \{g \neq 0\}$, and $q_0 \leq 0$ near $\operatorname{supp} g$;*
2. *for each bicharacteristic $\gamma(t)$ with $\gamma(0) \in \operatorname{supp} a \cap \{p_0 = 0\}$, there exists $T > 0$ such that $\gamma(-T) \in \{b \neq 0\}$ and $\gamma([-T,0]) \subset \{g \neq 0\}$, and $q_0 \geq 0$ near $\operatorname{supp} g$.*

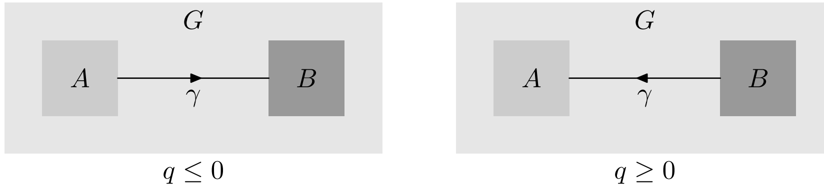

FIGURE 1. Assumptions of Theorem 2. We estimate Au by Bu and $G(P-iQ)u$.

Then, there exists a constant C such that for all u,

$$\|Au\|_{L^2} \leq C(\|Bu\|_{L^2} + h^{-1}\|G(P-iQ)u\|_{L^2}) + O(h^\infty)\|u\|_{L^2}. \tag{3.5}$$

If $Bu = O(h^\infty)$ and $Pu = 0$, we see that $Au = O(h^\infty)$. This means that for $q_0 = 0$, Theorem 2 can be reformulated as follows: the wavefront set (in the sense of [EvZw, Section 8.4]) of any solution of $Pu = 0$ lies in $\{p_0 = 0\}$ and is invariant under the Hamiltonian flow of p_0. One can also say that *regularity propagates along the bicharacteristics* in both directions. Then for $q_0 \leq 0$, regularity propagates backwards along the bicharacteristics, while for $q_0 \geq 0$, it propagates forward. The operator Q, called *complex absorbing operator*, is important in our application in Section 4 because it fixes the direction of propagation. To see how the sign of q affects the direction of propagation, one can consider the one-dimensional example with $P = hD_x$, $Q = q(x) \leq 0$ everywhere, $a(x)$ supported near $x = 0$, and $b(x)$ supported near $x = 1$. Then the solutions to $(P - iQ)u = 0$ are multiples of

$$u(x) = \exp\Big(-h^{-1}\int_0^x q(y)\,dy\Big);$$

we see that $\|Au\| \lesssim \|Bu\|$, but not the other way round. Note also that the assumptions of the theorem still hold for $q = Ch$ for some constant C (as the principal symbol of Q is then zero), and then $\|Au\| \sim e^C\|Bu\|$; this explains why the escape function we construct below has to grow sufficiently fast (condition (3) in Lemma 3.1).

We now prove Theorem 2. We operate under the assumption (1); assumption (2) is handled, for example, by multiplying both P and Q by -1. We can moreover assume that (1) holds for bicharacteristics with $\gamma(0) \in \operatorname{supp} a$, not just $\gamma(0) \in \operatorname{supp} a \cap \{p_0 = 0\}$; indeed, we can write $a = a_1 + a_2$, where (1) holds for each γ with $\gamma(0) \in \operatorname{supp} a_1$, and $a_2 \subset \{p_0 \neq 0\}$ and thus Theorem 1 applies to it. First, we reduce Theorem 2 to the following estimate:

$$\|Au\|_{L^2} \leq C(\|Bu\|_{L^2} + h^{-1}\|G(P-iQ)u\|_{L^2} + h^{1/2}\|Gu\|_{L^2}) + O(h^\infty)\|u\|_{L^2}. \tag{3.6}$$

Indeed, we can prove the following family of estimates by induction in n:

$$\|Au\|_{L^2} \leq C(\|Bu\|_{L^2} + h^{-1}\|G(P-iQ)u\|_{L^2} + h^{n/2}\|Gu\|_{L^2}) + O(h^\infty)\|u\|_{L^2}. \tag{3.7}$$

