Hörmander-Kashiwara and Maslov indices

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Abstract

Given a finite-dimensional symplectic vector space, we first study the fundamentals group of the space of symplectomorphisms and the space of Lagrangians to construct Maslov indices of loops in these spaces. Then we introduce Hörmander-Kashiwara index of a Lagrangian triple and use it to define Maslov indices counting the number of intersections of two paths of Lagrangians or symplectomorphisms. This paper largely follows [1], with more emphasis on the construction of Maslov and Hörmander-Kashiwara indices and proving their properties.

1 Lagrangians, complex structures, and symplectomorphisms

Recall that a **symplectic vector space** is a finite-dimensional vector space E together with a bilinear, skew-symmetric form $\omega : E \times E \to \mathbb{R}$ such that the kernel

$$\ker \omega = \{x \in E \mid \omega(x, y) = 0 \text{ for all } y \in E\}$$

is trivial. If $(E_j, \omega_j)_{j=1,2}$ are two symplectic vector spaces, then a **symplectomorphism** is a linear isomorphism $T : E_1 \to E_2$ such that $\omega_1 = T^*\omega_2$, meaning that $\omega_1(x, y) = \omega_2(Tx, Ty)$ for all $x, y \in E_1$. We denote by $\mathrm{Sp}(E)$ the group of all symplectomorphisms $E \to E$.

Note that ω can also be regarded as a linear map $\omega^* : E \to E^*$ given by the formula

$$\langle \omega^*(x), y\rangle = \omega(x, y),\ \ x, y \in E.$$

The triviality of $\ker \omega$ means that ω^* is a linear isomorphism.

Given a subspace $L \subset E$, we may define its 'orthogonal complement' with respect to ω by

$$L^\omega = \{x \in E \mid \omega(x, y) = 0 \text{ for all } y \in L\}.$$

We see immediately that $L \subset (L^\omega)^\omega$. On the other hand, L^ω is the preimage under ω^* of the annihilator of L; therefore,

$$\dim L^\omega = \dim E - \dim L,\ \ (L^\omega)^\omega = L.$$

We say that L is **isotropic** if $L \subset L^\omega$ and **Lagrangian** if $L = L^\omega$. We denote by $\mathrm{Lag}(E)$ the set of all Lagrangian subspaces of E. In studying these, we will often use the following

Lemma 1. *Suppose that $M_1, \ldots, M_n \subset E$ is a finite (possibly empty) set of isotropic subspaces and that L_0 is an isotropic subspace of E that is transversal to each M_j (henceforth we say that two subspaces of E are* ***transversal*** *if they have trivial intersection). Then there exists $L \in Lag(E)$ containing L_0 and transversal to each M_j.*

Proof. Since E is finite-dimensional, there exists a maximal isotropic subspace $L \subset E$ containing L_0 and transversal to each M_j. We claim that it is Lagrangian. Indeed, suppose that $L \neq L^\omega$. Consider the subspaces $L_j = L + (M_j \cap L^\omega) \subset L^\omega$; since both L and M_j are isotropic, L_j is isotropic as well. If $L_j = L^\omega$ for some j, then $L^\omega = L_j \subset L_j^\omega = (L^\omega)^\omega = L$, which is false. Therefore, each L_j is a proper subspace of L^ω and we can find a vector $x \in L^\omega$ such that $x \notin L_j$, $x \notin L$. Then the space $L \oplus \mathbb{R}x$ is an isotropic subspace properly containing L and transversal to each M_j, a contradiction. □

In particular, there exists at least one Lagrangian subspace $L \subset E$; together with the formula for the dimension above it yields that $\dim E = 2 \dim L$ must be even. We may use Lagrangian subspaces to prove that all symplectic spaces of given dimension are symplectomorphic:

Proposition 1. *1. Let L be any finite-dimensional vector space. Then the formula*

$$\omega_L((x, \alpha), (y, \beta)) = \langle \alpha, y \rangle - \langle \beta, x \rangle$$

defines a symplectic form ω_L on $L \oplus L^*$. The subspaces L and L^* are Lagrangian.

