Worksheet 25: Higher order linear ODE and systems of ODE

1. NS&S, 6.1.2.

Solution: The interval is $(0, \infty)$, because that's where the coefficient $\sqrt{x}$ is well-defined and continuous.

2. NS&S, 6.1.17.

Solution: We substitute $y = x^k$ into the equation to yield

$$\begin{aligned} &x^3 y''' - 3x^2 y'' + 6xy' - 6y \\ &= (k^3 - 6k^2 + 11k - 6)x^k = (k-1)(k-2)(k-3)x^k. \end{aligned}$$

Therefore, x, x^2, x^3 solve the considered equation. Next, the Wrosnkian is

$$W(x, x^2, x^3) = \det \begin{bmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{bmatrix} = 2x^3.$$

Since the Wronskian is not identically equal to zero, the functions $\{x, x^2, x^3\}$ form a linearly independent set and thus a fundamental system of solutions for our equation.

3. NS&S, 6.1.21.

Solution: (a) The general solution is $y = \ln x + c_1 x + c_2 x \ln x + c_3 x (\ln x)^2$, where $c_1, c_2, c_3 \in \mathbb{R}$ are arbitrary. (b) We find

$$\begin{gathered} y'(x) = x^{-1} + c_1 + c_2(1 + \ln x) + c_3(2 + \ln x) \ln x, \\ y''(x) = -x^{-2} + c_2 x^{-1} + 2c_3 x^{-1}(1 + \ln x); \\ 3 = y(1) = c_1, \\ 3 = y'(1) = 1 + c_1 + c_2, \\ 0 = y''(1) = -1 + c_2 + 2c_3; \end{gathered}$$

solving this system of linear equations in c_1, c_2, c_3, we find

$$c_1 = 3, \ c_2 = -1, \ c_3 = 1;$$

therefore, the solution to our initial-value problem is

$$y = \ln x + 3x - x \ln x + x(\ln x)^2.$$

4. NS&S, 6.2.2.

Solution: The auxiliary equation is $r^3 - 3r^2 - r + 3 = 0$; the roots are $r = -1, 1, 3$. The fundamental system is $\{e^{-x}, e^x, e^{3x}\}$; the general solution is $c_1 e^{-x} + c_2 e^x + c_3 e^{3x}$.

5. NS&S, 6.2.13.

Solution: The auxiliary equation is $r^4 + 4r^2 + 4 = 0$; the solutions are $r = \pm i\sqrt{2}$, both with multiplicity 2. The fundamental system is

$$\{\cos(\sqrt{2}x), \sin(\sqrt{2}x), x\cos(\sqrt{2}x), x\sin(\sqrt{2}x)\};$$

the general solution is

$$y = c_1 \cos(\sqrt{2}x) + c_2 \sin(\sqrt{2}x) + c_3 x \cos(\sqrt{2}x) + c_4 x \sin(\sqrt{2}x).$$

6. NS&S, 6.2.17.

Solution: The auxiliary equation is

$$(r+4)(r-3)(r+2)^3(r^2+4r+5)^2 r^5 = 0;$$

the roots are $r = -4, 3$ (multiplicity 1), $r = -2+i, -2-i$ (multiplicity 2), $r = -2$ (multiplicity 3), $r = 0$ (multiplicity 5); the fundamental system is

$$\{e^{-4x}, e^{3x}, e^{-2x}\cos x, e^{-2x}\sin x, xe^{-2x}\cos x, xe^{-2x}\sin x,$$
$$e^{-2x}, xe^{-2x}, x^2e^{-2x}, 1, x, x^2, x^3, x^4\}.$$

7.* Draw the trajectories $(y(t), y'(t))$ on the plane $\mathbb{R}^2$, where $y(t)$ solves the following initial-value problems:

(a) $y'' - y = 0$, $y(0) = 1$, $y'(0) = 0$;
(b) $y'' - y = 0$, $y(0) = 1$, $y'(0) = 1$;
(c) $y'' - y = 0$, $y(0) = 1$, $y'(0) = -1$;

(d) $y'' + y = 0$, $y(0) = 1$, $y'(0) = 0$;
(e) $y'' + y = 0$, $y(0) = 0$, $y'(0) = 0$;
(f) $y'' + 2y' + 2y = 0$, $y(0) = 1$, $y'(0) = 0$.

Solution: (a) We have

$$\begin{bmatrix} y(t) \\ y'(t) \end{bmatrix} = e^t \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} + e^{-t} \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix};$$

since $e^t \cdot e^{-t} = 1$, the trajectory, taken in the coordinate system induced by the vectors $(1/2, 1/2)$ and $(-1/2, -1/2)$, lies on a hyperbola.

(b) We have

$$\begin{bmatrix} y(t) \\ y'(t) \end{bmatrix} = e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix};$$

the trajectory consists of all positive multiples of the vector $(1, 1)$ and thus is a ray.

(c) We have

$$\begin{bmatrix} y(t) \\ y'(t) \end{bmatrix} = e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix};$$

the trajectory is a ray, as in (b).

(d) We have

$$\begin{bmatrix} y(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix};$$

since $\cos^2 t + \sin^2 t = 1$ for all t, the trajectory is a circle.

(e) We have

$$\begin{bmatrix} y(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix};$$

the trajectory is a single point. (This is called a stationary point.)

(f) We have

$$\begin{bmatrix} y(t) \\ y'(t) \end{bmatrix} = e^t \cos t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^t \sin t \begin{bmatrix} -1 \\ -2 \end{bmatrix}.$$

The coordinates of this vector with respect to the basis $\{(1, 0), (-1, -2)\}$ are $\vec{v}(t) = (e^t \cos t, e^t \sin t)$. We can write $\vec{v}(t) = e^t(\cos t, \sin t)$; we see that its length is e^t, while its polar angle is t. Therefore, the trajectory of $\vec{v}(t)$, and thus of the original vector, is a logarithmic spiral.

