Worksheet 26: PDE and Fourier method

0. Given the functions

$$u(x,t) = e^{-t}\sin x,$$
$$v(x,t) = \cos t \sin x,$$

calculate the derivatives

$$\frac{\partial u}{\partial t},\ \frac{\partial^2 u}{\partial x^2},\ \frac{\partial^2 v}{\partial t^2},\ \frac{\partial^2 v}{\partial x^2}.$$

Explain why u solves the following initial/boundary value problem for the heat equation:

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t),\ 0 < x < \pi,\ t > 0;$$
$$u(0,t) = u(\pi,t) = 0,\ t > 0;$$
$$u(x,0) = \sin(x),\ 0 < x < \pi,$$

while v solves the following initial/boundary value problem for the wave equation:

$$\frac{\partial^2 v}{\partial t^2}(x,t) = \frac{\partial^2 v}{\partial x^2}(x,t),\ 0 < x < \pi,\ t > 0;$$
$$v(0,t) = v(\pi,t) = 0,\ t > 0;$$
$$v(x,0) = \sin x,\ \frac{\partial v}{\partial t}(x,0) = 0,\ 0 < x < \pi.$$

Describe the behavior of the functions $u(t,\cdot)$ and $v(t,\cdot)$ as time goes on.

Solution: We find

$$\frac{\partial u}{\partial t} = -e^{-t}\sin x = \frac{\partial^2 u}{\partial x^2};$$
$$\frac{\partial v}{\partial t} = -\cos t \sin x = \frac{\partial^2 v}{\partial x^2}.$$

It remains to verify the boundary and initial conditions for u and v. The shape of the profile (for fixed t and varying x) of the functions u and v stays the same (in the shape of $\sin x$); however, the function u will exponentially fast go to zero, while the function v will bounce back and forth with period 2π.

1. Find all eigenvalues λ and the corresponding eigenfunctions for the boundary value problem (see also problem 2)

$$y''(x) + \lambda y(x) = 0,\ 0 < x < \pi;$$
$$y'(0) = 0,\ y'(\pi) = 0.$$

Solution: Assume that $\{y_1(x), y_2(x)\}$ is a fundamental system of solutions to the equation $y''(x) + \lambda y(x)$. (Both y_1 and y_2 depend on λ.) The general solution is then $c_1 y_1(x) + c_2 y_2(x)$ for c_1, c_2 arbitrary constants; the boundary conditions are satisfied if the following system of equations on c_1, c_2 holds:

$$\begin{aligned} 0 = y'(0) = c_1 y_1'(0) + c_2 y_2'(0), \\ 0 = y'(\pi) = c_1 y_1'(\pi) + c_2 y_2'(\pi). \end{aligned} \tag{1}$$

This system has a nonzero solution if and only if

$$\det \begin{bmatrix} y_1'(0) & y_2'(0) \\ y_1'(\pi) & y_2'(\pi) \end{bmatrix} = 0. \tag{2}$$

Now, the auxiliary equation is $r^2 + \lambda = 0$. We consider the following cases:

Case 1: $\lambda < 0$. Put $r = \sqrt{-\lambda} > 0$. We find $y_1(x) = e^{rx}, y_2(x) = e^{-rx}$, and (2) turns into

$$r^2(e^{r\pi} - e^{-r\pi}) = 0,$$

which cannot be true for $r > 0$.

Case 2: $\lambda = 0$. We find $y_1(x) = 1$, $y_2(x) = x$, and the equation (2) is satisfied. Solving (1), we get $c_1 \in \mathbb{R}$, $c_2 = 0$; therefore, $y = 1$ is an eigenfunction for this eigenvalue.

Case 3: $\lambda > 0$. Put $s = \sqrt{\lambda} > 0$. We find $y_1(x) = \cos(sx)$, $y_2(x) = \sin(sx)$; (2) turns into

$$0 = \sin(s\pi).$$

This equation is solved for $s = k$ a positive integer; the corresponding value of λ is $\lambda = k^2$. Solving (1), we get $c_1 \in \mathbb{R}$, $c_2 = 0$; therefore, $y = \cos(kx)$ is an eigenfunction for this eigenvalue.

Therefore, the eigenvalues for our problem are $\lambda = k^2$, $k \in \mathbb{Z}$, $k \geq 0$, and the corresponding eigenfunctions are $\cos(kx)$.