2. If $T : L \to M$ is a linear isomorphism, then $T \oplus T^{-*} : (L \oplus L^*, \omega_L) \to (M \oplus M^*, \omega_M)$ is a symplectomorphism. (Henceforth we denote $T^{-*} = (T^*)^{-1}$.)

3. Suppose that (E, ω) is a symplectic vector space and $L, M \in \mathrm{Lag}(E)$ are transversal to each other, so that $E = L \oplus M$. Then the map $\Phi : M \to L^*$ given by $\Phi(x) = \omega^*(x)|_L$ is a linear isomorphism, and the map $1 \oplus \Phi : E = L \oplus M \to (L \oplus L^*, \omega_L)$ is a symplectomorphism.

Now, we study compatible complex structures. A linear transformation $J : E \to E$ is called a **complex structure** on E iff $J^2 = -1$. If we put $ix = J(x)$ for $x \in E$, then E becomes a complex vector space. A complex structure J is called **ω-compatible** if the formula

$$g(v, w) = \omega(v, Jw)$$

defines a symmetric positive definite inner product on J. (Note that the symmetricity condition is equivalent to J being a symplectomorphism.) For a compatible complex structure J, we may define a positive definite Hermitian inner product on E by the formula

$$h(v, w) = g(v, w) - i\omega(v, w).$$

Conversely, if E is a finite-dimensional complex vector space and h is a positive definite Hermitian inner product on E, then the formula

$$\omega(v, w) = -\Im h(v, w)$$

defines a symplectic form on E compatible with the complex structure. We denote by $\mathcal{J}(E, \omega)$ the set of all compatible complex structures. Since all symplectic spaces of same dimension are symplectomorphic (and we have examples of Hermitian spaces in all dimensions), this set is nonempty.

Finally, we establish the identities

$$\mathcal{J}(E) \simeq \frac{\mathrm{Sp}(E)}{U(E)},\ \mathrm{Lag}(E) \simeq \frac{U(E)}{O(L)}$$

making $\mathcal{J}(E)$ and $\mathrm{Lag}(E)$ into homogeneous spaces (and thus smooth manifolds).

First, fix $J_0 \in \mathcal{J}(E, \omega)$; let g and h be the corresponding real and Hermitian inner products. Since $\omega = -\Im h$, the Lie group $U(E) = U(E, J_0)$ of all unitary transformations on E with respect to h is a closed subgroup of the Lie group $\mathrm{Sp}(E)$. The first identity above is now given by

Proposition 2. *The group* $\mathrm{Sp}(E)$ *acts on* $\mathcal{J}(E)$ *by the formula*

$$T \cdot J = TJT^{-1},\ T \in \mathrm{Sp}(E),\ J \in \mathcal{J}(E).$$

This action is transitive and the stabilizer of J_0 *is exactly* $U(E, J_0)$*.*

Proof. We only need to prove that the action is transitive. For that, consider $J_1, J_2 \in \mathcal{J}(E)$ and the corresponding Hermitian inner products h_1, h_2. Then (E, J_1, h_1) and (E, J_2, h_2) are Hermitian spaces of the same dimension; there exists some isomorphism $T : (E, J_1, h_1) \to (E, J_2, h_2)$. Since T is $\mathbb{C}$-linear, we get $TJ_1 = J_2T$; since T preserves Hermitian inner products, we get $T \in \mathrm{Sp}(E)$. □

To study the structure of $\mathcal{J}(E)$, we will need the following fact from the theory of Lie groups:

Lemma 2. *(Polar decomposition) Define the Lie algebra* $\mathfrak{sp}(E)$ *as the space of all linear transformations* $T : E \to E$ *such that* $\omega(Tx, y) + \omega(x, Ty) = 0$ *for all* $x, y \in E$*. Fix* $J \in \mathcal{J}(E)$ *and let*

$$\mathfrak{p} = \{T \in \mathfrak{sp}(E) \mid T^t = T\}.$$

Here T^t *is the transpose with respect to the inner product* g*. Then the map* $U(E) \times \mathfrak{p} \to \mathrm{Sp}(E)$ *given by*

$$(A, B) \mapsto A \cdot \exp B$$

is a diffeomorphism.

