Worksheet 24: Second order linear ODE

We will work with the space $C^\infty(\mathbb{R})$, which consists of functions $y : \mathbb{R} \to \mathbb{R}$ that have derivatives of all orders. (We call such functions **smooth**.)

1–3. Write the general solution for each of the following equations:

$$y'' + 2y' = 0, \tag{1}$$
$$y'' + 2y' + y = 0, \tag{2}$$
$$y'' + 2y' + 2y = 0. \tag{3}$$

Answers: (1) $c_1 + c_2 e^{-2t}$ (2) $c_1 e^{-t} + c_2 t e^{-t}$ (3) $c_1 e^{-t} \cos t + c_2 e^{-t} \sin t$.

4. Use the Wronskian to prove that the functions

$$y_1(t) = 1,\ y_2(t) = e^{-2t} \tag{4}$$

are linearly independent as elements of $C^\infty(\mathbb{R})$.

Solution: We compute

$$W(y_1, y_2)(t) = \det \begin{bmatrix} 1 & e^{-2t} \\ 0 & -2e^{-2t} \end{bmatrix} = -2e^{-2t}.$$

Since it is nonzero, the functions y_1 and y_2 are linearly independent.

5.* Define the linear transformation $T : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ by the formula

$$T(y) = y'' + 2y',\ y \in C^\infty(\mathbb{R}). \tag{5}$$

(a) Explain why the set of all smooth solutions to the equation (1) is equal to the kernel $\operatorname{Ker} T$. Conclude that it is a subspace of $C^\infty(\mathbb{R})$.

(b) Explain why the set $\{1, e^{-2t}\}$ is a basis of $\operatorname{Ker} T$. Find the dimension of $\operatorname{Ker} T$.

(c)* Use Theorem 4.2.1 to prove that the linear transformation $S : \operatorname{Ker} T \to \mathbb{R}^2$ defined by

$$S(y) = \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix},\ y \in \operatorname{Ker} T,$$

is invertible.

Solution: (a) The kernel $\operatorname{Ker} T$ consists of all functions $y \in C^\infty(\mathbb{R})$ such that $T(y) = 0$; the latter is exactly the equation (1). The kernel of any linear transformation is a subspace.

(b) The set $\{1, e^{-2t}\}$ is linearly independent, by problem 4. It spans $\operatorname{Ker} T$, by problem 1. Therefore, it is a basis of $\operatorname{Ker} T$ and the dimension of $\operatorname{Ker} T$ is equal to 2.

(c) Stating that $S(y)$ is invertible is the same as saying that it is 1-to-1 and onto; this means that for every $(Y_0, Y_1) \in \mathbb{R}^2$, there exists unique $y \in \operatorname{Ker} T$ such that $S(y) = (Y_0, Y_1)$. Recalling what the transformations S and T are, this can be reformulated as follows: for every $Y_0, Y_1 \in \mathbb{R}$, there exists a unique solution y to the equation (1) such that $y(0) = Y_0$ and $y(1) = Y_1$.

6. Solve the initial value problem for the equation (1), with the initial conditions

$$y(0) = 0,\ y'(0) = 1. \tag{6}$$

Solution: The general solution for (1) is $y = c_1 + c_2 e^{-2t}$; the initial conditions yield

$$\begin{aligned} 0 &= y(0) = c_1 + c_2, \\ 1 &= y'(0) = -2c_2. \end{aligned}$$

Solving this system of linear equations, we find $c_1 = 1/2$, $c_2 = -1/2$, and $y = (1 - e^{-2t})/2$.

7. Solve the boundary value problem for the equation

$$y'' + y = 0 \tag{7}$$

with the boundary conditions

$$y(0) = 1,\ y(\pi/2) = 0.$$

Solution: The general solution for (7) is $y = c_1 \cos t + c_2 \sin t$; the boundary conditions yield

$$1 = y(0) = c_1,\ 0 = y(1) = c_2.$$

Therefore, $y = \cos t$.

8.* Find all values of $T \in \mathbb{R}$ for which the boundary value problem for the equation (7) with the conditions

$$y(0) = 0,\ y(T) = 0 \tag{8}$$

has a nontrivial (nonzero) solution.

Solution: The general solution for (7) is $y = c_1 \cos t + c_2 \sin t$; the boundary conditions yield

$$0 = y(0) = c_1, \;\; 0 = y(T) = c_1 \cos T + c_2 \sin T.$$

Substituting $c_1 = 0$ into the second equation, we get $0 = c_2 \sin T$. A nontrivial solution to the system above exists if and only if $\sin T = 0$; that is, if $T = \pi k$ for some integer k.

9. NS&S, 4.4.27.

Answer: $(A_3t^3 + A_2t^2 + A_1t + A_0)t\cos(3t) + (B_3t^3 + B_2t^2 + B_1t + B_0)t\sin(3t)$.

10. Determine the form of a trial solution to the following equation. Do not solve.

$$y'' + 2y' + y = \cos^2 t + te^{-t}.$$

Answer: We write $\cos^2 t = (1 + \cos(2t))/2$; then the trial solution is $A + B\cos(2t) + C\sin(2t) + (D_1t + D_2)te^{-t}$.

11. Find the general solution to the equation

$$y'' - y = e^t + \cos t.$$

Solution: The general solution to the corresponding homogeneous equation is $c_1e^t + c_2e^{-t}$. The trial solution is

$$y = Ate^t + B\cos t + C\sin t;$$

we find

$$y'' - y = 2Ae^t - 2B\cos t - 2C\sin t;$$

therefore, $A = 1/2, B = -1/2, C = 0$, and the general solution to the inhomogeneous equation is

$$y = \frac{1}{2}te^t - \frac{1}{2}\cos t + c_1e^t + c_2e^{-t}.$$

12.* This problem provides an explanation of the method of undetermined coefficients using abstract vector spaces. Consider for example the equation

$$y'' - 2y' + y = \cos t.$$

(a) Let V be the subspace of $C^\infty(\mathbb{R})$ consisting of all functions of the form

$$A\cos t + B\sin t,\ \ A, B \in \mathbb{R}.$$

Prove that $\mathcal{B} = \{\cos t, \sin t\}$ is a basis of V.

(b) Show that for each $y \in V$, the function $y'' - 2y' + y$ lies in V. (This property of exponentials, polynomials, and trigonometric functions is what actually determines which right-hand sides the method of undetermined coefficients can handle.)

(c) Define the linear transformation $T : V \to V$ by the formula $T(y) = y'' - 2y' + y$. Find the matrix A of T in the basis $\mathcal{B}$.

(d) Show that the matrix A is invertible. Use coordinate vectors to find $y \in V$ solving the equation $T(y) = \cos t$.

Solution: (a) $\mathcal{B}$ is linearly independent, for example by Wronskian computation. It spans V by the definition of V.

(b) A direct computation shows that for each $y \in V$, its derivative lies in V. Using this fact twice, we get that $y'' \in V$; since V is a subspace, $y'' - 2y' + y \in V$ as a linear combination of y, y', y''.

(c) We have $T(\cos t) = 2\sin t$, $T(\sin t) = -2\cos t$; therefore,

$$A = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}.$$

(d) The matrix A is invertible since $\det A = 4 \neq 0$, and the equation $T(y) = \cos t$ is equivalent to

$$A[y]_{\mathcal{B}} = [\cos t]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Solving this, we find $[y]_{\mathcal{B}} = (0, -1/2)$ and thus $y = -1/2 \sin t$.