Scattering by (some) rotating black holes

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September 20, 2010

Detecting black holes

A black hole is an object whose gravitational field is so strong that not even light can escape.

Since we cannot observe the electromagnetic radiation of black holes, how to detect them?

Indirect methods

- Use the effect of the gravitational field of the black hole on nearby objects, such as stars
- Do not provide accurate information about the parameters of the black hole, such as mass or angular momentum

We want to get more information about a particular black hole…

Gravitational waves

Theory

- Gravitational waves are perturbations of the curvature of the spacetime, caused by a major cosmic event, such as creation or merging of black holes
- Their frequencies, called quasi-normal modes, depend only on the black hole itself, not on the perturbation

Practice

- Indirect evidence that gravitational waves exist: Hulse–Taylor binary system (1993 Nobel Prize)
- Gravitational wave detectors: GEO 600, LIGO, MiniGRAIL, VIRGO, . . .

Laser Interferometer Gravitational-Wave Observatory

Quasi-normal modes

Quasi-normal modes (QNMs) are the frequencies of the gravitational waves emitted by a black hole.

Properties of QNMs

- They are complex numbers: real part = rate of oscillation, negative imaginary part = rate of exponential decay
- They characterize the black hole much like the electromagnetic spectrum characterizes a star

Benefits of computing QNMs and detecting gravitational waves

- Precise information about any particular black hole
- One more verification of general relativity

Mathematics of black holes

There are many works by physicists on quasi-normal modes; however, there have been only a handful of attempts to put these works on a mathematical foundation: Bachelot '91, Bachelot–Motet-Bachelot '93, Sá Barreto–Zworski '97, Bony–Häfner '07, Melrose–Sá Barreto–Vasy '08, ...

A black hole is represented as a Lorentzian metric on a 4D spacetime; gravitational waves (in the simplest case of scalar perturbations) are approximated by solutions to the wave equation

$$\Box u = 0$$

We study solutions to this equation for large time using scattering theory.

Scattering theory strategy

- Take the Fourier transform in time: $\Box u = 0$ becomes

$$P(\omega)\hat{u}(\omega) = f(\omega), \ \omega \in \mathbb{R}$$

  where $P(\omega)$ is a certain operator on the space slice, and f depends on the initial conditions.

- Prove the existence of a meromorphic family $R(\omega), \ \omega \in \mathbb{C}$, of operators on the space slice, such that

$$\hat{u}(\omega) = R(\omega)f(\omega).$$

  This family is called the scattering resolvent. It is a right inverse to $P(\omega)$, with outgoing boundary conditions.

Scattering theory strategy, continued

- Study the distribution of poles of $R(\omega)$, also known as resonances
- Use contour deformation and estimates on $R(\omega)$ in the nonphysical half-plane to obtain the asymptotic resonance decomposition as $t \to \infty$:

$$u(t,x) \sim \sum_j t^{k_j} e^{-it\omega_j} u_j(x)$$

  Here ω_j are resonances.
- Conclude that Quasi-Normal Modes = Resonances

Schwarzschild–de Sitter black hole

The scattering theory strategy has been implemented by Sá Barreto–Zworski and Bony–Häfner in the case of Schwarzschild–de Sitter metric, corresponding to a spherically symmetric black hole with positive cosmological constant.

Sá Barreto–Zworski '97

- Used the theorem of Mazzeo–Melrose '87 to construct the scattering resolvent $R(\omega)$
- Used semiclassical analysis and complex scaling to show that QNMs approximately lie on a lattice

Bony–Häfner '07

- Proved an estimate on $R(\omega)$ for bounded $\operatorname{Im}\omega$
- Established the resonance decomposition

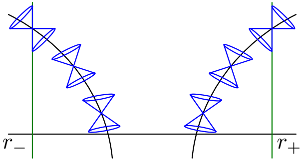
The next logical step after Schwarzschild–de Sitter is to study the rotating black hole given by the Kerr–de Sitter metric. This is the object of study of the presented research; the goals are:

- Construct the scattering resolvent and establish its connection to the wave equation
- Make the physicists' definitions of QNMs rigorous
- Study the asymptotic distribution of QNMs and compare it with the physicists' results
- Establish a resonance decomposition of linear waves

The paper [D '10] achieves the first two goals, and makes partial progress on the last one; namely, exponential local energy decay of solutions to the wave equation.