2.* Prove that problem 1 has no eigenvalues $\lambda < 0$, using the following method: assume that $y(x)$ is an eigenfunction with $\lambda < 0$. Using the equation, integration by parts, and boundary conditions, show that

$$0 = \int_0^\pi (y''(x) + \lambda y(x))y(x)\,dx = \int_0^\pi -y'(x)^2 + \lambda y(x)^2\,dx.$$

Explain why this leads to a contradiction.

Solution: Assume that $y(x)$ is an eigenfunction with $\lambda < 0$. We use the integration by parts formula

$$\int_0^\pi u'(x)v(x)\,dx = u(x)v(x)|_{x=0}^\pi - \int_0^\pi u(x)v'(x)\,dx$$

for $u = y'$ and $v = y$, to get

$$\int_0^\pi y''(x)y(x)\,dx = -\int_0^\pi (y'(x))^2\,dx,$$

since $y'(x)y(x) = 0$ both at $x = 0$ and at $x = \pi$ due to boundary conditions. Combining this with the equation $y'' + \lambda y = 0$, we get

$$\int_0^\pi -y'(x)^2 + \lambda y(x)^2\,dx = 0.$$

Since $\lambda < 0$, the expression under the integral is nonpositive. Therefore, if its integral is zero, this expression is identically zero. We then get $\lambda y(x)^2 \equiv 0$; since $\lambda \neq 0$, it follows that $y(x) \equiv 0$, a contradiction with $y(x)$ being an eigenfunction.

3. Using separation of variables, solve the following initial/boundary value problem for the heat equation:

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t),\ 0 < x < \pi,\ t > 0;$$
$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(\pi,t) = 0,\ t > 0;$$
$$u(x,0) = 2 + \cos x - \cos(3x),\ 0 < x < \pi.$$

Find the limit of $u(x,t)$ as $t \to +\infty$ and explain your results from a physical point of view.

Solution: We look for solutions in the form $u = T(t)X(x)$; plugging this into the equation, we get $T'(t)X(x) = T(t)X''(x)$, or $T'/T = X''/X = -\lambda$, where λ is a constant. Now, X needs to satisfy the boundary conditions $X'(0) = X'(\pi) = 0$; in other words, it is an eigenfunction for problem 1. Therefore, $\lambda = k^2$, where k is a nonnegative integer, and we can take $X = \cos(kx)$. The corresponding function T has the form $c_k e^{-k^2 x}$ and satisfies $T(0) = c_k$.

Therefore, the function $u = 2$ solves our problem with initial data $u(x, 0) = 2$; the function $u = e^{-t}\cos x$ solves our problem with initial data $u(x, 0) = \cos x$, and the function $u = -e^{-9t}\cos(3x)$ solves our problem with initial data $u(x, 0) = -\cos(3x)$; adding these up, we get the following solution to the original problem:

$$u(x,t) = 2 + e^{-t}\cos x - e^{9t}\cos(3x).$$

The limit of this expression as $t \to +\infty$ is equal to 2; this reflects the physical observation that, once you insulate a heated rod, after a large time the temperature everywhere in the rod will become the same.

4. Using separation of variables, solve the following initial/boundary value problem for the wave equation:

$$\frac{\partial^2 u}{\partial t^2}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t),\ 0 < x < \pi,\ t > 0;$$
$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(\pi,t) = 0,\ t > 0;$$
$$u(x,0) = 1,\ \frac{\partial u}{\partial t}(x,0) = \cos x,\ 0 < x < \pi.$$

Solution: We argue similarly to the previous problem, trying to find solutions of the form $T(t)X(x)$. We get $T''/T = X''/X = -\lambda$. The equation $X''/X = -\lambda$ is solved exactly as in the previous problem, yielding $\lambda = k^2$ with $k \geq 0$ an integer. The corresponding solution to the equation $T''/T = -\lambda$ is $a_k\cos(kt) + b_k\sin(kt)$, with $a_k, b_k \in \mathbb{R}$, and it has $T(0) = a_k$, $T'(0) = kb_k$.

Therefore, the function $u = 1$ solves our problem with initial data $u(x, 0) = 1$, $\frac{\partial u}{\partial t}(x, 0) = 0$; the function $u = \sin t \cos x$ solves our problem with initial data $u(x, 0) = 0$, $\frac{\partial u}{\partial t}(x, 0) = \cos x$. Adding up these two solutions, we get the following solution to the original problem:

$$u(x,t) = 1 + \sin t \cos x.$$