For $n = 1$, (3.7) is just (3.6); knowing (3.7) for all n immediately gives (3.5). So assume that (3.7) holds for some n and all operators satisfying the assumptions of Theorem 2, and we want to prove it for $n+1$ and some operators A, B, G. Since $\operatorname{supp} a \cap \{p_0 = 0\}$ is closed

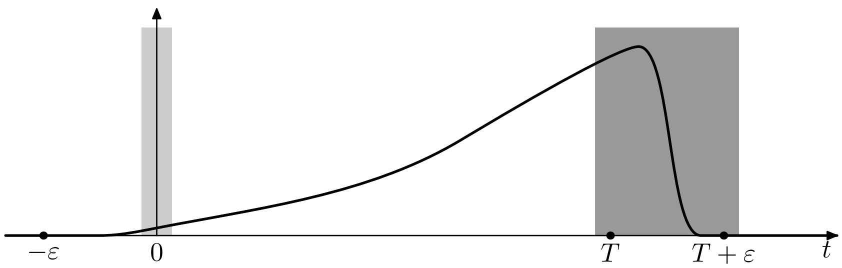

FIGURE 2. An escape function $f_1(t)$, with the ligher and darker shaded regions corresponding to A and B, respectively.

and $\{b \neq 0\}$ is open, we can find G' such that the assumptions of Theorem 2 hold both for (A, B, G') and (G', B, G). Then we apply (3.7) for n and the operators (A, B, G') and (3.6) for the operators (G', B, G) to estimate the term $\|G'u\|_{L^2}$ on the right-hand side of (3.7), and get the required estimate (with $\|G'(P - iQ)u\|_{L^2} \leq C\|G(P - iQ)u\|_{L^2} + O(h^\infty)\|u\|_{L^2}$ by Theorem 1).

It now remains to prove (3.6). For that, we use the positive commutator argument.[5] The key component is the *escape function*:

Lemma 3.1. *For each $R > 0$, there exists a function $f \in C_0^\infty(\mathbb{R}^{2n})$ such that:*

(1) $f \geq 0$ *everywhere and* $f > 0$ *on* $\operatorname{supp} a$*;*
(2) $\operatorname{supp} f \subset \{g \neq 0\}$*;*
(3) $H_{p_0} f \geq Rf$ *near the complement of* $\{b \neq 0\}$.

Proof. We use a local construction and a covering argument, similarly to [DaVa2, Lemma 4.3]. Take a point $\rho \in \operatorname{supp} a$ and let $\gamma(t) = \exp(tH_{p_0})\rho$ be the bicharacteristic starting at ρ. By the dynamical assumption, there exists $T > 0$ such that $\gamma(T) \in \{b \neq 0\}$ and $\gamma([0, T]) \subset \{g \neq 0\}$. We now introduce the system of coordinates[6]

$$(t, \zeta) \in [-\varepsilon, T + \varepsilon] \times \{|\zeta| < \varepsilon\}$$

in a neighborhood U of $\gamma([0, T])$ such that $\gamma(t)$ has coordinates $(t, 0)$ and the Hamiltonian vector field H_{p_0} corresponds to ∂_t. We shrink U (and decrease ε) so that $U \subset \{g \neq 0\}$ and each point with coordinates (t, ζ), where $t \in [T, T + \varepsilon)$, lies in $\{b \neq 0\}$. Now, take a function $f_1(t) \in C_0^\infty(-\varepsilon, T+\varepsilon)$ such that $f_1 \geq 0$, $f_1(0) > 0$, and $\partial_t f_1 \geq Rf_1$ on $(-\varepsilon, T]$. (See Figure 2; note that near the 'touchdown' $t = -\varepsilon/2$ we can make f_1 behave like $e^{-1/(t-\varepsilon/2)}$

[5]When there is no Q, one can prove propagation of singularities using Egorov's theorem; see for example [HöIV, Section 26.1] for the microlocal setting and [EvZw, Section 13.3] for the semiclassical setting. We use the positive commutator argument because it is more robust and in particular applies when the complex absorbing operator Q is present.