Therefore, each $J_0 \in \mathcal{J}(E)$ induces a diffeomorphism $\mathfrak{p} \to \mathcal{J}(E)$. It follows that $\mathrm{Sp}(E)$ is connected and $\mathcal{J}(E)$ is contractible.

Finally, we establish the identity for $\mathrm{Lag}(E)$. Let $L \in \mathrm{Lag}(E)$ and fix $J \in \mathcal{J}(E)$; then we get

Proposition 3. *1. A basis of L is orthonormal with respect to the Euclidean inner product iff it is an orthonormal complex basis of E with respect to the Hermitian inner product.*

2. The group $U(E)$ acts transitively on $\mathrm{Lag}(E)$ with stabilizer $O(L)$, the group of orthogonal transformations on L.

In particular, $\mathrm{Lag}(E)$ is connected.

2 Maslov indices

We know from the previous section that $\mathrm{Sp}(E)$ is homotopy equivalent to $U(E)$. It follows that their fundamental groups are isomorphic. However, the fundamental group of $U(E)$ can be found using the following fact from the theory of Lie groups:

Lemma 3. *The map $\det : U(E) \to \mathbb{S}^1$ mapping each matrix to its determinant induces an isomorphism of fundamental groups*

$$\pi_1(U(E)) \to \pi_1(\mathbb{S}^1) = \mathbb{Z}.$$

Composing this map with the isomorphism $\pi_1(\mathrm{Sp}(E)) \to \pi_1(U(E))$ (induced by the projection $\mathrm{Sp}(E) \to U(E)$ given to us by polar decomposition), we get an isomorphism

$$\mu : \pi_1(\mathrm{Sp}(E)) \to \mathbb{Z}.$$

Moreover, since $\mathcal{J}(E)$ is connected, this map does not depend on J. (The maps $\mathrm{Sp}(E) \to \mathbb{S}^1$ induced by two different complex structures are homotopic.) Note that det is the determinant of a complex matrix; in fact, the real determinant of an element of $\mathrm{Sp}(E)$ is always 1. The image under the map above of a loop in $\mathrm{Sp}(E)$ is called the **Maslov index** of this loop of symplectomorphisms.

The fundamental group of $\mathrm{Lag}(E)$ is also $\mathbb{Z}$, as we see from the following

Proposition 4. *The map $\det^2 : U(E) \to \mathbb{S}^1$ induces a map $\mathrm{Lag}(E) \to \mathbb{S}^1$ (which we also denote by $\det^2$). This in turn induces an isomorphism*

$$\mu : \pi_1(\mathrm{Lag}(E)) \to \pi_1(\mathbb{S}^1) = \mathbb{Z}.$$

Proof. The first statement follows from the identity $\mathrm{Lag}(E) \simeq U(E)/O(L)$ and the fact that det takes values ± 1 on $O(L)$; it remains to prove that μ is an isomorphism. For that, identify E with $\mathbb{C}^n$ and consider the path $A(t) = \mathrm{diag}(e^{i\pi t}, 1, \dots, 1) \in U(E)$, $0 \le t \le 1$. Take $L_0 = \mathbb{R}^n \in \mathrm{Lag}(E)$ and consider $L(t) = A(t)L_0$; then $L(0) = L(1)$. By definition we see that $\mu(L(t)) = 1 \in \mathbb{Z}$, so μ is epi.

