Worksheet 3: matrix-vector multiplication

1–2. Using Theorem 4 on page 43, verify whether the given sets of vectors span $\mathbb{R}^3$:

$$\{\vec{a}_1, \vec{a}_2, \vec{a}_3\},\ \vec{a}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix},\ \vec{a}_2 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix},\ \vec{a}_3 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}; \tag{1}$$

$$\{\vec{a}_1, \vec{a}_2\},\ \vec{a}_1 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix},\ \vec{a}_2 = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}. \tag{2}$$

Solution to problem 1: We write

$$A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 1 \\ 1 & 2 & 3 \end{bmatrix}.$$

then we need to find out whether A has a pivot position in each row. We perform row reductions to bring A to the following REF:

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix};$$

we see now that there is no pivot in row 3. Therefore, the vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3$ do not span the whole $\mathbb{R}^3$. (As a matter of fact, they span a plane. To understand why this is true, use the fact that $\vec{a}_1 + \vec{a}_2 = \vec{a}_3$.

Solution to problem 2: No, since two vectors can never span the whole $\mathbb{R}^3$. (A 3×2 matrix cannot have a pivot position in each row.)

3. Let

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},\ \vec{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ \vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Compute $A\vec{u}$, $A\vec{v}$, and $A(\vec{u}+\vec{v})$. Verify that $A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v}$. Draw the vectors $\vec{u}, \vec{v}, \vec{u}+\vec{v}$ on one set of axes and the vectors $A\vec{u}, A\vec{v}, A(\vec{u}+\vec{v})$ on another set of axes.

Answer:

$$A(\vec{u}) = \begin{bmatrix} -1 \\ 0 \end{bmatrix},\ A(\vec{v}) = \begin{bmatrix} -1 \\ 1 \end{bmatrix},\ A(\vec{u}+\vec{v}) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

4. Lay, 1.4.2.

Answer: We cannot multiply because the number of columns in the matrix is not equal to the number of rows in the vector.

5. Compute the product

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

first by representing the answer as a linear combinationof the columns of the matrix, and then by using the row-vector rule.

Solution: The answer is

$$2\begin{bmatrix} 2 \\ 0 \end{bmatrix} + (-1)\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 2\cdot 2 + 1\cdot(-1) \\ 0\cdot 2 + 3\cdot(-1) \end{bmatrix}.$$

6. Lay, 1.4.10.

Answer:

$$x_1\begin{bmatrix} 8 \\ 5 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix};\ \begin{bmatrix} 8 & -1 \\ 5 & 4 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}.$$

7. Lay, 1.4.24.

Solution: (a) True, see Theorem 3
(b) True, see Example 2
(c) True, see Theorem 3
(d) True, see the box before Example 2
(e) False, see the warning after Theorem 4
(f) True, see Theorem 3

8. Let

$$A = \begin{bmatrix} \vec{a}_1 & \dots & \vec{a}_p \end{bmatrix}, \ \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Prove that $A\vec{e}_1 = \vec{a}_1$.

Solution: Use the definition of $A\vec{e}_1$ as a linear combination of the columns of A.

9. Lay, 1.4.34. (Hint: think what the RREF of A should be.)

Solution: The RREF of A should have no free variables. Therefore, there should be a pivot in each of the 3 columns of A. However, since A has 3 rows, this implies that there is a pivot in each row; it remains to use Theorem 4 on page 43.

10. Lay, 1.4.35.

Solution: Put $\vec{x} = \vec{x}_1 + \vec{x}_2$; then

$$A\vec{x} = A(\vec{x}_1 + \vec{x}_2) = A\vec{x}_1 + A\vec{x}_2 = \vec{y}_1 + \vec{y}_2 = \vec{w}.$$