[6]Of course, such coordinates exist only when ρ is not a critical point of p_0 (in most applications, p_0 has no critical points on $\{p_0 = 0\}$). The opposite case is handled trivially, as then $\rho \in \{b \neq 0\}$.

and we see that $\partial_t f_1(t) \gg f_1(t)$.). Also, take nonnegative $\chi(\zeta)$ supported in $\{|\zeta| < \varepsilon\}$ and such that $\chi(0) = 1$. Then the function

$$f_\rho = f_1(t)\chi(\zeta) \in C_0^\infty(U),$$

extended by zero outside U, is nonnegative, satisfies properties (2) and (3) of the statement of this Lemma, and is nonzero in a neighborhood U_ρ of ρ. It remains to use compactness of $\operatorname{supp} a$, cover it by finitely many sets U_ρ, and take the sum of the corresponding functions f_ρ. □

Now, let $F = \operatorname{Op}_h(f)$. Let us write $P = \operatorname{Re} P + i \operatorname{Im} P$, where by the adjoint property of semiclassical quantization [EvZw, Theorem 4.1(ii)] $\operatorname{Re} P = \frac{1}{2}(P + P^*)$ is formally self-adjoint and still has principal symbol p_0 and $\operatorname{Im} P = \frac{1}{2i}(P - P^*)$ is $O(h)$. We now calculate (with $\langle\cdot,\cdot\rangle$ denoting the L^2-inner product)

$$\begin{aligned}\operatorname{Im}\langle (P - iQ)u, F^*Fu\rangle \\ = \frac{1}{2i}(\langle(\operatorname{Re} P)u, F^*Fu\rangle - \langle F^*Fu, (\operatorname{Re} P)u\rangle) - \operatorname{Re}\langle (Q - \operatorname{Im} P)u, F^*Fu\rangle \\ = \frac{1}{2i}\langle [F^*F, \operatorname{Re} P]u, u\rangle - \operatorname{Re}\langle F\widetilde{Q}u, Fu\rangle,\end{aligned}$$

where $\widetilde{Q} = Q - \operatorname{Im} P$ still has principal symbol $q_0 \geq 0$ on $\operatorname{supp} f \subset \operatorname{supp} g$. Then it follows from sharp Gårding inequality (2.3) (by applying it to $\operatorname{Op}_h(\chi)\widetilde{Q}$, with χ equal to 1 near $\operatorname{supp} f$ and supported inside $\{g \neq 0\}$, and using that $F(1 - \operatorname{Op}_h(\chi)) = O(h^\infty)$) that for some constant C_0 independent of the choice of F,

$$\operatorname{Re}\langle \widetilde{Q}Fu, Fu\rangle \geq -C_0 h\|Fu\|_{L^2}^2 + O(h^\infty)\|u\|_{L^2}^2.$$

Now,

$$\operatorname{Re}\langle F\widetilde{Q}u, Fu\rangle = \operatorname{Re}\langle \widetilde{Q}Fu, Fu\rangle + \operatorname{Re}\langle [F, \widetilde{Q}]u, Fu\rangle;$$

the last term on the right-hand side equals $\langle \operatorname{Re}(F^*[F,\widetilde{Q}])u, u\rangle$, where $T = F^*[F,\widetilde{Q}] = \frac{h}{i}\operatorname{Op}_h(f\{f,q\}) + O(h^2)$. Since $f\{f,q\}$ is real-valued, we know that $\operatorname{Re} T = O(h^2)$ and its symbol is supported in $\{g \neq 0\}$. Therefore, for h small enough

$$\operatorname{Re}\langle F\widetilde{Q}u, Fu\rangle \geq -C_0 h\langle F^*Fu, u\rangle - O(h^2)\|Gu\|_{L^2}^2 - O(h^\infty)\|u\|_{L^2}^2$$

and

$$h^{-1}\operatorname{Im}\langle (P - iQ)u, F^*Fu\rangle \geq \left\langle\left(\frac{i}{2h}[\operatorname{Re} P, F^*F] - C_0F^*F\right)u, u\right\rangle - O(h)\|Gu\|_{L^2}^2 - O(h^\infty)\|u\|_{L^2}^2.$$