Kerr–de Sitter metric

$$g = -\rho^2\Big(\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta}\Big) - \frac{\Delta_\theta \sin^2\theta}{(1+\alpha)^2\rho^2}(a\,dt - (r^2+a^2)\,d\varphi)^2 + \frac{\Delta_r}{(1+\alpha)^2\rho^2}(dt - a\sin^2\theta\,d\varphi)^2.$$

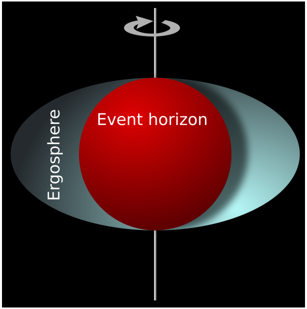


Two radial timelike geodesics, with light cones shown

Here a is the angular momentum; $\rho(r,\theta)$ and $\Delta_\theta(\theta)$ are nonzero functions, and $\Delta_r(r)$ is a fourth degree polynomial. The metric is defined on $\mathbb{R}_t \times (r_-, r_+) \times \mathbb{S}^2_{\theta,\varphi}$, where $r_\pm$ are two roots of the equation $\Delta_r = 0$. The surfaces $\{r = r_\pm\}$ are event horizons.

Features of the metric

- Positive cosmological constant
- Two asymptotically hyperbolic event horizons
- Stationary (∂_t is a Killing field)
- Invariant under axial rotation (∂_φ is a Killing field)
- The field ∂_t is not timelike inside the two ergospheres, located close to the event horizons. Inside the ergospheres, the operator $P(\omega)$ is not elliptic



Picture courtesy of Wikipedia.

Existence and exponential decay

Theorem

Fix a compact region $K \subset (r_-, r_+) \times \mathbb{S}^2$. If the angular momentum a is small enough, depending on K, then:

- $\mathbf{1}_K R(\omega) \mathbf{1}_K$, *where* $R(\omega)$ *is the scattering resolvent* $R(\omega)$, *is a meromorphic family of operators on* $\mathbb{C}$.
- *There are no resonances in* $\{\operatorname{Im}\omega \geq 0,\ \omega \neq 0\}$.
- *There is a resonance free strip* $\{-\nu < \operatorname{Im}\omega < 0\}$.
- *Any solution u to the wave equation with initial data in* $H_0^{3/2+\varepsilon}(K) \oplus H_0^{1/2+\varepsilon}(K)$ *and orthogonal to the resonant state at zero has* $\|u\|_{L^2(K)} \leq Ce^{-\nu t}$ *as* $t \to +\infty$.

The presence of the compact set K can be interpreted as construction of the resolvent away from the ergospheres.

Decay on black holes

There are numerous results on decay of linear waves on both spherically symmetric and rotating black holes:
Bony–Häfner '07, '10, Dafermos–Rodnianski '07, '08, '09, Donninger–Schlag–Soffer '09, Finster–Kamran–Smoller–Yau '09, Marzuola–Metcalfe–Tataru–Tohaneanu '08, Tataru '09, Tataru–Tohaneanu '08...

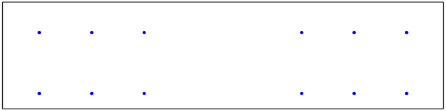
However, most of these results deal with the case of zero cosmological constant, when there is an asymptotically flat infinity. In this case, the global meromorphy of the scattering resolvent $R(\omega)$ is unlikely, and the rate of decay is only polynomial in time.

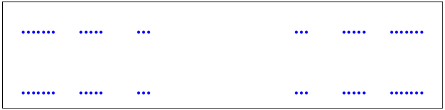
Distribution of resonances (work in progress)

- $a = 0$: QNMs lie asymptotically on a lattice [Sá Ba–Zw]

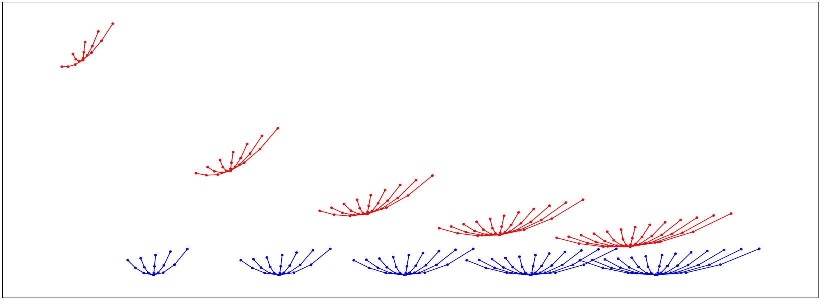
$$\omega \sim [\pm(l+1/2) - i(m+1/2)]\frac{\sqrt{1-9\Lambda M^2}}{3\sqrt{3}M}; \; l, m = 0, 1 \ldots \quad (1)$$

  Spherical symmetry $\rightarrow$ each QNM has multiplicity $2l+1$
- $a \neq 0$: analogue of the Zeeman effect: each QNM in (1) splits into $2l+1$ QNMs, each corresponding to its own value of the φ-angular momentum in the range $-l, \ldots, l$.





Comparison with physicists (work in progress)



For $\Lambda = 0$, $l = 2, \ldots, 6$, $m = 0$, $a = 0, 0.1, 0.2, 0.3$, we compare our first degree approximation of QNMs, given by a certain Bohr–Sommerfeld condition, with QNMs computed in by Berti–Cardoso–Starinets (see `http://phy.olemiss.edu/~berti/qnms.html`). Each line on the graph displays the QNMs for fixed angular momentum and varying a.

