Worksheet 28: Heat equation and review of PDE

1. Find the Fourier cosine series of the function $f(x) = x$, $0 < x < \pi$. (I believe I did it in class once — try to find it in your notes if you do not want to calculate.)

Answer:

$$f(x) \sim \frac{\pi}{2} - \frac{4}{\pi} \sum_{j=1}^{\infty} \frac{\cos((2j-1)x)}{2j-1}.$$

2. Use the result of problem 1 to find the formal solution for the following problem for the heat equation with inhomogeneous boundary conditions. What is the limit of this solution as $t \to +\infty$?

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},\ 0 < x < \pi,\ t > 0;$$
$$u(0,t) = 0,\ u(\pi,t) = 1,\ t > 0;$$
$$u(x,0) = \sin(2x) + 5\sin(3x),\ x > 0.$$

Answer:

$$u(x,t) = \frac{x}{\pi} - \frac{4}{\pi^2} \sum_{j=1}^{\infty} \frac{\sin((2j-1)x)}{2j-1} + e^{-4t}\sin(2x) + 5e^{-9t}\sin(3x).$$

3. Describe the function to which the Fourier cosine series of the function $f(x) = x$, $0 < x < \pi$, converges, and sketch its graph.

Solution: The 2π-periodic extension of the function $\tilde{f}(x) = |x|$, $-\pi \le x \le \pi$.

4. Describe the function to which the Fourier sine series of the function $f(x) = x$, $0 < x < \pi$, converges, and sketch its graph.

Solution: The 2π-periodic extension of the function $\tilde{f}(x) = x$, $-\pi \le x \le \pi$.