8. NS&S, 9.1.2.

Answer:

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

9. NS&S, 9.1.8.
Solution: We divide the equation by $1 - t^2$ to get

$$y'' - \frac{2t}{1-t^2} y' + \frac{2}{1-t^2} y = 0;$$

we can rewrite it as

$$\begin{bmatrix} y \\ y' \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -\frac{2}{1-t^2} & \frac{2t}{1-t^2} \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix}.$$

10. NS&S, 9.1.11.
Answer:

$$\begin{bmatrix} x \\ x' \\ y \\ y' \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -3 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ x' \\ y \\ y' \end{bmatrix}.$$

11. NS&S, 9.4.20.
Solution: We compute the Wronskian:

$$W = \det \begin{bmatrix} 3e^{-t} & e^{4t} \\ 2e^{-t} & -e^{4t} \end{bmatrix} = -5e^{3t};$$

since it is not identically zero, the given vectors form a fundamental solution set and the general solution is given by

$$c_1 e^{-t} \begin{bmatrix} 3 \\ 2 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

12. NS&S, 9.5.12.
Solution: The characteristic polynomial is $\lambda^2 - 2\lambda - 35$; the eigenvalues are $-5, 7$. A basis for the eigenspace for $\lambda = -5$ is $\{(-1, 2)\}$; a basis for the eigenspace for $\lambda = 7$ is $\{(1, 2)\}$. Therefore, a fundamental system of solutions is

$$\left\{ e^{-5t} \begin{bmatrix} -1 \\ 2 \end{bmatrix}, e^{7t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}.$$

13. NS&S, 9.5.17.

Solution: (a) We verify that $A\vec{u}_1 = 2\vec{u}_1$ and $A\vec{u}_2 = -2\vec{u}_2$. (As a harder exercise, try to prove that the matrix $A/2$ is actually the standard matrix of a certain reflection in $\mathbb{R}^2$.)

(b) The solution is $-e^{2t}\vec{u}_1$. The trajectory consists of all negative multiples of $\vec{u}_1$ and is a ray (starting from the origin, but not containing it).

(c) Similar to (b).

(d) The solution is $-e^{2t}\vec{u}_1 + e^{-2t}\vec{u}_2$; the coordinates of a point on the trajectory in the basis $\{\vec{u}_1, \vec{u}_2\}$ are $(e^{2t}, e^{-2t}) = (f(t), g(t))$. Since $f(t) \cdot g(t) = 1$ for all t, we see that these coordinates lie on a hyperbola.

14. NS&S, 9.5.31.

Solution: The eigenvalues are $-2, 4$; a basis of the eigenspace for $\lambda = -2$ is $\{(-1, 1)\}$ and a basis of the eigenspace for $\lambda = 4$ is $\{(1, 1)\}$. The general solution is

$$y = c_1 e^{-2t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Plugging in the initial condition, we get

$$c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix};$$

solving this vector equation, we get $c_1 = -1, c_2 = 2$; the solution to the initial value problem is

$$y = e^{-2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + e^{4t} \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

15.* NS&S, 9.5.37.

Solution: (a) Direct verification, using the fact that A is upper triangular.

(b) Take $\vec{u}_1 = (1, 0, 0)$; then $A\vec{u}_1 = 2\vec{u}_1$. We calculate

$$\vec{x}_1'(t) = (e^{2t}\vec{u}_1)' = 2e^{2t}\vec{u}_1 = 2\vec{x}_1(t) = A\vec{x}_1(t).$$

Therefore, $\vec{x}_1(t)$ is a solution to the equation $\vec{x}' = A\vec{x}$.

(c) Using that $A\vec{u}_1 = 2\vec{u}_1$, we get

$$\vec{x}_2'(t) = 2te^{2t}\vec{u}_1 + e^{2t}(\vec{u}_1 + 2\vec{u}_2);$$
$$A\vec{x}_2(t) = 2te^{2t}\vec{u}_1 + e^{2t}A\vec{u}_2;$$

for these two vector-valued functions to be equal, we need

$$\vec{u}_1 + 2\vec{u}_2 = A\vec{u}_2,$$

or $(A-2I)\vec{u}_2 = \vec{u}_1$. We solve this equation for $\vec{u}_2$ (recalling that $\vec{u}_1 = (1, 0, 0)$) and get

$$\vec{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad c \in \mathbb{R}.$$

We are free to choose any value of c (this will add a multiple of the solution found in (b) to our solution); we pick $c = 0$ and then $\vec{u}_2 = (0, 1, 0)$.

(d) Using that $A\vec{u}_1 = 2\vec{u}_1$ and $A\vec{u}_2 = 2\vec{u}_2 + \vec{u}_1$, we get

$$\begin{aligned} \vec{x}_3'(t) &= t^2 e^{2t}\vec{u}_1 + te^{2t}(\vec{u}_1 + 2\vec{u}_2) + e^{2t}(\vec{u}_2 + 2\vec{u}_3); \\ A\vec{x}_3(t) &= t^2 e^{2t}\vec{u}_1 + te^{2t}(2\vec{u}_2 + \vec{u}_1) + e^{2t}A\vec{u}_3; \end{aligned}$$

for these two vector-valued functions to be equal, we need

$$\vec{u}_2 + 2\vec{u}_3 = \vec{u}_3,$$

or $(A - 2I)\vec{u}_3 = \vec{u}_2$. Similarly to the above, we can solve for $\vec{u}_3$; one possible solution is $\vec{u}_3 = (0, 0, 1)$.