Worksheet 27: Fourier series

Full Fourier series: if f is a function on the interval $[-\pi, \pi]$, then the corresponding series is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx);$$
$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx,$$
$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx.$$

Fourier cosine and sine series: if f is a function on the interval $[0, \pi]$, then the corresponding cosine series is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx);$$
$$a_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \cos(nx)\, dx,$$

and the corresponding sine series is

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin(nx);$$
$$b_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin(nx).$$

Convergence theorem for full Fourier series: if f is a piecewise differentiable function on $[-\pi, \pi]$, then its Fourier series converges at every point. The sum of the series is computed as follows:

1. Forget about what the function f looks like outside of the interval $[-\pi, \pi]$. After all, the formulas for the coefficients only feature the values of f on that interval.

2. Continue f periodically from $[-\pi, \pi]$ to the whole real line; let $\tilde{f}$ be the resulting function.

3. The sum of the Fourier series at the point x is equal to $\tilde{f}(x)$, if $\tilde{f}$ is continuous at x; otherwise, it is equal to $(\tilde{f}(x+) + \tilde{f}(x-))/2$.

Convergence for Fourier cosine series: forget about what f looks like outside of $[0, \pi]$, then extend f as an even function to $[-\pi, \pi]$, then use the above algorithm. Same approach works for sine series, except that you extend f to $[-\pi, \pi]$ as an odd function.

1.* This problem shows an alternative way of proving that the functions $\sin(kx)$, $k \in \mathbb{Z}$, $k > 0$, form an orthogonal set in $C[0, \pi]$.

(a) Assume that u and v are eigenfunctions of the following problem:

$$\begin{gathered} u''(x) + \lambda u(x) = 0, \ 0 < x < \pi; \\ u(0) = u(\pi) = 0; \\ v''(x) + \mu v(x) = 0, \ 0 < x < \pi; \\ v(0) = v(\pi) = 0, \end{gathered}$$

where λ and μ are two real numbers. Integrate by parts twice and use the boundary conditions to show that

$$\int_0^\pi u''(x)v(x)\,dx = \int_0^\pi u(x)v''(x)\,dx.$$

Use the differential equations satisfied by u and v to show that

$$(\lambda - \mu)\int_0^\pi u(x)v(x)\,dx = 0.$$

(b) Take $u(x) = \sin(kx)$, $v(x) = \sin(lx)$, for k, l positive integers and $k \neq l$. Verify that these functions satisfy the conditions of part (a) for certain λ and μ, and conclude that

$$\int_0^\pi \sin(kx)\sin(lx)\,dx = 0.$$

Solution: (a) We have

$$\int_0^\pi u''(x)v(x)\,dx = u'(x)v(x)|_{x=0}^{\pi} - \int_0^\pi u'(x)v'(x)\,dx;$$
$$\int_0^\pi u(x)v''(x)\,dx = u(x)v'(x)|_{x=0}^{\pi} - \int_0^\pi u'(x)v'(x)\,dx.$$

Since $u(0) = u(\pi) = v(0) = v(\pi)$, the boundary terms vanish and we get

$$\int_0^\pi u''(x)v(x)\,dx = \int_0^\pi u(x)v''(x)\,dx.$$

Next, $u''(x) = -\lambda u(x)$ and $v''(x) = -\mu v(x)$; substituting this into the equation above, we get

$$-\lambda \int_0^\pi u(x)v(x)\,dx = -\mu \int_0^\pi u(x)v(x)\,dx.$$

(b) The functions $u(x)$ and $v(x)$ satisfy the equations of (a) for $\lambda = k^2$ and $\mu = l^2$; therefore,

$$(k^2 - l^2)\int_0^\pi u(x)v(x)\,dx = 0.$$

Since $k^2 \neq l^2$, the functions u and v are orthogonal.

2. Find the Fourier sine series for the function

$$f(x) = x(\pi - x),\ 0 < x < \pi.$$

Solution: Integrate by parts:

$$\begin{aligned}
b_k &= \frac{2}{\pi}\int_0^\pi x(\pi - x)\sin(kx)\,dx \\
&= -\frac{2}{\pi k}\int_0^\pi x(\pi - x)\,d(\cos(kx)) \\
&= -\frac{2}{\pi k}x(\pi - x)\cos(kx)|_{x=0}^{\pi} + \frac{2}{\pi k}\int_0^\pi (\pi - 2x)\cos(kx)\,dx \\
&= \frac{2}{\pi k^2}\int_0^\pi \pi - 2x\,d(\sin(kx)) \\
&= \frac{2}{\pi k^2}(\pi - 2x)\sin(kx)|_{x=0}^{\pi} + \frac{4}{\pi k^2}\int_0^\pi \sin(kx)\,dx \\
&= -\frac{4}{\pi k^3}\cos(kx)|_{x=0}^{\pi} = \frac{4}{\pi k^3}[1 - (-1)^k].
\end{aligned}$$

