Worksheet 9: Inverses, invertibility, and determinants

1. Find the inverse of the matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Answer:

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

2–3. Determine if the following matrices are invertible. Do not compute the inverses.

$$\begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 3 \\ 0 & 2 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}.$$

Answers: (2) Yes, as the matrix has 3 pivot positions (3) No, as it is not a square matrix

4. Assume that A and B are square matrices such that $AB = I$. Can you prove that $AB = BA$?

Solution: Yes. Indeed, by IMT (k), the matrix A is invertible. Multiplying both sides of the equation $AB = I$ by A^{-1} to the left, we get $B = A^{-1}$. Then $AB = BA = I$, so A and B commute.

5. Assume that A is a square matrix and the columns of A^2 are linearly dependent. Prove that the columns of A are linearly dependent.

Solution: We argue by contradiction. Assume that the columns of A^2 are linearly dependent, yet the columns of A are linearly independent. Then by IMT (e), the matrix A is invertible. Therefore, $A^2 = A \cdot A$ is invertible; by IMT (e), the columns of A^2 are linearly independent, a contradiction.

6. Lay, 2.3.23.

Solution: See the back of the book.

7. Compute $\det A$ and state whether A is invertible:

$$A = \begin{bmatrix} 1 & -5 & -4 \\ 0 & 3 & 4 \\ -3 & 6 & 0 \end{bmatrix}.$$

Answer: Use the cofactor expansion along the second row:

$$\begin{aligned} \det A &= -0 \cdot \det \begin{bmatrix} -5 & -4 \\ 6 & 0 \end{bmatrix} + 3 \cdot \det \begin{bmatrix} 1 & -4 \\ -3 & 0 \end{bmatrix} - 4 \cdot \det \begin{bmatrix} 1 & -5 \\ -3 & 6 \end{bmatrix} \\ &= 0 - 3 \cdot 12 - 4(-9) = 0. \end{aligned}$$

Therefore, A is not invertible.

8. Use determinants to find all t for which the vectors $(1, 2)$ and $(t, t+3)$ are linearly independent.

Solution: The vectors in question are linearly independent if and only if

$$0 \neq \det \begin{bmatrix} 1 & t \\ 2 & t+3 \end{bmatrix} = 3 - t.$$

Therefore, the answer is $t \neq 3$.

9. Use determinants to find all λ for which the matrix

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} - \lambda I_2$$

is not invertible.

Solution: The matrix in question is not invertible if and only if

$$0 = \det \begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix} = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1).$$

Therefore, the answer is $\lambda = -1, 3$.

10. Compute the determinant of the 2×2 matrix with columns

$$\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \vec{v} = \begin{bmatrix} t \\ 1 \end{bmatrix}.$$

Here $t \in \mathbb{R}$ is some number. Draw the vectors $\vec{u}$ and $\vec{v}$ and use plane geometry to compute the area of the parallelogram spanned by these vectors.

Answer: The determinant is equal to 1. To prove that the area is also equal to 1, multiply the length of the side of the parallelogram between 0 and $\vec{u}$ by the distance from $\vec{v}$ to this side.

100.* (Neumann series) Let A be a square matrix and consider the series

$$\sum_{j \geq 0} A^j = I + A + A^2 + A^3 + \ldots$$

(a) Assume that the series converges to a matrix B (in the sense that each of its entries converges). Prove that $B = (I - A)^{-1}$.

(b) Assume that A is **nilpotent**; that is, $A^m = 0$ for some positive integer m. Use part (a) to prove that $I - A$ is invertible.