PSEUDODIFFERENTIAL OPERATORS AND ELLIPTIC REGULARITY

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In this talk, we will use the algebra of **pseudodifferential operators** in one of its basic applications, namely to prove the following **elliptic regularity** result:

Theorem 1. *Let M be a compact manifold, E and F be two (smooth) vector bundles over M, and $P : C^\infty(M;E) \to C^\infty(M;F)$ be a differential operator (with smooth coefficients) of order n. Assume that P is* ***elliptic****, in the sense defined in the next section. Then:*

1. The operator P (or rather its relevant extension) is Fredholm $W_2^n(M;E) \to L^2(M;F)$. Here W_2^n is a Sobolev space (introduced below). (The question of the index of P is extremely interesting, but falls outside of this talk.)

2. The kernel of P consists of smooth sections.

3. For any $f \in L^2(M;F)$, there exists $u \in W_2^n(M;E)$ such that $Pu - f$ is smooth.

An example of an elliptic operator would be the Laplace-Beltrami operator on a Riemannian manifold (see the end of the next section).

This theorem follows immediately from the following fact:

Theorem 2. *Under the conditions of the above theorem, there exists a (non-differential) operator Q on M, called the* ***parametrix****, with the following properties:*

1. Q is bounded $L^2(M;F) \to W_2^n(M;E)$.

2. The restriction of Q to $C^\infty(M;F)$ is continuous $C^\infty(M;F) \to C^\infty(M;E)$ (in the naturally defined topologies, not in the restriction topologies).

3. We have $PQ = 1 - K_1$ and $QP = 1 - K_2$, where K_j are smoothing operators; i.e., they are continuous $L^2(M;F) \to C^\infty(M;F)$ and $W_2^n(M;E) \to C^\infty(M;E)$, respectively.

Indeed, smoothing operators are compact (for example, by Arzelà-Ascoli theorem; note, however, that not every compact operator is smoothing); therefore, $P : W_2^n(M;E) \to L^2(M;F)$ has an almost inverse and thus is Fredholm. Now, assume that $Pu = 0$; then, $u = K_2 u \in C^\infty$. Finally, if $f \in L^2(M;F)$ and $u = Qf$, then $K_1 f = f - Pu$ is smooth. (Note that P is a parametrix for Q; therefore, an analogue of Theorem 1 holds for Q. Also, the above is still true if we replace $P : W_2^n \to L^2$ by $P : W_2^s \to W_2^{s-n}$ for any real s.)

The operator Q lies in the class of **pseudodifferential operators**. Before we can define these, however, let us review differential operators.

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1. Differential operators

Let M be a manifold, E and F be vector bundles over M, and denote by $\operatorname{Diff}^n(M;E,F)$ the space of differential operators of order at most n acting $C^\infty(M;E)\to C^\infty(M;F)$. Denote by $\operatorname{Diff}(M)$ the category whose objects are vector bundles over M and whose morphisms are differential operators (possibly of infinite order) between sections of corresponding bundles; then this category is filtered by the order of the operator (which can be given by Grothendieck's definition). We will mostly concentrate on operators of finite order. Let us recall some properties of this category:

(Local) If $u\in C^\infty(M;E)$ and $P\in\operatorname{Diff}^n(M;E,F)$, then $\operatorname{supp}(Pu)\subset\operatorname{supp}u$;

(Symb) Let $S^n_P(M;E,F)$ be the space of all smooth sections of the pullback of $\operatorname{Hom}(E,F)$ under the projection map $T^*M\to M$ that are polynomials of degree $\leq n$ on each fiber. These spaces together form a filtered category Symb_P, whose objects are again vector bundles over M. Then there exists a (canonically defined) linear isomorphism

$$\sigma_n:\operatorname{Diff}^n(M;E,F)/\operatorname{Diff}^{n-1}(M;E,F)\to S^n_P(M;E,F)/S^{n-1}_P(M;E,F).$$

(Note that the quotient in the right-hand side is just the space of homogeneous polynomials of degree n.) The map σ_n is called the **symbol map of order** n. It is functorial in the sense that, if $A\in\operatorname{Diff}^n(M;E,F)$ and $B\in\operatorname{Diff}^m(M;F,G)$, then $\sigma_{n+m}(BA)=\sigma_m(B)\sigma_n(A)$. (In other words, the associated graded categories of Diff and Symb_P are canonically equivalent.

