Worksheet 5: linear independence

1–4. Is the following set of vectors linearly independent? If it is linearly dependent, find a linear dependence relation. For each vector in the set, find whether it lies in the set spanned by the other vectors.

$$\{\vec{a}_1, \vec{a}_2\},\ \vec{a}_1 = \begin{bmatrix} -1 \\ 4 \end{bmatrix},\ \vec{a}_2 = \begin{bmatrix} -2 \\ -8 \end{bmatrix}; \tag{1}$$

$$\{\vec{a}_1, \vec{a}_2\},\ \vec{a}_1 = \begin{bmatrix} -1 \\ 4 \end{bmatrix},\ \vec{a}_2 = \begin{bmatrix} 2 \\ -8 \end{bmatrix}; \tag{2}$$

$$\{\vec{a}_1, \vec{a}_2, \vec{a}_3\},\ \vec{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},\ \vec{a}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},\ \vec{a}_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}; \tag{3}$$

$$\{\vec{a}_1, \vec{a}_2, \vec{a}_3\},\ \vec{a}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix},\ \vec{a}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix},\ \vec{a}_3 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}. \tag{4}$$

Answers: 1. Linearly independent; no vector is in the span of the other vector.

2. Linearly dependent, with a relation $2\vec{a}_1 + \vec{a}_2 = 0$. Therefore, $\vec{a}_1 = -\vec{a}_2/2 \in \mathrm{Span}(\vec{a}_2)$ and $\vec{a}_2 = -2\vec{a}_1 \in \mathrm{Span}(\vec{a}_1)$.

3. Linearly dependent, with a relation $\vec{a}_1 - \vec{a}_2 + \vec{a}_3 = 0$. Therefore, $\vec{a}_1 = \vec{a}_2 - \vec{a}_3 \in \mathrm{Span}(\vec{a}_2, \vec{a}_3)$; similarly, both $\vec{a}_2$ and $\vec{a}_3$ lie in the span of other vectors.

4. Linearly dependent (for example, because there are more vectors than dimensions). The vectors $\vec{a}_1$ and $\vec{a}_3$ are multiples of each other, so $\vec{a}_1 \in \mathrm{Span}(\vec{a}_2, \vec{a}_3)$ and $\vec{a}_3 \in \mathrm{Span}(\vec{a}_1, \vec{a}_2)$. However, $\vec{a}_2 \notin \mathrm{Span}(\vec{a}_1, \vec{a}_3)$. (Drawing the three vectors could be helpful.)

5. Consider the vectors

$$\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix},\ \vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

and let $A = [\vec{u}\ \vec{v}]$.

(a) Prove that for each $\vec{b}$, the equation $A\vec{x} = \vec{b}$ has unique solution.

(b) Use part (a) to prove that each $\vec{b} \in \mathbb{R}^2$ can be represented as a linear combination of $\vec{u}$ and $\vec{v}$ in a unique way. (A set $\{\vec{u}, \vec{v}\}$ with this property is called a **basis**.)

Solution: (a) Since A has a pivot in each row, the equation $A\vec{x} = \vec{b}$ can be solved for every $\vec{b}$. Since A has a pivot in each column, the solution to this equation is unique.

(b) We can rewrite $A\vec{x} = \vec{b}$ as $\vec{b} = x_1\vec{u} + x_2\vec{v}$. The statement we need to prove is now identical to (a).

6. True or false:

(a) If a set contains the zero vector, it is linearly dependent.

(b) If $\vec{z} \in \text{Span}(\vec{x}, \vec{y})$, then $\{\vec{x}, \vec{y}, \vec{z}\}$ is linearly dependent.

(c) If a 2×2 matrix has linearly independent columns, then its columns span $\mathbb{R}^2$.

(d) If $\vec{x}, \vec{y} \in \mathbb{R}^3$ and $\vec{x}$ is not a multiple of $\vec{y}$, then $\{\vec{x}, \vec{y}\}$ is linearly independent.

Solution: (a) True, see Theorem 9

(b) True, see Theorem 7

(c) True, since the matrix has to have two pivot positions.

(d) False, as we can have $\vec{x} \neq 0$ and $\vec{y} = 0$. ($\vec{y}$ will be a multiple of $\vec{x}$ in this case, so there is no contradiction with Theorem 7.)