To see that μ is mono, consider a loop $L(t) \in \mathrm{Lag}(E)$ such that $\det^2(L(t))$ is contractible; we have to show that $L(t)$ is contractible. Choose a (continuous) path $A(t) \in U(E)$ such that $L(t) = A(t)L_0$; then $A(0)^{-1}A(1) \in O(L_0)$. Multiplying $A(t)$ on the right by a suitable path in $O(L_0)$, we may assume that either $A(1) = A(0)$ or $A(1) = A(0)\operatorname{diag}(-1, 1, \ldots, 1)$. In the first case A is a loop in $U(E)$ and $\det^2 A(t)$ is contractible; it follows that $A(t)$ is contractible and thus $L(t) = A(t)L_0$ is contractible. In the second case we have $A(t) = B(t)\operatorname{diag}(e^{i\pi t}, 1, \ldots, 1)$, where $B(t)$ is a loop in $U(E)$. It follows that the loop $e^{2i\pi t}\det^2 B(t)$ is contractible, which is impossible because the loop $\det^2 B(t)$ on $\mathbb{S}^1$ corresponds to an even integer. □

Since both $\mathrm{Lag}(E)$ and $\mathcal{J}(E)$ are connected, the homomorphism μ defined above does not depend on J or L_0. The image of a loop of Lagrangians under this map is also called Maslov index. The two indices can be related by the following

Proposition 5. *If $A(t)$ is a loop of symplectomorphisms and $L(t)$ is a loop of Lagrangians, then*

$$\mu(A(t)L(t)) = \mu(L(t)) + 2\mu(A(t)).$$

Proof. We may replace A by a homotopic loop to get $A(t) \in U(E)$. Now, we have $\det^2(A(t)L(t)) = \det^2 A(t)\det^2 L(t)$; the proposition follows. □

We will now try to generalize Maslov indices to paths that are not necessarily loops. But first, let us reduce symplectomorphisms to Lagrangians:

Proposition 6. *Let E^- be the vector space E with the symplectic form $-\omega$. Let $A : E \to E$ be a linear transformation and consider its graph*

$$\Gamma_A = \{(Ax, x) \mid x \in E\} \subset E^- \oplus E.$$

Then:

1. $A \in \mathrm{Sp}(E)$ iff $\Gamma_A \in \mathrm{Lag}(E^- \oplus E)$.

2. It $A(t)$ is a loop of symplectomorphisms and $\Gamma(t) = \Gamma_{A(t)}$ is the corresponding loop in $\mathrm{Lag}(E^- \oplus E)$, then

$$\mu(\Gamma(t)) = 2\mu(A(t)).$$

Proof. 2. If $\Delta \subset E^- \oplus E$ is the diagonal (equivalently, the graph of the identity transformation), then we have $\Gamma(t) = (A(t) \oplus 1)\Delta$, so by the previous proposition $\mu(\Gamma(t)) = 2\mu(A(t) \oplus 1)$. Now, it follows from the definition of Maslov index that $\mu(A(t) \oplus 1) = \mu(A(t))$. □

Before making general definitions, let us consider the space $E = \mathbb{C} = \mathbb{R}^2$ with the symplectic form $\omega(z_1, z_2) = \Im(\bar{z}_1 z_2)$ induced by the canonical Hermitian product. Each one-dimensional subspace is Lagrangian, so $\mathrm{Lag}(E) \simeq \mathbb{S}^1$. Consider the path $L(t) = \mathbb{R}e^{i\pi t}$, $0 \le t \le 1$; then $\mu(L(t)) = 1$. Suppose now that we wish to define Maslov index of the subpath of $L(t)$ given by $a \le t \le b$, where

$0 \le a \le b \le 1$. For that, we may consider a fixed line, say $M = \{\Re z = 0\}$, and count the (signed) number of times that $L(t)$ intersects M. Then we will put $\mu = 0$ if $1/2 \notin [a, b]$ and $\mu = 1$ if $1/2 \in (a, b)$. This answer can be expressed as follows: if the endpoints of the path in question are transversal to M, then we can augment it to a loop by a path of Lagrangians transversal to M at each point and then calculate the Maslov index of this loop. (The question of what to do if either a or b equals $1/2$ is more subtle and will be dealt with in the following section. In our example, each intersection with M will count as a half if it lies at an endpoint.)