Ingredients

- Instead of Mazzeo–Melrose theorem, use Teukolsky separation of variables and a customized version of complex scaling
- Obtain resolvent estimates in the low energy regime and use them to prove the meromorphy of the resolvent
- Normally hyperbolic trapping $\rightarrow$ use the result of Wunsch–Zworski '10 to get a resonance free strip

Problems

- The angular operator given by the separation of variables is nonselfadjoint and depends on ω
- Complex scaling fails at low energy $\rightarrow$ use analyticity to get boundary conditions away from the event horizons

Separation of variables

The operator $P(\omega)$ is invariant under the axial rotation $\varphi \mapsto \varphi + s$. Take $k \in \mathbb{Z}$ and let $\mathcal{D}'_k = \mathrm{Ker}(D_\varphi - k)$ be the space of functions with angular momentum k; then

$$\rho^2 P(\omega)|_{\mathcal{D}'_k} = P_r(\omega, k) + P_\theta(\omega)|_{\mathcal{D}'_k},$$

where P_r is a differential operator in r and P_θ is a differential operator on $\mathbb{S}^2$. For $a = 0$, P_θ is independent of ω and is just the negative Laplace–Betrami operator on the round sphere.

$$P_r(\omega, k) = D_r(\Delta_r D_r) - \frac{(1+\alpha)^2}{\Delta_r}((r^2 + a^2)\omega - ak)^2,$$

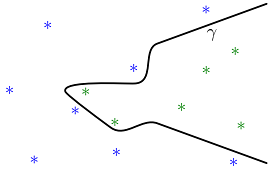
$$P_\theta(\omega) = \frac{1}{\sin\theta} D_\theta(\Delta_\theta \sin\theta D_\theta) + \frac{(1+\alpha)^2}{\Delta_\theta \sin^2\theta}(a\omega\sin^2\theta - D_\varphi)^2.$$

Separation of variables, continued

We need to invert the operator $\rho^2 P(\omega)|_{\mathcal{D}'_k} = P_r(\omega, k) + P_\theta(\omega)|_{\mathcal{D}'_k}$

Problems for $a \neq 0$

- P_θ is not self-adjoint $\rightarrow$ complete system of eigenfunctions?
- P_θ depends on ω



Solution

For each $\lambda \in \mathbb{C}$, construct $R_r(\omega, k, \lambda) = (P_r(\omega, k) + \lambda)^{-1}$ and $R_\theta(\omega, \lambda) = (P_\theta(\omega) - \lambda)^{-1}$ and write

$$R(\omega)|_{\mathcal{D}'_k} = \frac{1}{2\pi i} \int_\gamma R_r(\omega, k, \lambda) \otimes R_\theta(\omega, \lambda)|_{\mathcal{D}'_k} \, d\lambda.$$

Here γ is a contour separating the poles of R_r from those of R_θ.

Nonstandard complex contour deformation

After a Regge–Wheeler change of variables $r \to x$ mapping $r_\pm \mapsto \pm\infty$, the operator $P_r + \lambda$ is roughly equivalent to

$$P_x = D_x^2 + V(x; \omega, \lambda, k), \ \ V(x) \sim (\omega - ak)^2 - \lambda e^{\mp x}, \ \pm x \gg 1.$$

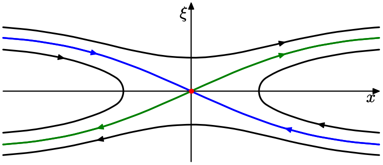
We need to study the scattering problem for P_x in the low energy regime $|\lambda| \gg |\omega|^2 + |ak|^2$.

- Standard complex scaling fails (no ellipticity near $x = \pm\infty$).
- Let u be an outgoing solution to $P_x u = f \in L^2_{\mathrm{comp}}$; extend it analytically to a neighborhood of $\mathbb{R}$ in $\mathbb{C}$.
- Use semiclassical analysis on two circles to get control on u at two distant, but fixed, points $z_\pm \in \mathbb{C}$.
- Formulate a BVP for the restriction of u to a certain contour between z_- and z_+, and get $\|u\| \lesssim |\lambda|^{-1}\|f\|$.

Trapping

The trapping in our situation is normally hyperbolic. It features:

- incoming tail (codim=1)
- outgoing tail (codim=1)
- trapped set (codim=2)



Example of normally hyperbolic trapping in 1D

Wunsch–Zworski '10: for normally hyperbolic trapping and under suitable assumptions at the boundary, there is a resonance free strip, with a polynomial resolvent estimate.
We use the method of Wunsch–Zworski together with complex scaling to get the resonance free strip and exponential local energy decay in our case.

Thank you for your attention!