There are some naturally arising modules over differential operators. Namely, let LCVec be the category of locally convex Hausdorff topological vector spaces. Then one can define the functors C^∞ and C^∞_0 from the category $\operatorname{Diff}(M)$ to LCVec in the following way. For a vector bundle E over M, let $C^\infty(M;E)$ be the space of all smooth sections of E and let $C^\infty_0(M;E)$ be the space of smooth sections of E with compact support. Each differential operator is mapped to the corresponding linear operator acting on the space C^∞ or C^∞_0.

Assume that $M=\mathbb{R}^n$ and E is the trivial one-dimensional bundle. Then any operator $A\in\operatorname{Diff}^n(M;E)$ can be written as

$$A=\sum_{|\alpha|\leq n}a_\alpha(x)D_x^\alpha,$$

where $D=\frac{1}{i}\partial$. The principal symbol is then given by

$$\sigma(A)(x,\xi)=\sum_{|\alpha|=n}a_\alpha(x)\xi^\alpha.$$

We use the principal symbol instead of the complete symbol

$$a(x,\xi)=\sum_\alpha a_\alpha(x)\xi^\alpha$$

because the former is invariant under changes of variables as a function on the cotangent bundle, while the latter is not. The use of the cotangent bundle can be explained if we let E and F be trivial and assume that X is a vector field on M; then $X \in \mathrm{Diff}^1$ and, if $(p, v) \in T^*M$ with $p \in M$ and $v \in T_p^*M$, then $\sigma(X)(p, v) = \langle v, X_p \rangle$.

The use of D instead of ∂ is due to the following expression of differential operators using the Fourier transform. Recall that for u in the class $\mathcal{S}$ of Schwartz functions on $\mathbb{R}^n$ (i.e., functions all of whose derivatives decay faster than any power of x), the Fourier transform

$$\hat{u}(\xi) = \int e^{-i(x,\xi)} u(x)\, dx$$

also lies in $\mathcal{S}(\mathbb{R}^n)$ and we have the Fourier inversion formula

$$u(x) = (2\pi)^{-n} \int e^{i(x,\xi)} \hat{u}(\xi)\, d\xi.$$

Also, one can differentiate under the integral sign to get

$$D^\alpha u(x) = (2\pi)^{-n} \int e^{i(x,\xi)} \xi^\alpha \hat{u}(\xi)\, d\xi.$$

for any multiindex α. Then, if A is a differential operator and a is its complete symbol, we have

$$Au(x) = (2\pi)^{-n} \int e^{i(x,\xi)} a(x,\xi) \hat{u}(\xi)\, d\xi,\ u \in \mathcal{S}(\mathbb{R}^n). \tag{1.1}$$

(Another advantages of using D instead of ∂ include the principal symbol of the adjoint operator with respect to the *Hermitian* inner product being equal to the symbol of the original operator and positive differential operators having positive symbols.)

The notion of the symbol makes it possible to talk about **elliptic operators**. Namely, if $\sigma(A)$ is the principal symbol of the operator A, then A is elliptic if and only if for every nonzero p in the total space of T^*M, the homomorphism $\sigma(p)$ (of the fibers of E and F at the corresponding point of M) is invertible.

The basic example of an elliptic operator is the **Laplace's operator**

$$\Delta = \sum_{i=1}^n \partial_{x_i}^2$$

on $\mathbb{R}^n$ (with the trivial line bundles); its symbol is $\sigma(x, \xi) = -|\xi|^2$. A more general example is the Laplace-Beltrami operator (on functions) on a Riemannian manifold T^*M; its principal symbol maps each cotangent vector to negative the square of its length. However, if our manifold is, say, Lorentzian (the metric has signature $(n-1, 1)$), then the corresponding d'Alembert-Beltrami operator will not be elliptic. This can be seen on the **wave operator**

$$\Box = -\partial_{x_1}^2 + \sum_{i=2}^n \partial_{x_i}^2.$$

2. Distributions and Schwartz kernels

In this section, we define **distributions**, or **generalized functions**, and state some of their properties. For detailed information on distributions, the reader is referred to [Hö I] or [F-J].

The spaces $C^\infty(M;E)$ and $C_0^\infty(M;E)$ introduced in the previous sections are modules over differential operators. However, the natural topology of these spaces is extremely strong and the structure of their space is quite complicated (they only form a Fréchet spaces, not Banach or Hilbert ones). This is one of the reasons why it is often desirable to apply differentiation to more general classes of sections. The space of **distributions** is in a sense the largest space we would like to consider when studying linear differential equations; it contains most of other functional spaces as well as objects that are not sections, such as the delta density.