Now, let us go back to the general case and suppose that $M \in \mathrm{Lag}(E)$. Denote by $\mathrm{Lag}(E, M)$ the set of all Lagrangians L that are transversal to M; that is, $L \cap M = 0$. This is an open subset of $\mathrm{Lag}(E)$. In fact, it can be identified with a linear space:

Proposition 7. *Let $L_0 \in \mathrm{Lag}(E, M)$ and let $L \subset E$ be a subspace of half dimension. Then:*

1. L is transversal to M iff it is the graph of a map $T_L : L_0 \to M$.

2. If $S_L : L_0 \times L_0 \to \mathbb{R}$ is the bilinear form given by $S_L(x, y) = \omega(T_L(x), y)$, then L is Lagrangian iff S_L is symmetric. The form S_L uniquely determines T_L and thus L; in fact, we have $S_L(x, y) = \langle \Phi(T_L(x)), y\rangle$, where $\Phi : M \to L_0^*$ is the isomorphism from Proposition 1.

3. $\ker S_L = L_0 \cap L$.

We see in particular that $\mathrm{Lag}(E, M)$ is contractible, which lets us make

Definition 1. *Suppose that $L(t)$ is a path of Lagrangians whose endpoints are transversal to M. Augment it to a loop $L'(t)$ by a path between endpoints lying completely in $\mathrm{Lag}(E, M)$; each two of these loops are homotopic. Now, define the Maslov index $[L(t) : M] = \mu(L'(t))$.*

3 Hörmander-Kashiwara index

In the end of the previous section, we constructed Maslov index $[L(t) : M]$ for a path of Lagrangians $L(t)$ that is transversal to M at the endpoints. This construction does not work if we drop the transversality assumption; in this section we will generalize Maslov indices to any paths of Lagrangians. (A useful application, for example, would be to compute Maslov index with respect to the diagonal $\Delta \subset E^- \oplus E$ of a path of symplectomorphisms starting at the identity.) As we saw earlier, in the case $E = \mathbb{C}$ Maslov index can be interpreted as the (signed) number of intersections of $L(t)$ with M. However, in higher dimensions the set $\mathrm{Lag}(E) \setminus \mathrm{Lag}(E, M)$ is not a submanifold; it has different portions corresponding to different dimensions of $L(t) \cap M$. One way to get around this is to perturb $L(t)$ so that it only intersects the regular part of $\mathrm{Lag}(E) \setminus \mathrm{Lag}(E, M)$ and prove that the resulting number of intersections does not depend on the pertrubation. However, we will instead use the following algebraic invariant:

Definition 2. *Let* $L_1, L_2, L_3 \in \mathrm{Lag}(E)$. *Consider the symmetric form* Q *on* $L_1 \oplus L_2 \oplus L_3$ *given by*

$$Q((x_1, x_2, x_3), (y_1, y_2, y_3)) = \omega(x_1, y_2) + \omega(y_1, x_2) \\ +\omega(x_2, y_3) + \omega(y_2, x_3) + \omega(x_3, y_1) + \omega(y_3, x_1).$$

Define the ***Hörmander-Kashiwara index of the Lagrangian triple*** (L_1, L_2, L_3) *as*

$$s(L_1, L_2, L_3) = \mathrm{Sig}\, Q \in \mathbb{Z},$$

where Sig *denotes the signature (the dimension of the positive eigenspace minus the dimension of the negative eigenspace).*

It follows from the definition that $s(L_1, L_2, L_3)$ is anti-symmetric under permutations of the arguments and $s(A(L_1), A(L_2), A(L_3)) = s(L_1, L_2, L_3)$ for any symplectomorphism A.

