Worksheet 6: Matrix transformations

1–6. Given the transformation $T : \mathbb{R}^a \to \mathbb{R}^b$,

(a) (for problems 1–2) Find the values a and b for which the transformation is well defined.

(b) Determine whether T is 1-to-1.

(c) Determine whether T is onto.

(d) (for problems 2–6) Try to come up with a geometric description of the transformation T. For that, first find a formula for the coordinates of $T\vec{x}$ with $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, in terms of x_1 and x_2. Next, pick several vectors $\vec{x}$ (be sure to include the zero vector) and plot them on one set of axes, while plotting $T\vec{x}$ on another set of axes.

$$T(\vec{x}) = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix} \vec{x}, \tag{1}$$

$$T(\vec{x}) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \vec{x}, \tag{2}$$

$$T(\vec{x}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \vec{x}, \tag{3}$$

$$T(\vec{x}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \vec{x}, \tag{4}$$

$$T(\vec{x}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \vec{x}, \tag{5}$$

$$T(\vec{x}) = \vec{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{6}$$

Beware that (6) is not a matrix transformation! So, you will need to use the definitions on page 87 to solve (b) and (c) for it. If you don't know how to do this, go on to problem 7.

Answers: 1. (a) $a = 3, b = 2$ (b) No (c) Yes

2. (a) $a = b = 2$ (this is the case for problems 2–6) (b) Yes (c) Yes (d) dilation by a factor of 2

3. (b) Yes (c) Yes (d) reflection with respect to the x_1 axis

4. (b) No (c) No (d) orthogonal projection onto the x_1 axis

5. (b) Yes (c) Yes (d) reflection with respect to the line $x_1 = x_2$

6. (b) Yes, as $\vec{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \vec{y} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ implies that $\vec{x} = \vec{y}$ by subtracting $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ from both sides

(c) Yes, as for any $\vec{b} \in \mathbb{R}^2$, there is $\vec{x} \in \mathbb{R}^2$ such that $\vec{b} = \vec{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$; in fact, take $\vec{x} = \vec{b} - \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

(d) shift by the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

7. Give a geometric description of the range of the transformation T from problem 4 and of the solution set of the equation $T\vec{x} = \vec{0}$.

Answer: The range is the x_1 axis, while the solution set of the equation $T\vec{x} = \vec{0}$ is the x_2 axis.