Let Dens be the line bundle of densities over M. For any $u \in C_0^\infty(M;\mathrm{Dens})$, we can define the integral

$$\int_M u \in \mathbb{C}.$$

Now, let E be a vector bundle over M and E^* be the dual bundle. If $u \in C^\infty(M;E)$ and $\phi \in C_0^\infty(M;\mathrm{Dens}\otimes E^*)$, then we can define $\phi u \in C_0^\infty(M;\mathrm{Dens})$ and

$$\langle u,\phi\rangle = \int_M \phi u.$$

So, if we define the space of distributions $\mathcal{D}'(M;E)$ as the space dual to $C_0^\infty(M;\mathrm{Dens}\otimes E^*)$ (where the latter is equipped with certain non-metrizable topology), then the formula above gives a canonical embedding of $C^\infty(M;E)$ into $\mathcal{D}'(M;E)$. (On $\mathcal{D}'$, we will always consider the weak topology.) More generally, one can take u to be any locally integrable section. (In this talk, by 'locally integrable sections' we actually mean equivalence classes of locally integrable section by the relation of coinciding almost everywhere; otherwise, we would not have an embedding.) Note that the dual to $C_0^\infty(M)$ will be $\mathcal{D}'(M;\mathrm{Dens})$.

However, the space $\mathcal{D}'$ has elements that are not given by sections. An example is the following delta density. Fix $p \in M$ and define $\delta \in \mathcal{D}'(M;\mathrm{Dens})$ by

$$\langle \delta,\phi\rangle = \phi(p),\ \ \phi \in C_0^\infty(M).$$

Indeed, if δ was given by some section u on M, then, since $\langle\delta,\phi\rangle = 0$ for $\phi \in C_0^\infty(M\setminus p)$, we would have $u = 0$ almost everywhere on $M \setminus p$, which would yield $u \equiv 0$. However, with all the variety of distributions possible, the space C^∞ is still dense in $\mathcal{D}'$ (with weak topology on the latter).

Let us mention several important operations on distributions. Most of them are defined by duality. First of all, if $U \subset M$ is open, then the natural embedding $C_0^\infty(U) \to C_0^\infty(M)$ induces the projection operator $\mathcal{D}'(M) \to \mathcal{D}'(U)$. Using a partition of unity, one can show that distributions form a sheaf.

Next, let $P \in \mathrm{Diff}(M; E, F)$ be a differential operator. Then there exists the adjoint differential operator $P^t \in \mathrm{Diff}(M; \mathrm{Dens} \otimes F^*, \mathrm{Dens} \otimes E^*)$ such that for $u \in C^\infty(M; E)$ and $v \in C_0^\infty(M; \mathrm{Dens} \otimes F^*)$, we have

$$\langle v, Pu \rangle = \langle P^t v, u \rangle.$$

For example, if P is the multiplication operator by a smooth function, then P^t is the multiplication by the same function. On the other hand, if, for example, $M = \mathbb{R}^n$, the bundle Dens is trivialized using the standard volume form, and $P = \partial/\partial x_i$, then one can use integration by parts to show that $P^t = -\partial/\partial x_i$. Now, we use the adjoint operator to extend P to an operator $\mathcal{D}'(M; E) \to \mathcal{D}'(M; F)$ by the rule

$$\langle Pu, \phi \rangle = \langle u, P^t \phi \rangle,\ u \in \mathcal{D}'(M; E),\ \phi \in C_0^\infty(M; \mathrm{Dens} \otimes F^*).$$

This definition makes the space of distributions into a module over the algebra of differential operators (in contrast with, say, sections with a fixed number of derivatives, which can only be differentiated finitely many times).

One can define the **support** of a distribution $u \in \mathcal{D}'(M; E)$ as the minimal closed set $\mathrm{supp}\, u \subset M$ such that $u|_{M \setminus \mathrm{supp}\, u} = 0$. The space $\mathcal{E}'(M; E)$ of distributions with compact support is dual to $C^\infty(M; \mathrm{Dens} \otimes E^*)$. Also, one can define the **singular support** as the minimal closed set $\mathrm{sing\,supp}\, u \subset M$ such that $u|_{M \setminus \mathrm{sing\,supp}\, u} \in C^\infty$.