To derive further properties, suppose that $M \in \mathrm{Lag}(E)$ is transversal to all L_j. Take $L_0 \in \mathrm{Lag}(E, M)$; then by Proposition 7 we may associate to each L_j a symmetric bilinear form S_j on L_0. Consider the isomorphisms $T_j : L_0 \to L_j$ given by the fact that each L_j is the graph of a map $L_0 \to M$; then

$$(T_1 \oplus T_2 \oplus T_3)^* Q(L_1, L_2, L_3) = \begin{pmatrix} 0 & S_1 - S_2 & S_3 - S_1 \\ S_1 - S_2 & 0 & S_2 - S_3 \\ S_3 - S_1 & S_2 - S_3 & 0 \end{pmatrix}.$$

Now, if the linear isomorphism T on L_0^3 is given by $T(v_1, v_2, v_3) = (v_2 + v_3, v_3 + v_1, v_1 + v_2)$, then $T^* Q = \mathrm{diag}(S_2 - S_3, S_3 - S_1, S_1 - S_2)$; therefore,

$$s(L_1, L_2, L_3) = \mathrm{Sig}(S_1 - S_2) + \mathrm{Sig}(S_2 - S_3) + \mathrm{Sig}(S_3 - S_1).$$

From here, we get

Proposition 8. *1. (Cocycle identity) For each* $L_1, L_2, L_3, L_4 \in \mathrm{Lag}(E)$,

$$s(L_2, L_3, L_4) - s(L_1, L_3, L_4) + s(L_1, L_2, L_4) - s(L_1, L_2, L_3) = 0.$$

2. If $L(t)$ *is a (continuous) path of Lagrangians transversal to* $L_1, L_2 \in \mathrm{Lag}(E)$ *at each point, then* $s(L_1, L_2, L(t))$ *is constant as a function of* t.

3. If $L_1(t)$, $L_2(t)$, $L_3(t)$ *are paths of Lagrangians pairwise transversal at each point, then* $s(L_1(t), L_2(t), L_3(t))$ *is constant as a function of* t.

Proof. 1. Follows directly from the formula above, if we select M transversal to all L_j.

2. It suffices to prove that $s(L_1, L_2, L(t))$ is locally constant. Take any t_0; then there exists $M \in \mathrm{Lag}(E)$ transversal to L_1, L_2, and $L(t_0)$. Since $\mathrm{Lag}(E; M)$ is an open subset of $\mathrm{Lag}(E)$, $L(t)$ is transversal to M for t close to t_0. Now, we get

$$s(L_1, L_2, L(t)) = \mathrm{Sig}(S_1 - S_2) + \mathrm{Sig}(S_2 - S(t)) + \mathrm{Sig}(S(t) - S_1)$$

near t_0; here S_j and S are the corresponding bilinear forms. Now, since $L(t)$ is transversal to L_1 and L_2, the forms $S_2 - S(t)$ and $S(t) - S_1$ are nonsingular; it follows that the signatures above do not depend on t.

3. This is proven similarly to the previous statement. □

Let us compute $s(L_1, L_2, L_3)$ when L_2 is transversal to both L_1 and L_3. Using a symplectomorphism, we may assume that $E = L \oplus L^*$, $L_1 = L$, and $L_2 = L^*$. As before, we may associate to L_3 a symmetric bilinear form S on L. Now, if T is a linear automorphism on L, then $A = T \oplus T^{-*}$ is a symplectomorphism on E; $A(L) = L$, $A(L^*) = L^*$, and $A(L_3)$ corresponds to the form $(T^{-1})^* S$. Therefore, we may use a symplectomorphism to make S diagonal in a chosen basis $e_1, \dots, e_n$ of L. If we consider the basis $e_1, \dots, e_n, e_1^*, \dots, e_n^*, e_1 + S(e_1), \dots, e_n + S(e_n)$ of $L \oplus L^* \oplus L_3$, where e^* is the basis of L^* dual to e, then the form $Q(L_1, L_2, L_3)$ splits into the following blocks Q_ε corresponding to each diagonal value $\varepsilon \in \{0, \pm 1\}$ of S:

$$Q_\varepsilon = \begin{pmatrix} 0 & -1 & \varepsilon \\ -1 & 0 & 1 \\ \varepsilon & 1 & 0 \end{pmatrix}.$$