Choose $R > C_0$ in Lemma 3.1. The operator in parentheses on the right-hand side has principal symbol

$$fH_{p_0}f - C_0f^2,$$

which is nonnegative near the complement of $\{b \neq 0\}$. Then, for large enough constant C_1, the function

$$fH_{p_0}f - C_0f^2 + C_1|b|^2 - C_1^{-1}f^2 \tag{3.8}$$

is nonnegative everywhere. We may always shrink b so that $\operatorname{supp} b \subset \{g \neq 0\}$ and the dynamical assumption still holds. Then the function from (3.8) is supported inside $\{g \neq 0\}$; applying again sharp Gårding inequality, we get

$$h^{-1}\operatorname{Im}\langle (P-iQ)u, F^*Fu\rangle \geq C_1^{-1}\|Fu\|_{L^2}^2 - C_1\|Bu\|_{L^2}^2 - O(h)\|Gu\|_{L^2}^2 - O(h^\infty)\|u\|_{L^2}^2.$$

This converts (using Theorem 1 and the fact that the symbol of F^*F is supported inside $\{g \neq 0\}$) to the estimate

$$\|Fu\|_{L^2}^2 \leq C(\|Bu\|_{L^2}^2 + h^{-1}\|G(P-iQ)u\|_{L^2}\cdot\|Fu\|_{L^2} + h\|Gu\|_{L^2}^2) + O(h^\infty)\|u\|_{L^2}^2.$$

From here, it follows that

$$\|Fu\|_{L^2} \leq C(\|Bu\|_{L^2} + \|G(P-iQ)u\|_{L^2} + h^{1/2}\|Gu\|_{L^2}) + O(h^\infty)\|u\|_{L^2};$$

using that $\|Au\|_{L^2} \leq C\|Fu\|_{L^2} + O(h^\infty)\|u\|_{L^2}$ by Theorem 1, we get (3.6) and thus finish the proof of Theorem 2.

4. Complex scaling and nontrapping estimate

We now explain how propagation of singularities implies the nontrapping estimate (1.6). First of all, we perform semiclassical rescaling: for the obstacle case, let $P = -h^2\Delta$ on $\mathcal{E}$ with Dirichlet boundary conditions, and assume that h is chosen so that $h\omega = 1 + h\sigma$, with $\sigma = O(1)$. (Recall that $|\operatorname{Im}\omega|$ is bounded by ν in the nontrapping estimate. The case $\operatorname{Re}\omega < 0$ is handled similarly.) Then, if ω is not a resonance, $f \in C_0^\infty$, and $u = R(\omega)f$, then u is the unique solution to the *scattering problem*

$$Pu = h^2 v,\ u \text{ is outgoing.} \tag{4.1}$$

(For the obstacle case, one needs to assume the Dirichlet boundary condition on u; as remarked before, we will avoid dealing with boundary value problems and billiard ball flow and study a model problem with no boundary.) If ω is a resonance, then there exists a nonzero solution to (4.1) with zero right-hand side. The outgoing condition acts like a boundary condition at infinity. We will not state this condition in general, noting only the following: (1) for $\operatorname{Im}\omega > 0$, being outgoing is equivalent to lying in L^2 of the whole space, and thus the scattering resolvent is equal to the usual resolvent given by spectral theory of the Laplacian (2) for $\operatorname{Im}\omega = 0$, the outgoing condition is known as *Sommerfeld radiation condition*, see [Ta, Section 9.1] (3) for the obstacle case, the outgoing condition means that outside of some ball $B(0,R)$, $u = R_0(\omega)\tilde v$, with $\tilde v \in C_0^\infty(\mathbb{R}^3)$ and $R_0(\omega)$ the free resolvent from (1.5).

