Worksheet 8: Matrix algebra and inverses

1. Given the matrices

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix},$$

compute the following expressions:

$$B + AB, \; B + BA, \; B^T B, \; BB^T + 2I_2.$$

Here I_2 is the 2×2 identity matrix. If an expression is undefined, explain why.

Answer: $B + BA$ is undefined since the product BA is undefined (B has 3 columns, while A has only two rows). Next,

$$B + AB = \begin{bmatrix} 2 & 2 & 4 \\ 5 & 3 & 8 \end{bmatrix},$$

$$B^T B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix},$$

$$BB^T + 2I_2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}.$$

2. Can you derive the following facts from the properties of matrix operations (assuming that all operations are well defined)? If so, state clearly which properties you use.

(a) $(AB)^T = A^T B^T$

(b) $(AA^T)^T = AA^T$

(c) $(A + B)(C + D) = AC + BD + BC + AD$

(d) If $CA = I_n$ and $AD = I_m$, then $C = D$ (hint: think about the product CAD)

Solution: (a) No; in fact, $(AB)^T = B^T A^T$, and A^T and B^T do not have to commute.

(b) Yes, by properties (15) and (12) (in the matrix operations handout)

$$(AA^T)^T = (A^T)^T A^T = AA^T.$$

(c) Yes, by properties (9) and (8):

$$(A+B)(C+D) = A(C+D) + B(C+D) = AC + AD + BC + BD.$$

(d) Yes, by properties (7) and (11):

$$C(AD) = CI_m = C;$$

on the other hand,

$$C(AD) = (CA)D = I_n D = D.$$

3. Lay, 2.1.19. (Hint: use the definition of the matrix product given at the beginning of page 110.)

Solution: See the back of the book.

4–6. Use the formula on page 119 to find the inverses of the following matrices or state that they are not invertible:

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$

$$\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}.$$

Answers:

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^{-1} = \frac{1}{3}\begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{ is not invertible,}$$

$$\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}^{-1} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}.$$

Note that in the last case, the inverse matrix to the matrix of rotation by ϕ degrees counterclockwise is the matrix of rotation by ϕ degrees clockwise.

7. Use the inverse found in exercise 4 to solve the equation

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Solution: We have

$$\vec{x} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix}.$$

8. Use invertibility to prove that the equation

$$\begin{bmatrix} 100 & 99 \\ 101 & 100 \end{bmatrix} \vec{x} = \vec{b}$$

has a unique solution for each $\vec{b}$. (Hint: you do not need to compute the inverse here.)

Solution: We have

$$\det \begin{bmatrix} 100 & 99 \\ 101 & 100 \end{bmatrix} = 100^2 - 99 \cdot 101 = 1 \neq 0;$$

therefore, the matrix in question is invertible (Theorem 4 in 2.2). It follows that the equation in question has a unique solution for each right hand side (Theorem 5 in 2.2).

9. Lay, 2.2.18.

Solution: Multiply both sides of the equation by P^{-1} to the left and by P to the right; we get

$$P^{-1}AP = P^{-1}PBP^{-1}P = (P^{-1}P)B(P^{-1}P) = I_n B I_n = B.$$

10. Lay, 2.2.9, (a)–(d).

Answers: (a) True (b) False (c) True (d) True; see the solution guide for details.

11. Lay, 2.2.16.

Solution: Since we do not know that A is invertible, we cannot use the formula $(AB)^{-1} = B^{-1}A^{-1}$. Instead, put $C = AB$; multiplying both sides of this equation by B^{-1} to the right, we get $A = CB^{-1}$. Now, both C and B^{-1} are invertible; therefore, A is invertible and in fact, $A^{-1} = BC^{-1}$.

100.* (The center of the matrix algebra) Find all 2×2 matrices A such that for each 2×2 matrix B, $AB = BA$. (Hint: try taking matrices B that have element 1 at one position and 0 at all other positions.)

Answer: A has to be a multiple of the identity matrix.

101.* (A model of complex numbers) For $a, b \in \mathbb{R}$, define the 2×2 matrix $T(a, b)$ as

$$T(a,b) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}.$$

We associate to this matrix the complex number $a + ib$.

(a) Prove that $T(a, b) + T(c, d) = T(a + c, b + d)$ and relate this to the law of addition of complex numbers.

(b) Prove that (as a matrix product) $T(a, b)T(c, d) = T(c, d)T(a, b) = T(ac - bd, bc + ad)$ and relate this to the law of multiplication of complex numbers.

(c) Prove that $T(a, b)^T = T(a, -b)$ and relate this to complex conjugation.

(d) Prove that for $a^2 + b^2 > 0$, the matrix $T(a, b)$ is invertible and $T(a, b)^{-1} = T(a, -b)/(a^2 + b^2)$; relate this to inverses of complex numbers.

(e) If $a^2 + b^2 > 0$, put $r = \sqrt{a^2 + b^2}$ and show that $T(a, b)$ is equal to r times the matrix of a certain rotation. Relate this to the polar decomposition of complex numbers.