By computing the signature of each of such blocks, we get

$$s(L_1, L_2, L_3) = \operatorname{Sig} S.$$

We also see that a Lagrangian triple $L_1, L_2, L_3 \in \operatorname{Lag}(E)$ such that one of the Lagrangians is transversal to the other two is determined up to a symplectomorphism by the index $s(L_1, L_2, L_3)$. (As a matter of fact, any triple of Lagrangians is determined up to a symplectomorphism by the dimensions of their intersections and their index.)

We are now ready to define Maslov index of any two paths of Lagrangians. Before we do that, however, let us again consider the case $E = \mathbb{C}$. Suppose that $L_2 = \{\Re z = 0\}$ and $L_1(t)$, $a \leq t \leq b$, is a path of lines. We cannot *a priori* compute the intersection number $[L_1(t) : L_2]$ based on information at the endpoints; for example, one may let $L_1(t)$ be a loop winding around the origin arbitrarily many times. However, if we know that there exists a line M such that $L_1(t)$ is transversal to M at each t, then we may deduce the index $[L_1(t) : L_2]$ from the relative position of L_1, L_2, M at the endpoints $t = a, b$. For example, if $L_1(a) = \mathbb{R}(1 + i)$ and $L_1(b) = \mathbb{R}(1 - i)$, then $[L_1(t) : L_2]$ must equal 1.

Back to the general case. We first prove the following

Lemma 4. *Suppose that $L_1(t), L_2(t)$ are two paths of Lagrangians, $a \leq t \leq b$, and $M \in \operatorname{Lag}(E)$ is transversal to $L_1(t)$ and $L_2(t)$ for each t. Then the value*

$$[L_1(t) : L_2(t)] = \frac{1}{2}(s(L_1(a), L_2(a), M) - s(L_1(b), L_2(b), M))$$

does not depend on the choice of M.

Proof. Let M_1 and M_2 be two Lagrangians with the required property; then by the cocycle identity we have to prove that

$$\begin{aligned} & s(L_1(a), M_1, M_2) - s(L_2(a), M_1, M_2) \\ &= s(L_1(b), M_1, M_2) - s(L_2(b), M_1, M_2). \end{aligned}$$

This identity follows from Proposition 8. $\square$

Now, suppose that $L_1(t), L_2(t)$ are any two paths of Lagrangians, $a \le t \le b$. Consider a partition $a = t_0 < t_1 < \cdots < t_N = b$ such that there exist $M_j \in \mathrm{Lag}(E)$ transversal to both $L_1(t)$ and $L_2(t)$ for $t_{j-1} \le t \le t_j$. Then we may define

$$[L_1(t) : L_2(t)] = \frac{1}{2}\sum_{j=1}^{N} s(L_1(t_{j-1}), L_2(t_{j-1}), M_j) - s(L_1(t_j), L_2(t_j), M_j).$$

By the lemma above, it does not depend on M_j and the partition.