The construction above can be reformulated as follows. Consider the contravariant functor Adj on $\mathrm{Diff}(M)$ mapping each vector bundle E to the bundle $\mathrm{Dens} \otimes E^*$ and every differential operator to the adjoint operator. Also, let Dual be the contravariant functor on LCVec mapping every space to its dual (with weak topology). Then we can define functors $\mathcal{D}'(M)$ and $\mathcal{E}'(M)$ from $\mathrm{Diff}(M)$ to LCVec as follows:

$$\mathcal{D}'(M) = \mathrm{Dual} \circ C_0^\infty \circ \mathrm{Adj},$$
$$\mathcal{E}'(M) = \mathrm{Dual} \circ C^\infty \circ \mathrm{Adj}.$$

Next, we wish to study operators on manifolds. Let E and F be two vector bundles over M and consider a continuous operator $A : C_0^\infty(M; E) \to \mathcal{D}'(M; F)$. This is as little regularity as we can require from an operator. The Schwartz kernel theorem states that each such operator has a unique **Schwartz kernel** $K \in \mathcal{D}'(M \times M; F \times (\mathrm{Dens} \otimes E^*))$ such that

$$\langle Au, v \rangle = \langle K, v \otimes u \rangle,\ u \in C_0^\infty(M; E),\ v \in C_0^\infty(M; \mathrm{Dens} \otimes F^*).$$

Also, it is easy to see that every K as above defines an operator A. If $M = \mathbb{R}^n$, $E = F = \mathbb{R}$, and K is a function, then A is just an integral operator:

$$Au(x) = \int K(x, y) u(y).$$

The adjoint operator A^t has the kernel $K^t(x, y) = K(y, x)$ under the identification

$$\mathrm{Hom}(\mathrm{Dens} \otimes F^*, \mathrm{Dens} \otimes E^*) \simeq \mathrm{Hom}(E, F).$$

If A^t is continuous $C_0^\infty \to C^\infty$ (meaning that the image of A^t lies in C^∞ and A^t is continuous with the C^∞ topology on the target, which is stronger than the restriction of the $\mathcal{D}'$ topology to C^∞), then one can use duality to extend A to an operator $\mathcal{E}' \to \mathcal{D}'$. If A acts both $C_0^\infty \to C^\infty$ and $\mathcal{E}' \to \mathcal{D}'$, then it is called **regular**. Also, let us say that an operator A is **properly supported** if the support $\operatorname{supp} K$ of its Schwartz kernel has the following property: both projections $\pi_1, \pi_2 : M \times M \to M$ are proper when restricted to $\operatorname{supp} K$. A properly supported operator acts $C_0^\infty \to \mathcal{E}'$ and can be extended to act $C^\infty \to \mathcal{D}'$. We will see below that all differential operators are both regular and properly supported.

One reason why regular properly supported operators are important is because we can multiply them. More specifically, one can extend $\operatorname{Diff}(M)$ to the category $\operatorname{RegPS}(M)$ whose objects are vector bundles and morphisms are regular properly supported operators, and the functors C^∞, C_0^∞, $\mathcal{D}'$, and $\mathcal{E}'$ can be extended to act from this new category. Also, the functor Adj can be extended from $\operatorname{Diff}(M)$ to $\operatorname{RegPS}(M)$. This category has an ideal of **smoothing operators**; i.e., operators that can be extended to act continuously $\mathcal{E}' \to C^\infty$ (or rather, of properly supported smoothing operators); It can be proved that an operator A is smoothing if and only if its Schwartz kernel K is smooth.

If A is the identity operator on $C^\infty(M; E)$, then its kernel is given by the distribution $\delta(x-y) \in \mathcal{D}'(M \times M; E \times (\operatorname{Dens} \otimes E))$:

$$\langle \delta(x-y), \phi(x,y) \rangle = \int \operatorname{tr}(\phi(x,x)),\ \phi \in C^\infty(M \times M; (\operatorname{Dens} \otimes E^*) \times E).$$

Here $\operatorname{tr} : E^* \otimes E \to \mathbb{C}$ is the trace morphism. If A is a differential operator, then one can obtain its kernel by applying A in the x variable to $\delta(x-y)$. It is important to note that for a differential operator A, the support of its kernel K lies on the diagonal $\Delta_M \subset M \times M$. In fact, the following three facts are equivalent: (a) $\operatorname{supp} K \subset \Delta_M$ (b) $\operatorname{supp} Au \subset \operatorname{supp} u$ for any u (c) A is a differential operator, possibly of infinite order.