There are several methods of obtaining microlocal information from the outgoing condition, including (1) complex scaling the problem to emphasize the outgoing trajectories while dampening the incoming ones [SjZw], (2) radial points estimates [Va], and (3) considering instead the operator $P - iQ$ with $q > 0$ near infinity (see for example [WuZw]) and gluing it together with some other estimate at infinity [DaVa1]. Each of the methods above should produce the following effect: u is $O(h^\infty)$ near the *incoming* points near infinity; for a large compact set $K \subset \mathbb{R}^{2n}$ containing the trapped set, a point $(x, \xi) \in K$ is incoming if the backwards bicharacteristic starting at this point never enters K and goes to infinity. Then by propagation of singularities, u has to be microlocalized on the set of backwards trapped trajectories and the trajectories starting at the wavefront set of v. If the nontrapping assumption holds, then there are no backwards trapped trajectories and we can estimate u through v, thus getting the nontrapping estimate.[7]

We now give the proof of the nontrapping estimate in a model example of one-dimensional scattering by a compactly supported potential; see [Zw, Section 2] for a detailed description of this setting. Namely, assume that u solves the scattering problem

$$P(h)u = v_h,\ \ P(h) = h^2 D_x^2 + V(x) - (1 + h\sigma)^2.$$

Here $\sigma \in \mathbb{C}$ is bounded, $x \in \mathbb{R}$, and $V(x) \in C_0^\infty(\mathbb{R})$. We assume that both V and v are supported in some ball $(-X, X)$. We also assume that u satisfies the following outgoing condition:

$$u(x) = c_\pm e^{\pm i(1+h\sigma)x/h} \text{ for } \ \pm x \geq X,$$

where $c_\pm$ are some constants. (By theory of constant-coefficient ODE, on any interval outside of the support of V the function u is a linear combination of $e^{i(1+h\sigma)x/h}$ and $e^{-i(1+h\sigma)x/h}$.) The nontrapping estimate takes the form

$$\|u\|_{L^2(-X,X)} \leq Ch^{-1}\|v_h\|_{L^2}. \tag{4.2}$$

(The power of h is different from (1.6) because the operator $P(h)$ is h^2 times the original operator and thus we are considering $v_h = h^2 v$ instead of v.) To prove (4.2), we start by applying *complex scaling* as described in [Zw, Section 2.6]: using the holomorphy of u near infinity, we deform it to complex values of x. Namely, let $F(x)$ be a smooth function supported in $\{|x| \geq X\}$, equal to x for large values of x, and with $F'(x) \geq 0$ and $F'(x) > 0$ for $|x| > X$. Define the complex rescaled function

$$\tilde{u}(x) = \begin{cases} u(x), & |x| \leq X; \\ c_\pm e^{\pm i(1+h\sigma)(x+iF(x))/h}, & \pm x \geq X. \end{cases}$$

This is the result of first extending u to the complex plane $\mathbb{C}$ on $\{|x| > X\}$, and then restricting it to the contour $\Gamma = \{x + iF(x) \mid x \in \mathbb{R}\}$. Note that $\tilde{u}(x)$ is now exponentially decaying and thus lies in L^2 globally. We can deform the operator $P(h)$ into the complex as well (recalling that $V = 0$ when Γ is not on the real line) and then restrict it to Γ, to

[7] In fact, instead of the positive commutator method presented in Section 3 one often needs to use a closely related method of conjugating by the exponential of the escape function, for instance to get a logarithmic resonance free region [SjZw] or in certain estimates for mild trapping [WuZw], but we do not pursue this direction here.