We will now prove that if L_2 is constant and $L_1(t)$ is transversal to L_2 at the endpoints, then the so-defined index coincides with the Maslov index defined in the previous section. We start with

Proposition 9. *(Homotopy invariance) Suppose that $L_1(t,s)$ and $L_2(t,s)$ are families of Lagrangians continuously depending on $t \in [a,b]$, $s \in [c,d]$ with $L_i(a,s)$ and $L_i(b,s)$ independent of s. Then*

$$[L_1(\cdot, c) : L_2(\cdot, c)] = [L_1(\cdot, d) : L_2(\cdot, d)].$$

Proof. Assume first that there exists a partition $a = t_0 < t_1 < \cdots < t_N = b$ and a sequence of spaces $M_j \in \mathrm{Lag}(E)$ such that M_j is transversal to each $L_i(t,s)$ for $t_{j-1} \le t \le t_j$ and all s. Then we get

$$\begin{aligned} & 2([L_1(\cdot, c) : L_2(\cdot, c)] - [L_1(\cdot, d) : L_2(\cdot, d)]) \\ &= \sum_{j=1}^{N} s(L_1(t_{j-1}, c), L_2(t_{j-1}, c), M_j) - s(L_1(t_j, c), L_2(t_j, c), M_j) \\ &\quad - s(L_1(t_{j-1}, d), L_2(t_{j-1}, d), M_j) + s(L_1(t_j, d), L_2(t_j, d), M_j) \\ &= \sum_{j=1}^{N-1} (s(L_1(t_j, c), L_2(t_j, c), M_{j+1}) - s(L_1(t_j, d), L_2(t_j, d), M_{j+1})) \\ &\quad - (s(L_1(t_j, c), L_2(t_j, c), M_j) - s(L_1(t_j, d), L_2(t_j, d), M_j)). \end{aligned}$$

Now, each of the terms in the last sum is zero by Lemma 4 applied to the path $s \mapsto (L_1(t_j, s), L_2(t_j, s))$.

In the general case, we only need to prove that the function $s \mapsto [L_1(\cdot, s) : L_2(\cdot, s)]$ is locally constant. Consider any $s_0 \in [c,d]$; then for each $t_0 \in [a,b]$, there exists an open neighborhood U of (t_0, s_0) in $[a,b] \times [c,d]$ and a Lagrangian M that is transversal to $L_i(t,s)$ for all i and all $(t,s) \in U$. We may assume that U is a Cartesian product of two open intervals; taking a finite subcover of the cover of $[a,b] \times \{s_0\}$ by such U, we reduce to the case considered above. $\square$

Now, $[L_1(t) : L_2(t)]$ is clearly additive under addition of paths; moreover, if $L_1(t)$ and $L_2(t)$ are transversal to each other at each t, then $[L_1(t) : L_2(t)] = 0$ by Proposition 8. Therefore, it remains to prove that the homomorphism $\pi_1(\mathrm{Lag}(E)) \to \mathbb{Z}$ induced by $L(t) \mapsto [L(t) : L_2]$ (for any fixed L_2) is equal to μ. For that, it suffices to verify that both homomorphisms take the same value on a given non-contractible loop:

Proposition 10. *Let $E = \mathbb{C}^n$ and put $L_1(t) = \mathrm{diag}(e^{i\pi t}, e^{i\pi/4}, \dots, e^{i\pi/4})\mathbb{R}^n$ for $0 \le t \le 1$ and $L_2 = i\mathbb{R}^n$. Then the Maslov index defined in this section is*

$$[L_1(t) : L_2] = 1$$

and thus coincides with the Maslov index defined in the previous section.

Proof. The spaces $L_1(t)$ and L_2 are transversal for $t \neq 1/2$, so it suffices to compute the index on the interval $1/4 \le t \le 3/4$. Now, the space $M = \mathbb{R}^n$ is transversal to both M and $L(t)$, and by the reasoning above we get

$$s(L_1(1/4), L_2, M) = n, \;\; s(L_1(3/4), L_2, M) = n - 2.$$

Therefore, $[L_1(t) : L_2] = 1$ as required. □

References

[1] Eckhard Meinrenken, *Symplectic Geometry.* Lecture notes, University of Totonto, `http://www.math.toronto.edu/mein/teaching/lectures.html`