Finally, let us introduce Sobolev spaces. Let L^2_{loc} be the space of (equivalency classes of) sections that are locally in L^2. This space is embedded into $\mathcal{D}'$. If $s \geq 0$ is an integer, then we can define the Sobolev space $W^s_{2,\mathrm{loc}}$ as the space of all (equivalency classes of) sections $u \in L^2_{\mathrm{loc}}$ such that for any differential operator A of order no more than s, we have $Au \in L^2_{\mathrm{loc}}$. Similarly, one can define the spaces L^2_{comp} and $W^s_{2,\mathrm{comp}}$ containing sections with compact support. If M is compact, then $W^s_{2,\mathrm{loc}} = W^s_{2,\mathrm{comp}} = W^s_2$ is a Hilbert space. One can actually define the spaces above when s is any real number, and we have

$$\bigcap_s W^s_{\mathrm{loc}} = C^\infty,$$
$$\bigcup_s W^s_{\mathrm{loc}} = \mathcal{D}'.$$

3. Pseudodifferential calculus

In this section, we list certain properties of pseudodifferential operators (see, for example, [Hö III] or [G-S] for the proofs). Before we do that, however, we need to define the appropriate generalization of the symbol space S^n_P introduced for differential operators.

For $\lambda > 0$ let, τ_λ be the multiplication map by λ on the vector bundle T^*M. We say that a differential operator $A \in \mathrm{Diff}^n(T^*M \setminus 0)$ is **positively homogeneous** if $A\tau^*_\lambda = \tau^*_\lambda A$ for any $\lambda > 0$. Then we define the symbol space $S^s(M)$ as consisting of functions that have order of growth at most $|\xi|^s$ as $\xi \to \infty$ over compact subsets of M after applying any positively homogeneous differential operator. (Clearly, this definition applies to sections of the pull-back of any vector bundle under the projection map $T^*M \to M$.) It is easy to see that if a is a smooth section that is positively homogeneous of degree s (i.e., $\tau^*_\lambda a = \lambda^s a$ for any $\lambda > 0$), then $a \in S^s$.

For $M = \mathbb{R}^n$, a positively homogeneous operator has the form $\sum b_{\alpha\beta}(x,\xi)\partial^\alpha_x \partial^\beta_\xi$, where $b_{\alpha\beta}$ is homogeneous of degree $|\beta|$ in ξ. Therefore, we have

$$S^s(M;\mathbb{R}) = \{a(x,\xi) \in C^\infty(T^*M) \mid \forall \alpha,\beta,K\ \exists C_{\alpha\beta K} : |\partial^\alpha_x \partial^\beta_\xi a(x,\xi)| \le C_{\alpha\beta K}(1+|\xi|)^{s-|\beta|}, x \in K\}.$$

Here α and β are multiindices and $K \subset M$ is compact. The optimal constants $C_{\alpha\beta K}$ form a family of seminorms on $S^s(M)$ which make it into a Fréchet space. We see from the definition that $S^s(M)$ consists of functions that grow like $\langle\xi\rangle^s$, and their order of growth stays the same when differentiating in x and decreases by 1 when differentiating in ξ. In particular, if s is a nonnegative integer, then $S^n_P(M)$ embeds into $S^s(M)$ (which has already been proven above, since a polynomial is a sum of positively homogeneous functions).

Note that $S^s(M;E,F)$ form a commutative filtered algebra if $E = F = \mathbb{R}$. One can also consider the space $S^{-\infty}(M) = \bigcap_s S^s(M)$ consisting of symbols decaying rapidly in ξ with all their derivatives.

We are now ready to introduce pseudodifferential operators. First, let $M = \mathbb{R}^n$ and $E = \mathbb{R}$. In analogy with (1.1), for $a(x,\xi) \in S^s(M)$ define the operator $a(x,D)$ by the formula

$$a(x,D)u(x) = (2\pi)^{-n} \int e^{i(x,\xi)} a(x,\xi)\hat{u}(\xi)\, d\xi, u \in C^\infty_0(\mathbb{R}^n).$$

This operator is not properly supported, but one can show that the singular support of its Schwartz kernel lies on the diagonal and therefore it can be represented as the sum of a smoothing and a properly supported operator. A detailed study of the construction above, including how it behaves under changes of variables, yields the following abstract construction:

Let E and F be vector bundles over M. For every $s \in \mathbb{R}$, there is a class $\Psi^s(M;E,F)$ of regular properly supported operators $E \to F$, called **pseudodifferential operators of order** s, with the following properties:

(Alg) The product of an element of Ψ^s and an element of Ψ^t lies in Ψ^{s+t}. In other words, there is a subcategory Ψ of the category RegPS consisting of pseudodifferential operators. (Equivalently, the category RegPS is filtered by s; it should be noted, however, that there are significantly more operators in RegPS than those covered by the filtration.)