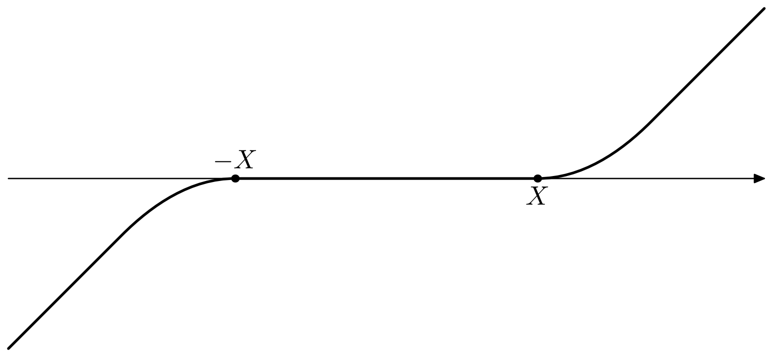

Figure 3. The contour used for complex scaling.

get

$$\widetilde{P}\widetilde{u}(x) = v_h,\ \widetilde{P} = \Big(\frac{hD_x}{1+iF'(x)}\Big)^2 + V(x) - (1+h\sigma)^2.$$

We can split $\widetilde{P}$ into two parts, roughly the real and the imaginary part :

$$\widetilde{P} = P_1 - iQ_1;$$
$$P_1 = \frac{1-F'(x)^2}{(1+F'(x)^2)^2}(hD_x)^2 + V(x) - 1 + O(h),$$
$$Q_1 = \frac{2F'(x)}{(1+F'(x)^2)^2}(hD_x)^2.$$

We see that both P_1 and Q_1 are semiclassical differential operators of order 2 of real principal type. In fact, the principal symbol of P_1 restricted to $[-X, X]$ is just

$$p(x,\xi) = \xi^2 + V(x) - 1,$$

and the principal symbol q_0 of Q_1 satisfies $q_0 \geq 0$ everywhere and $q_0 > 0$ on $\{|x| > X\}$. The nontrapping assumption in our situation in particular means that every bicharacteristic of p on $\{p = 0\}$ escapes to infinity as time goes to $-\infty$. In particular, this bicharacteristic reaches the elliptic set $\{q > 0\}$ of Q right after it crosses the lines $\{x = \pm X\}$; a little bit beyond these lines, the principal symbol of P_1 is still close to p. Therefore, we can find compactly supported functions $a(x,\xi)$ and $b(x,\xi)$ such that:

- $a \neq 0$ on $\{p^{-1}(0)\} \cap \{|x| \leq X\}$;
- $\operatorname{supp} b \subset \{|x| > X\}$;
- each backward bicharacteristic of the principal symbol of P_1 starting at $\operatorname{supp} a$ reaches $\{b \neq 0\}$.

We are now in position to apply Theorem 2; we get

$$\|A\tilde{u}\|_{L^2} \leq C(h^{-1}\|v_h\|_{L^2} + \|B\tilde{u}\|_{L^2}) + O(h^\infty)\|u\|_{L^2}.$$

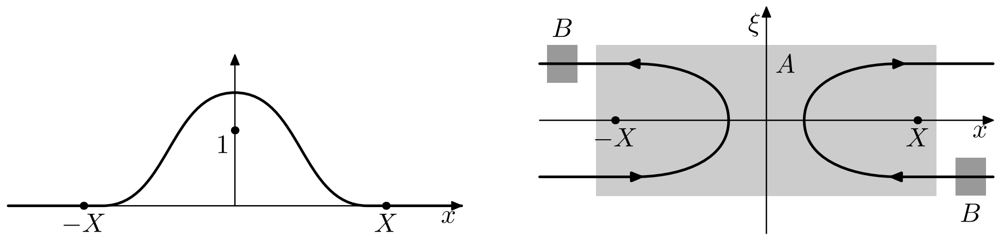

FIGURE 4. Left: an example of a potential $V(x)$ satisfying the nontrapping assumption. Right: the corresponding bicharacteristics, with the supports of a and b marked.

However, $\operatorname{supp} b$ lies inside the set $\{q > 0\}$; therefore, by Theorem 1

$$\|B\tilde{u}\|_{L^2} = O(h^\infty)\|u\|_{L^2}.$$

Combining these with a standard elliptic estimate away from $\{p - iq = 0\}$ (namely, a version of Theorem 1 allowing non-compactly supported symbols with estimates in semiclassical Sobolev classes), we get

$$\|\tilde{u}\|_{L^2} \leq Ch^{-1}\|v_h\|_{L^2},$$

which immediately translates to the required estimate (4.2).

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