Therefore, $b_k = 0$ for k even and $b_k = 8/(\pi k^3)$ for k odd; the Fourier series is

$$f(x) \sim \frac{8}{\pi} \sum_{j=1}^{\infty} \frac{\sin((2j-1)x)}{(2j-1)^3}.$$

3. Using the previous problem, find the formal solution to the problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},\ 0 < x < \pi,\ t > 0;$$
$$u(0,t) = u(\pi,t) = 0,\ t > 0;$$
$$u(x,0) = x(\pi - x),\ 0 < x < \pi.$$

Answer:

$$u(x,t) = \sum_{j=1}^{\infty} \frac{8}{\pi(2j-1)^3} e^{-(2j-1)^2 t} \sin((2j-1)x).$$

4. Find the Fourier cosine series for the function

$$f(x) = \pi - x,\ 0 < x < \pi.$$

Solution: We calculate

$$a_0 = \frac{2}{\pi} \int_0^{\pi} \pi - x\, dx = \pi,$$
$$a_k = \frac{2}{\pi} \int_0^{\pi} (\pi - x) \cos(kx)\, dx$$
$$= \frac{2}{\pi k} \int_0^{\pi} \pi - x\, d(\sin(kx))$$
$$= \frac{2}{\pi k} \int_0^{\pi} \sin(kx)\, dx = \frac{2}{\pi k^2}[1 - (-1)^k].$$

The corresponding Fourier series is

$$f(x) \sim \frac{\pi}{2} + \frac{4}{\pi} \sum_{j=1}^{\infty} \frac{\cos((2j-1)x)}{(2j-1)^2}.$$

5. Find the full (sine and cosine) Fourier series for the function

$$f(x) = \begin{cases} 0, & -\pi < x < 0; \\ 1, & 0 < x < \pi. \end{cases}$$

Solution: We calculate

$$a_0 = \frac{1}{\pi}\int_0^\pi dx = 1,$$
$$a_k = \frac{1}{\pi}\int_0^\pi \cos(kx)\,dx = 0,\ \ k > 0,$$
$$b_k = \frac{1}{\pi}\int_0^\pi \sin(kx)\,dx = \frac{1-(-1)^k}{k}.$$

Therefore, the Fourier series is

$$f(x) \sim \frac{1}{2} + \sum_{j=1}^{\infty} \frac{2}{2j-1}\sin((2j-1)x).$$

6. Assume that $f(x)$ is an odd function on the interval $[-\pi, \pi]$. Explain why the full Fourier series of f consists only of sines (in other words, why the coefficients next to the cosines are all zero).

Solution: We have

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,dx = 0,$$

since $\cos(kx)$ is even, $f(x)$ is odd, the product $f(x)\cos(kx)$ is odd and thus its integral over $[-\pi, \pi]$ is zero.

7. Using the Fourier series convergence theorem, find the functions to which the series in problems 2, 4, and 5 converge. Sketch their graphs.

Answers: For problem 2, the Fourier series converges to the 2π-periodic extension of the function

$$g(x) = \begin{cases} x(\pi - x), & 0 \le x \le \pi; \\ -x(\pi - x), & -\pi \le x \le 0. \end{cases}$$

For problem 4, the Fourier series converges to the 2π-periodic extension of the function $\pi - |x|$ from the segment $[-\pi, \pi]$.

For problem 5, the Fourier series converges to the 2π-periodic extension of the function

$$h(x) = \begin{cases} 0, & -\pi < x < 0; \\ 1, & 0 < x < \pi; \\ 1/2, & x \in \{-\pi, 0, \pi\}. \end{cases}$$

8. Assume that f is a function on the interval $[0, \pi]$ whose graph is symmetric with respect to the line $x = \pi/2$; in other words, $f(\pi - x) = f(x)$. If

$$f(x) \sim \sum_{k=1}^{\infty} b_k \sin(kx)$$

is the Fourier sine series of f, prove that $b_k = 0$ for even k. (Hint: write out the formula for b_k and make the change of variables $y = \pi - x$.)

Solution: Making the change of variables $y = \pi - x$, we get

$$\begin{aligned} b_k &= \frac{2}{\pi} \int_0^{\pi} f(x) \sin(kx)\, dx \\ &= \frac{2}{\pi} \int_0^{\pi} f(\pi - y) \sin(k(\pi - y))\, dy \\ &= \frac{2}{\pi} \int_0^{\pi} f(y)(-1)^{k+1} \sin(ky)\, dy = (-1)^{k+1} b_k. \end{aligned}$$

Therefore, for k even, $b_k = -b_k$ and thus $b_k = 0$.