(Adj) The adjoints of operators in $\Psi^s(M;E,F)$ lie in $\Psi^s(M;F^*\otimes\mathrm{Dens},E^*\otimes\mathrm{Dens})$; so, one can define the contravariant function Adj of $\Psi(M)$. (Equivalently, Adj respects the filtration of RegPS introduced above.)

(Smooth) The intersection $\Psi^{-\infty}(M;E,F)=\bigcap_s\Psi^s(M;E,F)$ is exactly the ideal of (properly supported) smoothing operators.

(Supp) The singular support of the Schwartz kernel of every $A\in\Psi^s(M;E,F)$ lies on the diagonal $\Delta_M\subset M\times M$. Equivalently, if $u\in\mathcal{D}'(M;E)$, then $\operatorname{sing\,supp}(Au)\subset\operatorname{sing\,supp}u$.

(Sob) If $W^t_{2,\mathrm{loc}}(M;E)$ is the space of distributions locally in the Sobolev class t (which can in fact be defined for any real t), then any $A\in\Psi^s(M;E,F)$ is bounded $W^t_{2,\mathrm{loc}}(M;E)\to W^{t-s}_{2,\mathrm{loc}}(M;F)$.

(Asymp) Assume that $A_j\in\Psi^{s_j}(M;E,F)$, $j\geq 0$, where the sequence s_j is monotonely decreasing and converges to $-\infty$. Then there exists an operator $A\in\Psi^{s_0}(M;E,F)$ that is an **asymptotic sum** of the family of A_j in the following sense:

$$\forall k,\ A-\sum_{j<k}A_j\in\Psi^{s_k}(M;E,F).$$

We denote $A\sim\sum_j A_j$. Such an operator A is unique modulo $\Psi^{-\infty}$.

(Symb) For every $s\in\mathbb{R}$, there exists a canonical principal symbol map

$$\sigma_s:\Psi^s(M;E,F)/\Psi^{s-1}(M;E,F)\to S^s(M;E,F)/S^{s-1}(M;E,F)$$

This map is a linear isomorphism for every s and it is functorial in the sense that for $A\in\Psi^s(M;E,F)$ and $B\in\Psi^t(M;F,G)$, we have $\sigma_{t+s}(BA)=\sigma_t(B)\sigma_s(A)$. Also, $\sigma(A^t)$ equals $\sigma(A)\circ(-1)$ under the identification $\mathrm{Hom}(E,F)\simeq\mathrm{Hom}(\mathrm{Dens}\otimes F^*,\mathrm{Dens}\otimes E^*)$, where (-1) is the antipodal map on T^*M (given by the vector bundle structure).

(Diff) If s is a nonnegative integer, then $\mathrm{Diff}^s(M;E,F)$ embeds into $\Psi^s(M;E,F)$ and the symbol maps agree.

Armed with these facts, we can perform the parametrix construction. Assume that $P\in\mathrm{Diff}^n(M;E,F)$ is elliptic and let $p\in S^n(M;E,F)$ be a representative of $\sigma_n(P)$. Then one can find a symbol $q_0\in S^{-n}(M;F,E)$ such that $pq_0-1,q_0p-1\in S^{-\infty}$. In fact, one can put $q_0=p^{-1}$ outside of a compact neighborhood of the zero section and use a cutoff function to continue it to the whole T^*M. (The fact that q_0 is a symbol can be verified by a straightforward calculation.) In fact, a symbol p is elliptic if and only if it is invertible modulo $S^{-\infty}$ (or even $S^{-\varepsilon}$ for $\varepsilon>0$); however, the definition of ellipticity given in the beginning of the talk only works for polynomial symbols.

Now, take $Q_0 \in \Psi^{-n}(M; F, E)$ with principal symbol q_0. Then

$$R = 1 - PQ_0$$

has zero principal symbol and thus belongs to $\Psi^{-1}(M; F, F)$. Now, we can take the asymptotic sum

$$T \sim \sum_{j \geq 0} R^j;$$

it can be seen immediately that

$$(1 - R)T = 1 \mod \Psi^{-\infty}.$$

Therefore, $Q = Q_0 T$ is a right inverse to T modulo $\Psi^{-\infty}$. Similarly, we get a left inverse. Then Q is a parametrix.

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